# PERIODIC SOLUTIONS IN CERTAIN CLASS OF 3- DIMENSION DISCONTINUOUS AUTONOMOUS SYSTEMS

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Abstract. In the present paper the linear oscillator in $R^3$ with $z$ =constant has been considered. The aim is to determine the necessary conditions for the persistence of periodic solutions under discontinuous perturbations. A new approach based on a computational method has been used. At the end we apply our method on an example.

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## 1. Introduction

Over the years, there has been growing interest and need for the modeling, analysis and control of non-smooth dynamical systems characterized by discontinuous charges in system properties. A particular case is represented by dynamical system, discontinuous with respect to the state variables. The first studies on discontinuous differential equations with based on geometrical theory, referred to Filippov ([1]). Here we use the definitions and notations that was used in [4]. We are faced with two different field, one of them is continuation theory and another is differential equations with discontinuous right hand side. Both fields have a large share in

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mathematical researches, for example see [2, 3]. For the purpose of this paper we need to introduce the Filippov's theory which we give in the follow:
Suppose that $D$ is a domain in $R^n$ and $h : D \rightarrow R$ whit $h \in C^r(D, R), r \geq 1$. Defining $V_1$ and $V_2$ and $\Sigma$ as:

$$V_1 = \{x \in R^n | h(x) < 0\}, \qquad V_2 = \{x \in R^n | h(x) > 0\},$$

and

$$\Sigma = \{x \in R^n | h(x) = 0\}.$$

Let $f_1 : D \rightarrow R^n$ and $f_2 : D \rightarrow R^n$ be $C^r, r \geqslant 1$, functions. we consider the differential equation:

$$\dot{x} = f(x), \qquad x(t_0) = x_0. \tag{1.1}$$

where:

(1) $f$ has the form

$$f(x) = \begin{cases} f_1(x) & x \in V_1 \\ f_2(x) & x \in V_2 \end{cases}$$

(2) The normal of the plane $\Sigma$, given by $n(x) = [Dh(x)]^T$ is chosen such that it always hold that $n(x) \neq 0$, for all $x \in \Sigma$.

(3) There exist functions $g_i : D_i \rightarrow R^n$ for $i = 1, 2$ with following properties:
   (a) $V_i \cup \Sigma \subseteq D_i$ for $i = 1, 2$
   (b) $g_1, g_2 \in C^r, r \geq 1$
   (c) $g_i = f_i$ on $V_i$ for $i = 1, 2$.

Fillipov has been shown that system (1.1) with above properties has a solution in sense of Fillipov definition of solution.

**Definition 1.1.** *Function $x : I \rightarrow R^2$ where $I$ is an interval in $R$ is a solution of differential inclusion $\dot{x} \in F(x)$ if $x$ be almost everywhere continuous and $\dot{x}(t) \in F(x(t))$ for almost all $t \in I$, where*

$$F(x) = \begin{cases} \{f_1(x)\} & x \in V_1 \\ \{(1-\lambda) f_1(x) + \lambda f_2(x) | \lambda \in (0, 1)\} & x \in \Sigma \\ \{f_2(x)\} & x \in V_2 \end{cases} \tag{1.2}$$

**Definition 1.2.** *Let $x : I \to R^n$ be a solution of system (1.1) and $x_\Sigma \in \Sigma$ be a point such that $x(t_\Sigma) = x_\Sigma$ for some $t_\Sigma \in I$, we say that the solution $x(t)$ crosses the hypersurface $\Sigma$ transversally at $x_\Sigma$ if*

$$n^T(x_\Sigma).g_1(x_\Sigma)n^T(x_\Sigma).g_2(x_\Sigma) > 0. \tag{1.3}$$

In fact the condition (1.3) is a necessary condition for transversal intersection, that in [4] considered as a definition for transversally.
(For more information about Fillipov's theory see [1]).

In this paper we consider periodic solutions of linear oscillator in the plain $z =$ *constant* and then we study the effect of 3-dimension discontinuous damping on their periodic solutions and introduce a method for persistence of some periodic solutions, also our method can compute the value of period of the periodic solution of perturbed system. This paper extend the B.Mehri's paper ([5]) for introducing the necessary conditions for existence of periodic solutions for discontinuous perturbed systems. In fact they use Implicit function theorem and use the Taylor expansion to finding their necessary conditions for existence of periodic solutions, here we use their idea to introduce a method for analyze a class of three dimension discontinuous system which has a pure imaginary pair and a simple zero eigenvalue in the linear part.
Consider three dimension discontinuous system

$$\begin{cases} \dot{x} = -y + P(x, y, z) \\ \dot{y} = x + Q(x, y, z) \\ \dot{z} = R(x, y, z) \end{cases} \tag{1.4}$$

where

$$P(x, y, z) = \begin{cases} P_1(x, y, z) & y > 0; \\ P_2(x, y, z) & y < 0. \end{cases}, \quad Q(x, y, z) = \begin{cases} Q_1(x, y, z) & y > 0; \\ Q_2(x, y, z) & y < 0. \end{cases}$$

$$R(x, y, z) = \begin{cases} R_1(x, y, z) & y > 0; \\ R_2(x, y, z) & y < 0. \end{cases}$$

and

$$\mathcal{P}_i^j(x, y, z) = r_{1,i}^j x^2 + r_{2,i}^j y^2 + r_{3,i}^j z^2 + r_{4,i}^j xy + r_{5,i}^j xz + r_{6,i}^j yz + r_{7,i}^j x^3 + r_{8,i}^j y^3 + r_{9,i}^j z^3$$

$$+r^j_{10,i}xy^2+r^j_{11,i}x^2y+r^j_{12,i}xz^2+r^j_{13,i}x^2z+r^j_{14,i}yz^2+r^j_{15,i}y^2z+r^j_{16,i}xyz,$$

for $j=1,2,3$ and $i=1,2$ where

$$\mathcal{P}^1=P,\qquad \mathcal{P}^2=Q,\qquad \mathcal{P}^3=R,\qquad r^1=\alpha,\qquad r^2=\beta,\qquad r^3=\gamma.$$

In [5] M.Bayat and B.Mehri consider smooth $P,Q$ and $R$ and found a necessary condition for the existence of periodic solution, but in this paper we consider $\begin{pmatrix} P & Q & R \end{pmatrix}^T$ as a discontinuous perturbation and our aim is to study the persistence of periodic solutions of system of linear oscillator

$$\begin{cases} \dot{x}=-y,\\ \dot{y}=x,\\ \dot{z}=0 \end{cases}$$

under discontinuous perturbation.
Our method also can compute the period of the periodic solutions. Moreover when the perturbation terms are complicated then the analytical methods don't have any performance, so in these cases the numerical methods will be important. In fact this is in our problem, so we must use the numerical method. Therefore in the next section we will introduce our method and will give Maple code for software computations.

## 2. The method

The method is based on implicit function theorem and the work of M. Bayat and B. Mehri ([5]), in fact by knowing the initial and final point of a part of a solution we make a system of nonlinear equations and research for fixed points of it, the existence of fixed points are equal to existence of periodic solutions. Moreover the time of period can be computed.
Obviously, the linear system in (1.4) is smooth and has only periodic solutions which are circles with centers are on the z-axis lying in a plain $z=constant$. Our approach is to assume that the full system has a periodic solution close to a circular orbit of the linear system in the plain $z=0$. Therefore, we use the implicit function theorem to establish the necessary condition in order the system (1.4) has closed solutions in the neighborhood of the origin.

Let $\varphi(0,\xi,\eta)(t)$ be that solution of (1.4) which has initial point $x=\xi, y=0, z=0$ at $t=0$ and end point $x=\eta, y=0, z=0$ at $t=\pi+\tau_1$. After a time approximately $\pi$, this solution reach to plain $\Sigma : y=0$. Therefore we must consider two case:

(1) Intersect transversally $\varphi(0,\xi,\eta)$ and $\Sigma$;
(2) Sliding mode,

In this paper we only consider first cases.

2.1. **Transversal case.** Here we consider case transversal intersection. By assuming transversal intersection of $\varphi$ by $\Sigma$, this solution will have made one cycle around the origin and will reach the point $(\xi,0,0)$ provided that the following equations are satisfied:

$$\begin{cases} F_1(\tau_1,\xi,\eta) = x_1(\pi+\tau_1,\xi,\eta) - \eta = 0, \\ G_1(\tau_1,\xi,\eta) = y_1(\pi+\tau_1,\xi,\eta) = 0, \\ H_1(\tau_1,\xi,\eta) = z_1(\pi+\tau_1,\xi,\eta) = 0, \end{cases} \tag{2.1}$$

$$\begin{cases} F_2(\tau_2,\xi,\eta) = x_2(\pi+\tau_2,\xi,\eta) - \xi = 0, \\ G_2(\tau_2,\xi,\eta) = y_2(\pi+\tau_2,\xi,\eta) = 0, \\ H_2(\tau_2,\xi,\eta) = z_2(\pi+\tau_2,\xi,\eta) = 0, \end{cases} \tag{2.2}$$

where $\varphi(\xi,\eta)(\pi+\tau_1) = (\eta,0,0) \in \Sigma$ and

$$\varphi(\xi,\eta)(t) = \begin{cases} (x_1(t,\xi,\eta), y_1(t,\xi,\eta), z_1(t,\xi,\eta)), & 0 \le t < \pi+\tau_1; \\ (x_2(t,\xi,\eta), y_2(t,\xi,\eta), z_2(t,\xi,\eta)), & \pi+\tau_1 \le t < 2\pi+\tau_1+\tau_2. \end{cases}$$

Also we must have

$$(\gamma_{1,1}\eta^2 + \gamma_{7,1}\eta^3)(\gamma_{1,2}\eta^2 + \gamma_{7,2}\eta^3) > 0$$

If we solve the system (2.1) by assuming that $\tau_1$ and $\eta$ are functions of $\xi$, say $\tau_1 = \tau_1(\xi)$ and $\eta = \eta(\xi)$, then the solution curve $\varphi(\xi,\eta)$, is closed if and only if

$$\begin{cases} F_2(\tau_2,\eta(\xi),\xi) = 0, \\ G_2(\tau_2,\eta(\xi),\xi) = 0, \\ H_2(\tau_2,\eta(\xi),\xi) = 0, \end{cases} \tag{2.3}$$

The period of this solution is $2\pi+\tau_1+\tau_2$. Hence we express the following theorem:

**Theorem 2.1.** *The necessary conditions for existence of periodic solution for discontinuous perturbed system (1.4) are equal to existence of solution* $(\xi, \tau_2)$ *for system (2.3).*

Therefore, it is important to investigate the asymptotic behavior of $F_2(\tau_2, \eta(\xi), \xi) = 0, G_2(\tau_2, \eta(\xi), \xi) = 0, H_2(\tau_2, \eta(\xi), \xi) = 0$ as $\xi \to 0$. Since the polynomials $P_i, Q_i, R_i$ for $i = 1, 2$ are of degree 3, the behavior will be done by computing the first three derivatives of $F_i(\tau_i, \xi, \eta), G_i(\tau_i, \xi, \eta)$ and $H_i(\tau_i, \xi, \eta)$ at $\xi = 0$.
Since $(0, 0, 0)$ is a fixed point for the system (1.4) so

$$F_1(0,0,0) = G_1(0,0,0) = H_1(0,0,0) = 0.$$

**Lemma 2.2.** *The first derivatives of* $F_1, G_1, H_1$ *at* $(0, 0, 0)$ *are given by*

$$\begin{array}{ll}
\frac{\partial F_1}{\partial \tau_1}(0,0,0) = \dot{x_1}(\pi,0,0) = 0, & \frac{\partial F_1}{\partial \xi}(0,0,0) = \frac{\partial x_1}{\partial \xi}(\pi,0,0), \\
 & \frac{\partial F_1}{\partial \eta}(0,0,0) = \frac{\partial x_1}{\partial \eta}(\pi,0,0) - 1 \\
\frac{\partial G_1}{\partial \tau_1}(0,0,0) = \dot{y_1}(\pi,0,0) = 0, & \frac{\partial G_1}{\partial \xi}(0,0,0) = \frac{\partial y_1}{\partial \xi}(\pi,0,0), \\
 & \frac{\partial G_1}{\partial \eta}(0,0,0) = \frac{\partial y_1}{\partial \eta}(\pi,0,0) \\
\frac{\partial H_1}{\partial \tau_1}(0,0,0) = \dot{z_1}(\pi,0,0) = 0, & \frac{\partial H_1}{\partial \xi}(0,0,0) = \frac{\partial z_1}{\partial \xi}(\pi,0,0), \\
 & \frac{\partial H_1}{\partial \eta}(0,0,0) = \frac{\partial z_1}{\partial \eta}(\pi,0,0)
\end{array}$$

*For convenience we put* $x_1 = x, y_1 = y$ *and* $z_1 = z$.
*Also the derivative of* $x_\xi, y_\xi$ *and* $z_\xi$ *satisfy*

$$\left\{\begin{array}{l}
\dot{x}_\xi = -y_\xi + \frac{\partial P_1}{\partial x}(x,y,z)x_\xi + \frac{\partial P_1}{\partial y}(x,y,z)y_\xi + \frac{\partial P_1}{\partial z}(x,y,z)z_\xi \\
\dot{y}_\xi = x_\xi + \frac{\partial Q_1}{\partial x}(x,y,z)x_\xi + \frac{\partial Q_1}{\partial y}(x,y,z)y_\xi + \frac{\partial Q_1}{\partial z}(x,y,z)z_\xi \\
\dot{z}_\xi = \frac{\partial R_1}{\partial x}(x,y,z)x_\xi + \frac{\partial R_1}{\partial y}(x,y,z)y_\xi + \frac{\partial R_1}{\partial z}(x,y,z)z_\xi
\end{array}\right. \tag{2.4}$$

*whit the initial condition* $x_\xi = 1, y_\xi = 0$ *and* $z_\xi = 0$ *at* $t = 0$ *and*

$$\left\{\begin{array}{l}
\dot{x}_\eta = -y_\eta + \frac{\partial P_1}{\partial x}(x,y,z)x_\eta + \frac{\partial P_1}{\partial y}(x,y,z)y_\eta + \frac{\partial P_1}{\partial z}(x,y,z)z_\eta \\
\dot{y}_\eta = x_\eta + \frac{\partial Q_1}{\partial x}(x,y,z)x_\eta + \frac{\partial Q_1}{\partial y}(x,y,z)y_\eta + \frac{\partial Q_1}{\partial z}(x,y,z)z_\eta \\
\dot{z}_\eta = \frac{\partial R_1}{\partial x}(x,y,z)x_\eta + \frac{\partial R_1}{\partial y}(x,y,z)y_\eta + \frac{\partial R_1}{\partial z}(x,y,z)z_\eta
\end{array}\right. \tag{2.5}$$

*whit the initial condition* $x_\eta = 1, y_\eta = 0$ *and* $z_\eta = 0$ *at* $t = \pi$.

**Proof:** The proof is easy, it is sufficient to differentiate (1.4) and (2.1) with respect to suitable variables.□

Putting $\eta = \xi = 0$ in systems (2.4) and (2.5) we get

$$\begin{cases} \dot{x}_\xi = -y_\xi \\ \dot{y}_\xi = x_\xi \\ \dot{z}_\xi = 0 \end{cases}, \qquad \begin{cases} \dot{x}_\eta = -y_\eta \\ \dot{y}_\eta = x_\eta \\ \dot{z}_\eta = 0 \end{cases}$$

Then with the above initial conditions, we have

$$\begin{array}{lll} x_\xi(t,0,0) = \cos(t), & y_\xi(t,0,0) = \sin(t), & z_\xi(t,0,0) = 0 \\ x_\eta(t,0,0) = -\cos(t), & y_\eta(t,0,0) = -\sin(t), & z_\eta(t,0,0) = 0 \end{array}$$

Hence,

$$\begin{array}{lll} \frac{\partial F_1}{\partial \xi}(0,0,0) = -1, & \frac{\partial G_1}{\partial \xi}(0,0,0) = 0, & \frac{\partial H_1}{\partial \xi}(0,0,0) = 0 \\ \frac{\partial F_1}{\partial \eta}(0,0,0) = 0, & \frac{\partial G_1}{\partial \eta}(0,0,0) = 0, & \frac{\partial H_1}{\partial \eta}(0,0,0) = 0 \end{array}$$

**Lemma 2.3.** *The second derivatives of $F_1, G_1$ and $H_1$ at $(0,0,0)$ are as follows*

(2.6)

$$\begin{array}{lll} \frac{\partial^2 F_1}{\partial \tau_1 \partial \xi}(0,0,0) = 0, & \frac{\partial^2 F_1}{\partial \tau_1 \partial \eta}(0,0,0) = 0, & \frac{\partial^2 F_1}{\partial \xi \partial \eta}(0,0,0) = \frac{\partial^2 x_1}{\partial \xi \partial \eta}(\pi,0,0) \\ \frac{\partial^2 G_1}{\partial \tau_1 \partial \xi}(0,0,0) = -1, & \frac{\partial^2 G_1}{\partial \tau_1 \partial \eta}(0,0,0) = 1, & \frac{\partial^2 G_1}{\partial \xi \partial \eta}(0,0,0) = \frac{\partial^2 y_1}{\partial \xi \partial \eta}(\pi,0,0) \\ \frac{\partial^2 H_1}{\partial \tau_1 \partial \xi}(0,0,0) = 0, & \frac{\partial^2 H_1}{\partial \tau_1 \partial \eta}(0,0,0) = 0, & \frac{\partial^2 H_1}{\partial \xi \partial \eta}(0,0,0) = \frac{\partial^2 z_1}{\partial \xi \partial \eta}(\pi,0,0) \\ \frac{\partial^2 F_1}{\partial \tau_1^2}(0,0,0) = 0, & \frac{\partial^2 F_1}{\partial \xi^2}(0,0,0) = \frac{\partial^2 x_1}{\partial \xi^2}(\pi,0,0), & \frac{\partial^2 F_1}{\partial \eta^2}(0,0,0) = \frac{\partial^2 x_1}{\partial \eta^2}(\pi,0,0), \\ \frac{\partial^2 G_1}{\partial \tau_1^2}(0,0,0) = 0, & \frac{\partial^2 G_1}{\partial \xi^2}(0,0,0) = \frac{\partial^2 y_1}{\partial \xi^2}(\pi,0,0), & \frac{\partial^2 G_1}{\partial \eta^2}(0,0,0) = \frac{\partial^2 y_1}{\partial \eta^2}(\pi,0,0) \\ \frac{\partial^2 H_1}{\partial \tau_1^2}(0,0,0) = 0, & \frac{\partial^2 H_1}{\partial \xi^2}(0,0,0) = \frac{\partial^2 z_1}{\partial \xi^2}(\pi,0,0), & \frac{\partial^2 H_1}{\partial \eta^2}(0,0,0) = \frac{\partial^2 z_1}{\partial \eta^2}(\pi,0,0) \end{array}$$

*Also the derivatives of $x_{\xi\eta}, y_{\xi\eta}$ and $z_{\xi\eta}$ satisfy*

(2.7)

$$\begin{cases} \dot{x}_{\xi\eta} = -y_{\xi\eta} - 2\alpha_{1,1}\cos^2(t) + (-2\alpha_{4,1} - \alpha_{6,1})\sin(t)\cos(t) - 2\alpha_{2,1}\sin^2(t) \\ \dot{y}_{\xi\eta} = x_{\xi\eta} - 2\beta_{1,1}\cos^2(t) + (-2\beta_{4,1} - \beta_{6,1})\sin(t)\cos(t) - 2\beta_{2,1}\sin^2(t) \\ \dot{z}_{\xi\eta} = -2\gamma_{1,1}\cos^2(t) + (-2\gamma_{4,1} - \gamma_{6,1})\sin(t)\cos(t) - 2\gamma_{2,1}\sin^2(t) \end{cases}$$

*with initial condition $x_{\xi\eta} = y_{\xi\eta} = z_{\xi\eta} = 0$ at $t = 0$ and*
*The derivatives of $x_{\xi\xi}, y_{\xi\xi}$ and $z_{\xi\xi}$ satisfy*

$$\begin{cases} \dot{x}_{\xi\xi} = -y_{\xi\xi} + 2\alpha_{1,1}\cos^2(t) + 2\alpha_{4,1}\sin(t)\cos(t) + 2\alpha_{2,1}\sin^2(t) \\ \dot{y}_{\xi\xi} = x_{\xi\xi} + 2\beta_{1,1}\cos^2(t) + 2\beta_{4,1}\sin(t)\cos(t) + 2\beta_{2,1}\sin^2(t) \\ \dot{z}_{\xi\xi} = 2\gamma_{1,1}\cos^2(t) + 2\gamma_{4,1}\sin(t)\cos(t) + 2\gamma_{2,1}\sin^2(t) \end{cases} \tag{2.8}$$

*with initial condition* $x_{\xi\xi} = y_{\xi\xi} = z_{\xi\xi} = 0$ *at* $t = 0$.
*The derivatives of* $x_{\eta\eta}, y_{\eta\eta}$ *and* $z_{\eta\eta}$ *satisfy*

$$(2.9) \qquad \begin{cases} \dot{x}_{\eta\eta} = -y_{\eta\eta} + 2\alpha_{1,1}\cos^2(t) + 2\alpha_{4,1}\sin(t)\cos(t) + 2\alpha_{2,1}\sin^2(t) \\ \dot{y}_{\eta\eta} = x_{\eta\eta} + 2\beta_{1,1}\cos^2(t) + 2\beta_{4,1}\sin(t)\cos(t) + 2\beta_{2,1}\sin^2(t) \\ \dot{z}_{\eta\eta} = 2\gamma_{1,1}\cos^2(t) + 2\gamma_{4,1}\sin(t)\cos(t) + 2\gamma_{2,1}\sin^2(t) \end{cases}$$

*with initial condition* $x_{\xi\xi} = y_{\xi\xi} = z_{\xi\xi} = 0$ *at* $t = \pi$.

**Proof:** Like lemma (2.2)The proof is easy, it is sufficient to differentiate (1.4) and (2.1) with respect to suitable variables.□

The systems (2.7), (2.8) and (2.9), can be solved easily because all of them are in the form $\dot{X} = AX + b(t)$. So by formulas in (2.6) we have

$$\begin{gathered} \frac{\partial^2 H_1}{\partial\xi\partial\eta}(0,0,0) = -\pi(\gamma_{1,1} + \gamma_{2,1}), \\ \frac{\partial^2 G_1}{\partial\xi\partial\eta}(0,0,0) = \frac{-4}{3}\alpha_{1,1} + \frac{2}{3}\beta_{6,1} + \frac{4}{3}\beta_{4,1} - \frac{8}{3}\alpha_{2,1} \\ \frac{\partial^2 F_1}{\partial\xi\partial\eta}(0,0,0) = \frac{2}{3}\alpha_{6,1} + \frac{4}{3}\alpha_{4,1} + \frac{8}{3}\beta_{2,1} + \frac{4}{3}\beta_{1,1}, \\ \frac{\partial^2 F_1}{\partial\xi^2}(0,0,0) = \frac{-8}{3}\beta_{2,1} - \frac{4}{3}\alpha_{4,1} - \frac{4}{3}\beta_{1,1} \\ \frac{\partial^2 G_1}{\partial\xi^2}(0,0,0) = \frac{4}{3}\alpha_{2,1} + \frac{4}{3}\alpha_{1,1} - \frac{4}{3}\beta_{4,1}, \\ \frac{\partial^2 H_1}{\partial\xi^2}(0,0,0) = \pi(\gamma_{1,1} + \gamma_{2,1}) \end{gathered}$$

$$\frac{\partial^2 F_1}{\partial\eta^2}(0,0,0) = 0, \quad \frac{\partial^2 G_1}{\partial\eta^2}(0,0,0) = 0, \quad \frac{\partial^2 H_1}{\partial\eta^2}(0,0,0) = 0$$

The third derivatives of $F_1, G_1$ and $H_1$ at $(0,0,0)$ are computed in a similar way and we will‘ have,

$$\begin{array}{lll} \frac{\partial^3 F_1}{\partial\tau_1^3}(0,0,0) = 0, & \frac{\partial^3 F_1}{\partial\tau_1^2\partial\xi}(0,0,0) = 1, & \frac{\partial^3 F_1}{\partial\tau_1^2\partial\xi}(0,0,0) = -1 \\ \frac{\partial^3 G_1}{\partial\tau_1^3}(0,0,0) = 0, & \frac{\partial^3 G_1}{\partial\tau_1^2\partial\xi}(0,0,0) = 0, & \frac{\partial^3 G_1}{\partial\tau_1^2\partial\xi}(0,0,0) = 0 \\ \frac{\partial^3 H_1}{\partial\tau_1^3}(0,0,0) = 0, & \frac{\partial^3 H_1}{\partial\tau_1^2\partial\xi}(0,0,0) = 0, & \frac{\partial^3 H_1}{\partial\tau_1^2\partial\xi}(0,0,0) = 0 \end{array}$$

$$\begin{array}{ll} \frac{\partial^3 F_1}{\partial\tau_1\partial\xi^2}(0,0,0) = \frac{2}{3}\alpha_{1,1} + \frac{4}{3}\beta_{4,1} - 2\alpha_{2,1}, & \frac{\partial^3 H_1}{\partial\tau_1\partial\xi^2}(0,0,0) = 2\gamma_{1,1} \\ \frac{\partial^3 G_1}{\partial\tau_1\partial\xi^2}(0,0,0) = -\frac{8}{3}\beta_{2,1} - \frac{4}{3}\alpha_{4,1} + \frac{2}{3}\beta_{1,1}, & \frac{\partial^3 F_1}{\partial\tau_1\partial\eta^2}(0,0,0) = 2\alpha_{1,1} \\ \frac{\partial^3 F_1}{\partial\tau_1\partial\xi\partial\eta}(0,0,0) = \frac{-2}{3}\alpha_{1,1} - \frac{1}{3}\alpha_{6,1} - \frac{4}{3}\beta_{6,1} + \frac{8}{3}\alpha_{2,1} - \frac{1}{3}\beta_{6,1}, & \frac{\partial^3 G_1}{\partial\tau_1\partial\eta^2}(0,0,0) = 2\beta_{1,1} \\ \frac{\partial^3 G_1}{\partial\tau_1\partial\xi\partial\eta}(0,0,0) = \frac{8}{3}\beta_{2,1} + \frac{4}{3}\alpha_{4,1} - \frac{2}{3}\beta_{1,1} + \frac{2}{3}\alpha_{6,1}, & \frac{\partial^3 H_1}{\partial\tau_1\partial\eta^2}(0,0,0) = 2\gamma_{1,1} \\ \frac{\partial^3 H_1}{\partial\tau_1\partial\eta\partial\xi}(0,0,0) = -2\gamma_{1,1} & \end{array}$$

$$\begin{array}{ll} \frac{\partial^3 F_1}{\partial\xi^2\partial\eta}(0,0,0) = \frac{\partial^3 x}{\partial\xi^2\partial\eta}(\pi,0,0), & \frac{\partial^3 F_1}{\partial\xi^3}(0,0,0) = \frac{\partial^3 x}{\partial\xi^3}(\pi,0,0) \\ \frac{\partial^3 G_1}{\partial\xi^2\partial\eta}(0,0,0) = \frac{\partial^3 y}{\partial\xi^2\partial\eta}(\pi,0,0), & \frac{\partial^3 G_1}{\partial\xi^3}(0,0,0) = \frac{\partial^3 y}{\partial\xi^3}(\pi,0,0) \\ \frac{\partial^3 H_1}{\partial\xi^2\partial\eta}(0,0,0) = \frac{\partial^3 z}{\partial\xi^2\partial\eta}(\pi,0,0), & \frac{\partial^3 H_1}{\partial\xi^3}(0,0,0) = \frac{\partial^3 z}{\partial\xi^3}(\pi,0,0) \end{array}$$

$$\begin{array}{ll} \frac{\partial^3 F_1}{\partial\eta^2\partial\xi}(0,0,0)=\frac{\partial^3 x}{\partial\eta^2\partial\xi}(\pi,0,0), & \frac{\partial^3 F_1}{\partial\eta^3}(0,0,0)=\frac{\partial^3 x}{\partial\eta^3}(\pi,0,0)\\ \frac{\partial^3 G_1}{\partial\eta^2\partial\xi}(0,0,0)=\frac{\partial^3 y}{\partial\eta^2\partial\xi}(\pi,0,0), & \frac{\partial^3 G_1}{\partial\eta^3}(0,0,0)=\frac{\partial^3 y}{\partial\eta^3}(\pi,0,0)\\ \frac{\partial^3 H_1}{\partial\eta^2\partial\xi}(0,0,0)=\frac{\partial^3 z}{\partial\eta^2\partial\xi}(\pi,0,0), & \frac{\partial^3 H_1}{\partial\eta^3}(0,0,0)=\frac{\partial^3 z}{\partial\eta^3}(\pi,0,0) \end{array}$$

Similar to what we do for $x_{\xi\xi}, y_{\xi\xi}, z_{\xi\xi}, x_{\eta\xi}, y_{\eta\xi}, z_{\eta\xi}, x_{\eta\eta}, y_{\eta\eta}$, and $z_{\eta\eta}$, we can do for $x_{\xi\xi\eta}, y_{\xi\xi\eta}, z_{\xi\xi\eta}, x_{\xi\eta\eta}$,
$y_{\xi\eta\eta}, z_{\xi\eta\eta}, x_{\xi\xi\xi}, y_{\xi\xi\xi}, z_{\xi\xi\xi}, x_{\eta\eta\eta}, y_{\eta\eta\eta}$ and $z_{\eta\eta\eta}$; but for stoppage from scrolling we don't write the systems for $x_{\xi\xi\eta}, y_{\xi\xi\eta}, z_{\xi\xi\eta}, x_{\xi\eta\eta}, y_{\xi\eta\eta}, z_{\xi\eta\eta}, x_{\xi\xi\xi}, y_{\xi\xi\xi}, z_{\xi\xi\xi}, x_{\eta\eta\eta}, y_{\eta\eta\eta}$ and $z_{\eta\eta\eta}$.

Now we have enough information to write the expansions of $F_1(\tau_1,\xi,\eta), G_1(\tau_1,\xi,\eta)$ and $H_1(\tau_1,\xi,\eta)$ near $(0,0,0)$:

$$F_1(\tau_1,\xi,\eta) = -\xi + a_1\xi^2 + a_2\xi\eta + a_3\xi^3 + a_4\eta^3 + a_5\eta\xi^2 + a_6\tau\xi^2 + a_7\xi\eta^2 +$$

$$a_8\tau\eta^2 + 3\xi\tau^2 - 3\eta\tau^2 + a_9\tau\xi\eta,$$

$$G_1(\tau_1,\xi,\eta) = b_1\xi^2 + b_2\xi\eta - 2\xi\tau + 2\eta\tau + b_3\xi^3 + b_4\eta^3 + b_5\eta\xi^2 + b_6\tau\xi^2 + b_7\xi\eta^2 + b_8\tau\eta^2 + b_9\tau\xi\eta,$$

$$H_1(\tau_1,\xi,\eta) = c_1\xi^2 + c_2\xi\eta + c_3\xi^3 + c_4\eta^3 + c_5\eta\xi^2 + c_6\tau\xi^2 + c_7\xi\eta^2 + c_8\tau\eta^2 + c_9\tau\xi\eta,$$

where $a_i, b_i$ and $c_i$, for $i = 1, ..., 9$ are dependent to $\alpha_{j,1}, \beta_{j,1}$ and $\gamma_{j,1}$ for $j = 1, ..., 16$. Similar to what we do for $F_1, G_1$ and $H_1$, we can do to write the expansion of $F_2(\tau_2,\xi,\eta), G_2(\tau_2,\xi,\eta)$ and $H_2(\tau_2,\xi,\eta)$, but since the problem is concurrent; it is sufficient that in expansion of $F_1(\tau_1,\xi,\eta), G_1(\tau_1,\xi,\eta)$ and $H_1(\tau_1,\xi,\eta)$ we replace $\xi$ by $\eta$, $\eta$ by $\xi$ and $r_{j,1}$ by $r_{j,2}$ for $r = \alpha, \beta, \gamma$ and $j = 1, ..., 16$, so we will have:

$$F_2(\tau,\xi,\eta) = -\eta + \bar{a}_1\eta^2 + \bar{a}_2\xi\eta + \bar{a}_3\eta^3 + \bar{a}_4\xi^3 + \bar{a}_5\eta^2\xi + \bar{a}_6\tau\eta^2 +$$

$$\bar{a}_7\xi^2\eta + \bar{a}_8\tau\xi^2 + 3\eta\tau^2 - 3\xi\tau^2 + \bar{a}_9\tau\xi\eta,$$

$$G_2(\tau,\xi,\eta) = \bar{b}_1\eta^2 + \bar{b}_2\xi\eta - 2\eta\tau + 2\xi\tau + \bar{b}_3\eta^3 + \bar{b}_4\xi^3 + \bar{b}_5\eta^2\xi + \bar{b}_6\tau\eta^2 + \bar{b}_7\xi^2\eta + \bar{b}_8\tau\xi^2 + \bar{b}_9\tau\xi\eta,$$

$$H_2(\tau,\xi,\eta) = \bar{c}_1\eta^2 + \bar{c}_2\xi\eta + \bar{c}_3\eta^3 + \bar{c}_4\xi^3 + \bar{c}_5\eta^2\xi + \bar{c}_6\tau\eta^2 + \bar{c}_7\xi^2\eta + \bar{c}_8\tau\xi^2 + \bar{c}_9\tau\xi\eta,$$

where $\bar{a}_i, \bar{b}_i$ and $\bar{c}_i$, for $i = 1, ..., 9$ are dependent to $\alpha_{j,2}, \beta_{j,2}$ and $\gamma_{j,2}$ for $j = 1, ..., 16$. For stoppage from scrolling we give the coefficients in the appendix.

Therefore the problem of persistency of periodic solution under discontinuous perturbation is equal to solving the nonlinear system of equation (2.3). Also by solving equation (2.3) we can compute the time of return to $\Sigma$.

**Theorem 2.4.** *Persistence of the periodic solutions of the linear system Eq.(1.4) are independent from coefficients $r_{i,j}$ for $r = \alpha, \beta, \gamma$, $i = 3, 9, 12, 13, 14, 15, 16$ and $j = 1, 2$.*

**proof:** The coefficients which we give in the appendix are independent from coefficients $r_{i,j}$ for $r = \alpha, \beta, \gamma$, $i = 3, 9, 12, 13, 14, 15, 16$ and $j = 1, 2$, so the conditions for persistency is independent from coefficients $r_{i,j}$ for $r = \alpha, \beta, \gamma$, $i = 3, 9, 12, 13, 14, 15, 16$ and $j = 1, 2$.
Since in these problems there exist many of parameters here we give a Maple code for doing our computations:

## 3. Example

Consider discontinuous differential equation:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{cases} \begin{pmatrix} -y + ax^2 \\ x + by^2 \\ x^2 + z^2 \end{pmatrix}, & y < 0 \\ \begin{pmatrix} -y + dx^2 \\ x + ry^2 \\ x^2 + z^2 \end{pmatrix}, & y > 0 \end{cases}$$

We have

$$F_1(0, \frac{1}{2}, \eta) = \frac{-1}{4} + \frac{-1}{3}b + \frac{4}{3}b\eta + \frac{-1}{18}a^2 - \frac{2}{9}b^2 + (\frac{1}{6}(-\frac{8}{3}a^2 - \frac{32}{3}b^2))\eta^3 +$$
$$(\frac{1}{8}(\frac{8}{3}a^2 + \frac{32}{3}b^2))\eta + \frac{1}{12}a$$
$$+(\frac{1}{4}(-\frac{7}{6}ba\pi - \frac{16}{9}a^2 - \frac{32}{9}b^2))\eta^2 + a\eta^2 - \frac{1}{2}\eta - \frac{1}{3}a\eta$$
$$G_1(0, \frac{1}{2}, \eta) = \frac{1}{6}a - \frac{2}{3}a\eta + \frac{1}{48}a^2\pi + \frac{1}{48}b^2\pi + \frac{1}{24}ab +$$
$$(\frac{1}{6}(-a^2\pi - b^2\pi + 2ab))\eta^3 + (\frac{1}{8}(-a^2\pi - b^2\pi - 2ab))\eta$$
$$+(\frac{1}{4}(\frac{1}{2}a\pi + \frac{2}{3}a^2\pi + b^2\pi - \frac{2}{9}ab))\eta^2$$
$$H_1(0, \frac{1}{2}, \eta) = \frac{1}{8}\pi - \frac{1}{2}\pi\eta + \frac{1}{72}a + \frac{1}{12}b\pi + (\frac{1}{6}(4b\pi - \frac{14}{3}a))\eta^3 +$$
$$(\frac{1}{8}(-\frac{2}{3}a - 4b\pi))\eta + (\frac{1}{4}(\frac{4}{3} + 2a + \frac{4}{3}b\pi))\eta^2 = 0.$$

If we solve equation $F_1(0, 1/2, \eta) = G_1(0, 1/2, \eta) = H_1(0, 1/2, \eta) = 0$ we find $a = -.1474028520, b = .1129653205$, and $\eta = -2.913376624$.

Now we have

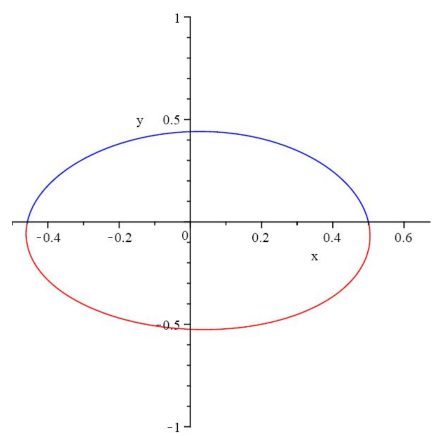


FIGURE 1. periodic solution for the system in $(x, y)$ space.

$$F_2(\tau, 1/2, -2.913376624) = 1.206688312 - 4.217835498r + 6.191224674d^2 + 32.30957725r^2 + 9.542222228d - 2.475597644rd\pi,$$

$$G_2(\tau, 1/2, -2.913376624) = 2.108917750d + 5.920974527d^2\pi + 6.628288139r^2\pi - 7.944215366dr + 1.060970419d\pi,$$

$$H_2(\tau, 1/2, -2.913376624) = 1.581688312\pi + 23.73348077d - 12.11609146r\pi +$$
$$11.65113998\tau + 2.829254450.$$

By Maple computation we find that for $r = -0.1399951418, d = -0.2715603496$, we have:

$$F_2(-0.5735020721, 1/2, -2.913376624) = G_2(-0.5735020721, 1/2, -2.913376624)$$
$$= H_2(-0.5735020721, 1/2, -2.913376624) = 0.$$

So for initial point $x = 1/2, y = 0, z = 0$ we have a periodic solution. Also we can see that the period of this solution is $2\pi - 0.5735020721$.

## 4. Conclusion

In this article by using a new numerical approach, the problem of persistence of periodic solution has been changed to existence of solution for a nonlinear algebraic system of equations. Since the number of parameters is much so any one can fined so many different conditions for existence of solutions for this system of equations (Like theorem 2.4). What is important is that the conditions for exist of solution for the system of equations is equivalent to the necessary conditions to persistence of periodic solution for perturbed linear oscillator.

## 5. Appendix1

$> restart$
$> for i from 1 to 2 for j from 1 to 16 input(alpha[j,i], beta[j,i], gamma[j,i])$
$> F[1] := (1/2 * \xi - 1/2 * \eta) * \tau^2 + (a[8] * \xi * \eta + a[6] * \xi^2 + 1, 1] * \eta^2) * \tau$
$- \xi + a[1] * \xi^2 + a[3] * \xi^3 + a[2] * \xi * \eta + a[7] * \xi * \eta^2 + a[4] * \eta^3 + a[5] * \xi^2 * \eta;$
$G[1] := (-\xi + b[6] * \xi^2 + \eta + \beta[1,1] * \eta^2 + b[8] * \eta * \xi) * \tau + b[1] * \xi^2 + b[3] * \xi^3$
$+ b[2] * \xi * \eta + b[7] * \xi * \eta^2 + b[4] * \eta^3 + b[5] * \eta * \xi^2;$
$H[1] := (\gamma[1,1] * \xi^2 + \gamma[1,1] * \eta^2 - 2 * \gamma[1,1] * \xi * \eta) * \tau + c[1] * \xi^2$
$+ c[3] * \xi^3 - 2 * c[1] * \xi * \eta + c[4] * \eta^3 + c[5] * \eta * \xi^2 + c[6] * \eta^2 * \xi;$
$F[2] := (1/2 * \xi - 1/2 * \eta) * \tau^2 + (a\bar{[8]} * \xi * \eta + a\bar{[6]} * \xi^2$
$+ 1, 1] * \eta^2) * \tau - \xi + a\bar{[1]} * \xi^2 + a\bar{[3]} * \xi^3 + a\bar{[2]} * \xi * \eta$
$+ a\bar{[7]} * \xi * \eta^2 + a\bar{[4]} * \eta^3 + a\bar{[5]} * \xi^2 * \eta;$
$G[2] := (-\xi + b\bar{[6]} * \xi^2 + \eta + \beta[1,1] * \eta^2 + b\bar{[8]} * \eta * \xi) * \tau$

$+ b\bar{[1]} * \xi^2 + b\bar{[3]} * \xi^3 + b\bar{[2]} * \xi * \eta + b\bar{[7]} * \xi * \eta^2$
$+ b\bar{[4]} * \eta^3 + b\bar{[5]} * \eta * \xi^2;$
$H[2] := (\gamma[1, 1] * \xi^2 + \gamma[1, 1] * \eta^2 - 2 * \gamma[1, 1] * \xi * \eta) * \tau$
$+ c\bar{[1]} * \xi^2 + c\bar{[3]} * \xi^3 - 2 * c\bar{[1]} * \xi * \eta + c\bar{[4]} * \eta^3$
$+ c\bar{[5]} * \eta * \xi^2 + c\bar{[6]} * \eta^2 * \xi;$
$> solve(G[1], \tau) = solve(H[1], \tau)(1);$
$> T := numer(lhs(1)) * denom(rhs(1)) - numer(rhs(1)) * denom(lhs(1));$
$> T := collect(simplify(T), \eta) :$
$> \eta := [fsolve(t[5] * x^5 + t[4] * x^4 + t[3] * x^3 + t[2] * x^2 + t[1] * x^1 + t[0] = 0, x)] :$
$> forifrom1tonops(etta)do$
$temp := [fsolve(F[1](\eta[i]) = 0)] :$
$ifnops(temp) > 0then$
$forjfrom1tonops(temp)do\tau[i, j] := temp[j] : end :$
$end :$
$end :$
$> forifrom1tonops(etta)do$
$tau[2] := fsolve(G[2](etta[i], \xi) = 0, \tau) :$
$ifabs(H[2](\eta[i], \xi, \tau[2])) < 10^{-8}andabs(F[2](\eta[i], \xi, \tau[2])) < 10^{-8}$
$thenprint\tau[2] : end :$
$end :$

## 6. Appendix2

Here we give the coefficients in the formulas of $F_1$, $F_2$, $G_1$, $G_2$, $H_1$ and $H_2$:
$a_1 = 1/2(-(8/3)\beta_{2,1} - (4/3)\alpha_{4,1} - (4/3)\beta_{1,1}),$

$a_2 = 1/2((4/3)\alpha_{6,1} + (8/3)\alpha_{4,1} + (16/3)\beta_{2,1} + (8/3)\beta_{1,1}),$

$a_3 = 1/6((20/9)\beta_{4,1}\alpha_{1,1} - (80/9)\alpha_{4,1}\beta_{2,1} - (28/9)\beta_{4,1}\alpha_{2,1} + (3/8)\alpha_{6,1}\gamma_{1,1}\pi - (1/3)\alpha_{1,1}\alpha 4, 1\pi + (1/6)\alpha_{4,1}\alpha_{2,1}\pi + (3/8)\beta_{5,1}\gamma_{1,1}\pi - (3/8)\alpha_{5,1}\gamma_{4,1}\pi - (9/8)\beta_{6,1}\gamma_{4,1}\pi + (1/3)\beta_{4,1}\beta_{1,1}\pi + (9/8)\beta_{5,1}\gamma_{2,1}\pi - (9/4)\alpha_{7,1}\pi + (3/2)\beta_{1,1}\alpha_{1,1}\pi - (9/4)\beta_{8,1}\pi + (1/6)\beta_{4,1}\beta_{2,1}\pi - (5/2)\alpha_{2,1}\beta_{2,1}\pi + (9/8)\alpha_{6,1}\gamma_{2,1}\pi - (32/3)\beta_{1,1}\beta_{2,1} - (3/4)\beta_{6,1}\gamma_{2,1} - (3/4)\beta_{11,1}\pi - (1/2)\alpha_{10,1}\pi - (3/4)\beta_{6,1}\gamma_{1,1} - (3/4)\alpha_{5,1}\gamma_{1,1} - (1/12)\beta_{4,1}\alpha_{4,1}\pi -$

$3\alpha_{5,1}\gamma_{2,1}((1/4)\pi^2-1/4)+(4/9)\beta_{4,1}^2-(16/9)\alpha_{4,1}^2-(32/3)\beta 2,1^2-(8/3)\alpha_{1,1}^2-(8/3)\beta_{1,1}^2+(8/3)\alpha_{2,1}^2-3\beta 6,1\gamma_{1,1}((1/4)\pi^2-1/4)-(40/9)\beta_{1,1}\alpha_{4,1}-3\alpha_{5,1}\gamma_{1,1}((1/4)\pi^2-1/4)-3\gamma_{2,1}\beta_{6,1}((1/4)\pi^2-1/4)-(3/4)\alpha_{5,1}\gamma_{2,1})$,

$a_4 = 1/6((4/3)\beta_{4,1}\alpha_{1,1}-(32/3)\alpha_{4,1}\beta_{2,1}-(3/8)\alpha_{6,1}\gamma_{1,1}\pi+(3/4)\alpha_{1,1}\alpha_{4,1}\pi+(3/4)\alpha_{4,1}\alpha_{2,1}\pi-(3/8)\beta_{5,1}\gamma_{1,1}\pi+(3/8)\alpha_{5,1}\gamma_{4,1}\pi-(3/2)\alpha_{5,1}\pi^2\gamma_{2,1}+(9/8)\beta_{6,1}\gamma_{4,1}\pi-(3/4)\beta_{4,1}\beta_{1,1}\pi-(9/8)\beta_{5,1}\gamma_{2,1}\pi+(9/4)\alpha_{7,1}\pi-(3/2)\beta_{1,1}\alpha_{1,1}\pi+(9/4)\beta_{8,1}\pi-(3/4)\beta_{4,1}\beta_{2,1}\pi+(3/2)\alpha_{2,1}\beta_{2,1}\pi-(9/8)\alpha_{6,1}\gamma_{2,1}\pi-(32/3)\beta_{1,1}\beta_{2,1}-(8/3)\alpha_{1,1}\alpha_{2,1}+(3/4)\beta_{6,1}\gamma_{2,1}+(3/4)\beta_{11,1}\pi+(3/4)\alpha_{10,1}\pi-(3/2)\beta_{6,1}\pi^2\gamma_{1,1}-(3/2)\alpha_{5,1}\pi^2\gamma_{1,1}+(3/4)\beta_{6,1}\gamma_{1,1}+(3/4)\alpha_{5,1}\gamma_{1,1}+3\alpha_{5,1}\gamma_{2,1}((1/4)\pi^2-1/4)+(4/3)\beta_{4,1}^2-(8/3)\alpha_{4,1}^2-(32/3)\beta_{2,1}^2-(8/3)\alpha_{1,1}^2-(8/3)\beta_{1,1}^2+(16/3)\alpha_{2,1}^2+3\beta_{6,1}\gamma_{1,1}((1/4)\pi^2-1/4)-(16/3)\beta_{1,1}\alpha_{4,1}+3\alpha_{5,1}\gamma_{1,1}((1/4)\pi^2-1/4)+3\gamma_{2,1}\beta_{6,1}((1/4)\pi^2-1/4)-(3/2)\beta_{6,1}\pi^2\gamma_{2,1}+(3/4)\alpha_{5,1}\gamma_{2,1})$

$a_5 = 1/2((1/3)\alpha_{1,1}\alpha_{6,1}\pi-(4/3)\beta_{4,1}\alpha_{1,1}+(32/3)\alpha_{4,1}\beta_{2,1}+(44/9)\beta_{4,1}\alpha_{2,1}-(3/8)\alpha_{6,1}\gamma_{1,1}\pi+(3/4)\alpha_{1,1}\alpha_{4,1}\pi+(7/12)\alpha_{4,1}\alpha_{2,1}\pi-(3/8)\beta_{5,1}\gamma_{1,1}\pi+(3/8)\alpha_{5,1}\gamma_{4,1}\pi+(3/4)\alpha_{5,1}\pi^2\gamma_{2,1}+(9/8)\beta_{6,1}\gamma_{4,1}\pi-(3/4)\beta_{4,1}\beta_{1,1}\pi-(9/8)\beta_{5,1}\gamma_{2,1}\pi+(9/4)\alpha_{7,1}\pi-(3/2)\beta_{1,1}\alpha_{1,1}\pi+(9/4)\beta_{8,1}\pi-(3/4)\beta_{4,1}\beta_{2,1}\pi+(11/6)\alpha_{2,1}\beta_{2,1}\pi-(9/8)\alpha_{6,1}\gamma_{2,1}\pi-(4/9)\beta_{4,1}\beta_{6,1}+(8/9)\beta_{1,1}\alpha_{6,1}+(32/3)\beta_{1,1}\beta_{2,1}+(16/9)\alpha_{1,1}\alpha_{2,1}-(1/6)\alpha_{2,1}\alpha_{6,1}\pi+(3/4)\beta_{11,1}\pi+(3/4)\alpha_{10,1}\pi+(3/4)\beta_{6,1}\pi^2\gamma_{1,1}+(3/4)\alpha_{5,1}\pi^2\gamma_{1,1}+(1/6)\beta_{1,1}\beta_{6,1}\pi+(3/8)\beta_{6,1}\gamma_{6,1}\pi+(8/9)\alpha_{2,1}\beta_{6,1}+(16/9)\beta_{2,1}\alpha_{6,1}+(8/9)\alpha_{4,1}\alpha_{6,1}-(8/9)\alpha_{1,1}\beta_{6,1}+(1/3)\beta_{2,1}\beta_{6,1}\pi-(4/3)\beta_{4,1}^2+(8/3)\alpha_{4,1}^2+(32/3)\beta_{2,1}^2+(8/3)\alpha_{1,1}^2+(8/3)\beta_{1,1}^2-(40/9)\alpha_{2,1}^2+(16/3)\beta_{1,1}\alpha_{4,1}+(1/8)\alpha_{5,1}\gamma_{6,1}\pi+(3/4)\beta_{6,1}\pi^2\gamma_{2,1}-(1/4)\beta_{4,1}\alpha_{6,1}\pi-(1/12)\alpha_{4,1}\beta_{6,1}\pi)$,

$a_6 = 1/6(2\alpha_{1,1}+4\beta_{4,1}-6\alpha_{2,1})$,

$a_7 = 1/2((1/3)\alpha_{1,1}\alpha_{6,1}\pi+(4/3)\beta_{4,1}\alpha_{1,1}-(40/9)\alpha_{4,1}\beta_{2,1}-(16/9)\beta_{4,1}\alpha_{2,1}+(3/8)\alpha_{6,1}\gamma_{1,1}\pi-(11/12)\alpha_{1,1}\alpha_{4,1}\pi-(11/12)\alpha_{4,1}\alpha_{2,1}\pi+(3/8)\beta_{5,1}\gamma_{1,1}\pi-(3/8)\alpha_{5,1}\gamma_{4,1}\pi-(1/4)\alpha_{5,1}\pi^2\gamma_{2,1}-(9/8)\beta_{6,1}\gamma_{4,1}\pi+(3/4)\beta_{4,1}\beta_{1,1}\pi+(9/8)\beta_{5,1}\gamma_{2,1}\pi-(9/4)\alpha_{7,1}\pi+(2/3)\beta_{1,1}\alpha_{1,1}\pi-(9/4)\beta_{8,1}\pi+(3/4)\beta_{4,1}\beta_{2,1}\pi-(3/2)\alpha_{2,1}\beta_{2,1}\pi+$

$(9/8)\alpha_{6,1}\gamma_{2,1}\pi + (4/9)\beta_{4,1}\beta_{6,1} - (8/9)\beta_{1,1}\alpha_{6,1} - (32/9)\beta_{1,1}\beta_{2,1} - (8/3)\alpha_{1,1}\alpha_{2,1} + (1/6)\alpha_{2,1}\alpha_{6,1}\pi + (1/4)\beta_{11,1}\pi - (3/4)\alpha_{10,1}\pi - (1/4)\beta_{6,1}\pi^2\gamma_{1,1} - (1/4)\alpha_{5,1}\pi^2\gamma_{1,1} - (1/6)\beta_{1,1}\beta_{6,1}\pi - (3/8)\beta_{6,1}\gamma_{6,1}\pi - (8/9)\alpha_{2,1}\beta_{6,1} - (16/9)\beta_{2,1}\alpha_{6,1} - (8/9)\alpha_{4,1}\alpha_{6,1} + (8/9)\alpha_{1,1}\beta_{6,1} + (8/45)\alpha_{1,1}\alpha_{4,1}^2 + (1/12)\beta_{4,1}\alpha_{4,1}\pi - (1/3)\beta_{2,1}\beta_{6,1}\pi + (4/9)\beta_{4,1}^2 - (1/2)\beta_{1,1}\pi - (4/3)\alpha_{4,1}^2 - (32/9)\beta_{2,1}^2 - (16/9)\alpha_{1,1}^2 - (8/9)\beta_{1,1}^2 + (16/9)\alpha_{2,1}^2 + (1/3)\alpha_{1,1}\alpha_{4,1}\alpha_{2,1}\pi - (1/6)\alpha_{1,1}\alpha_{4,1}\beta_{4,1}\pi - (20/9)\beta_{1,1}\alpha_{4,1} + (1/6)\alpha_{1,1}^2\alpha_{4,1}\pi - (1/8)\alpha_{5,1}\gamma_{6,1}\pi - (1/4)\beta_{6,1}\pi^2\gamma_{2,1} - (1/12)\beta_{4,1}\alpha_{6,1}\pi + (64/45)\alpha_{1,1}\alpha_{4,1}\beta_{2,1} + (1/12)\alpha_{4,1}\beta_{6,1}\pi - (7/6)\beta_{2,1}\alpha_{1,1}\pi + (56/45)\alpha_{1,1}\alpha_{4,1}\beta_{1,1}),$

$a_8 = 1/6(-4\alpha_{1,1} - 2\alpha_{6,1} - 8\beta_{4,1} + 16\alpha_{2,1} - 2\beta_{6,1}),$

$b_1 = 1/2((4/3)\alpha_{2,1} + (4/3)\alpha_{1,1} - (4/3)\beta_{4,1}),$

$b_2 = 1/2(-(8/3)\alpha_{1,1} + (4/3)\beta_{6,1} + (8/3)\beta_{4,1} - (16/3)\alpha_{2,1}),$

$b_3 = 1/6((5/2)\beta_{1,1}\beta_{2,1}\pi + (9/8)\gamma_{2,1}\beta_{6,1}\pi - (3/4)\beta_{5,1}\gamma_{1,1}(\pi^2-2) + (1/12)\beta_{4,1}^2\pi + (5/2)\beta_{1,1}^2\pi + (5/2)\alpha_{1,1}\alpha_{2,1}\pi + (2/3)\beta_{4,1}\alpha_{2,1}\pi + (1/6)\beta_{1,1}\alpha_{4,1}\pi + (9/8)\alpha_{6,1}\gamma_{4,1}\pi - (9/8)\alpha_{5,1}\gamma_{2,1}\pi - (5/6)\beta_{4,1}\alpha_{1,1}\pi - (3/8)\alpha_{5,1}\gamma_{1,1}\pi - (2/3)\alpha_{4,1}\beta_{2,1}\pi + (3/8)\beta_{6,1}\gamma_{1,1}\pi - (3/8)\beta_{5,1}\gamma_{4,1}\pi + (22/3)\alpha_{2,1}\beta_{2,1} + 2\alpha_{1,1}\beta_{2,1} + (1/6)\alpha_{4,1}^2\pi + (7/2)\alpha_{2,1}^2\pi + (22/9)\alpha_{4,1}\alpha_{2,1} + \alpha_{1,1}^2\pi + (3/4)\alpha_{11,1}\pi - (9/4)\beta_{7,1}\pi - 6\alpha_{1,1}\beta_{1,1} - (2/9)\alpha_{1,1}\alpha_{4,1} - (38/9)\alpha_{4,1}\beta_{4,1} - (98/9)\beta_{4,1}\beta_{2,1} - (10/3)\beta_{1,1}\alpha_{2,1} + \beta_{2,1}^2\pi + (14/9)\beta_{1,1}\beta_{4,1} + (3/2)\alpha_{6,1}\gamma_{1,1} + (9/4)\alpha_{8,1}\pi - (3/2)\beta_{5,1}\gamma_{2,1} + (3/2)\alpha_{6,1}\gamma_{2,1} + (3/4)\alpha_{6,1}\gamma_{2,1}(\pi^2-2) - (3/4)\beta_{5,1}\gamma_{2,1}(\pi^2-2) + (3/4)\alpha_{6,1}\gamma_{1,1}(\pi^2-2) - (1/2)\beta_{10,1}\pi - (3/2)\beta_{5,1}\gamma_{1,1}),$

$b_4 = 1/6((3/2)\alpha_{6,1}\pi^2\gamma_{1,1} - (5/2)\beta_{1,1}\beta_{2,1}\pi - (9/8)\gamma_{2,1}\beta_{6,1}\pi + (3/4)\beta_{5,1}\gamma_{1,1}(\pi^2 - 2) - (1/4)\beta_{4,1}^2\pi - (5/2)\beta_{1,1}^2\pi + (3/2)\alpha_{6,1}\pi^2\gamma_{2,1} - (5/2)\alpha_{1,1}\alpha_{2,1}\pi + (1/4)\beta_{4,1}\alpha_{2,1}\pi + (1/4)\beta_{1,1}\alpha_{4,1}\pi - (9/8)\alpha_{6,1}\gamma_{4,1}\pi + (9/8)\alpha_{5,1}\gamma_{2,1}\pi + (5/4)\beta_{4,1}\alpha_{1,1}\pi - (3/2)\beta_{5,1}\pi^2\gamma_{1,1} - (3/2)\beta_{5,1}\pi^2\gamma_{2,1} + (3/8)\alpha_{5,1}\gamma_{1,1}\pi + (5/4)\alpha_{4,1}\beta_{2,1}\pi - (3/8)\beta_{6,1}\gamma_{1,1}\pi + (3/8)\beta_{5,1}\gamma_{4,1}\pi + 12\alpha_{2,1}\beta_{2,1} + 2\alpha_{1,1}\beta_{2,1} - (1/4)\alpha_{4,1}^2\pi - (5/2)\alpha_{2,1}^2\pi + 4\alpha_{4,1}\alpha_{2,1} - \alpha_{1,1}^2\pi - (3/4)\alpha_{11,1}\pi + (9/4)\beta_{7,1}\pi - 6\alpha_{1,1}\beta_{1,1} - 6\alpha_{4,1}\beta_{4,1} - (14/3)\beta_{4,1}\beta_{2,1} - 8\beta_{1,1}\alpha_{2,1} - \beta_{2,1}^2\pi - (3/2)\alpha_{6,1}\gamma_{1,1} - (9/4)\alpha_{8,1}\pi + (3/2)\beta_{5,1}\gamma_{2,1} - (3/2)\alpha_{6,1}\gamma_{2,1} - (3/4)\alpha_{6,1}\gamma_{2,1}(\pi^2-2) + (3/4)\beta_{5,1}\gamma_{2,1}(\pi^2 -$

$2)-(3/4)\alpha_{6,1}\gamma_{1,1}(\pi^2-2)+(3/4)\beta_{10,1}\pi+(3/2)\beta_{5,1}\gamma_{1,1})$,

$b_5=1/2(-(1/3)\alpha_{2,1}\beta_{6,1}\pi-(5/2)\beta_{1,1}\beta_{2,1}\pi-(9/8)\gamma_{2,1}\beta_{6,1}\pi+(4/9)\alpha_{1,1}\alpha_{6,1}+(3/4)\beta_{5,1}\gamma_{1,1}(\pi^2-2)-(1/4)\beta_{4,1}^2\pi+(14/9)\beta_{4,1}\alpha_{6,1}-(5/2)\beta_{1,1}^2\pi+(1/3)\beta_{1,1}\alpha_{6,1}\pi-(5/2)\alpha_{1,1}\alpha_{2,1}\pi+(1/12)\beta_{4,1}\alpha_{2,1}\pi+(1/4)\beta_{1,1}\alpha_{4,1}\pi-(9/8)\alpha_{6,1}\gamma_{4,1}\pi+(9/8)\alpha_{5,1}\gamma_{2,1}\pi-(3/8)\alpha_{6,1}\gamma_{6,1}\pi+(5/4)\beta_{4,1}\alpha_{1,1}\pi+(4/9)\alpha_{4,1}\beta_{6,1}+(3/8)\alpha_{5,1}\gamma_{1,1}\pi+(5/4)\alpha_{4,1}\beta_{2,1}\pi-(3/8)\beta_{6,1}\gamma_{1,1}\pi+(3/8)\beta_{5,1}\gamma_{4,1}\pi+(14/9)\beta_{2,1}\beta_{6,1}-(94/9)\alpha_{2,1}\beta_{2,1}-2\alpha_{1,1}\beta_{2,1}-(1/4)\alpha_{4,1}^2\pi-(17/6)\alpha_{2,1}^2\pi-(14/9)\beta_{1,1}\beta_{6,1}-(32/9)\alpha_{4,1}\alpha_{2,1}-\alpha_{1,1}^2\pi-(3/4)\alpha_{11,1}\pi+(9/4)\beta_{7,1}\pi+6\alpha_{1,1}\beta_{1,1}+6\alpha_{4,1}\beta_{4,1}+14\beta_{4,1}\beta_{2,1}+(58/9)\beta_{1,1}\alpha_{2,1}-\beta_{2,1}^2\pi-(3/2)\alpha_{6,1}\gamma_{1,1}-(9/4)\alpha_{8,1}\pi+(3/2)\beta_{5,1}\gamma_{2,1}-(3/2)\alpha_{6,1}\gamma_{2,1}-(3/4)\alpha_{6,1}\gamma_{2,1}(\pi^2-2)+(1/8)\beta_{5,1}\gamma_{6,1}\pi+(3/4)\beta_{5,1}\gamma_{2,1}(\pi^2-2)+(1/4)\alpha_{4,1}\alpha_{6,1}\pi-(1/12)\beta_{4,1}\beta_{6,1}\pi-(3/4)\alpha_{6,1}\gamma_{1,1}(\pi^2-2)-(1/6)\alpha_{1,1}\beta_{6,1}\pi-(1/6)\beta_{2,1}\alpha_{6,1}\pi-(4/9)\alpha_{2,1}\alpha_{6,1}+(3/4)\beta_{10,1}\pi+(3/2)\beta_{5,1}\gamma_{1,1})$,

$b_6=1/6(-8\beta_{2,1}-4\alpha_{4,1}+2\beta_{1,1})$,

$b_7=1/2((1/3)\alpha_{2,1}\beta_{6,1}\pi+(1/4)\alpha_{6,1}\pi^2\gamma_{1,1}+(5/2)\beta_{1,1}\beta_{2,1}\pi+(9/8)\gamma_{2,1}\beta_{6,1}\pi-(16/45)\alpha_{1,1}\alpha_{4,1}\alpha_{2,1}-(4/9)\alpha_{1,1}\alpha_{6,1}+(1/4)\beta_{4,1}^2\pi-(14/9)\beta_{4,1}\alpha_{6,1}+(5/2)\beta_{1,1}^2\pi+(1/4)\alpha_{6,1}\pi^2\gamma_{2,1}+(1/3)\beta_{1,1}\alpha_{6,1}\pi+(8/45)\alpha_{1,1}\alpha_{4,1}\beta_{4,1}+(17/6)\alpha_{1,1}\alpha_{2,1}\pi-(1/4)\beta_{4,1}\alpha_{2,1}\pi+(1/3)\beta_{1,1}\alpha_{4,1}\pi+(9/8)\alpha_{6,1}\gamma_{4,1}\pi-(9/8)\alpha_{5,1}\gamma_{2,1}\pi+(3/8)\alpha_{6,1}\gamma_{6,1}\pi-(17/12)\beta_{4,1}\alpha_{1,1}\pi-(4/9)\alpha_{4,1}\beta_{6,1}-(1/4)\beta_{5,1}\pi^2\gamma_{1,1}-(1/6)\alpha_{1,1}\alpha_{4,1}^2\pi-(1/3)\alpha_{1,1}\alpha_{4,1}\beta_{2,1}\pi-(1/4)\beta_{5,1}\pi^2\gamma_{2,1}-(3/8)\alpha_{5,1}\gamma_{1,1}\pi-(5/6)\alpha_{4,1}\beta_{2,1}\pi+(3/8)\beta_{6,1}\gamma_{1,1}\pi-(3/8)\beta_{5,1}\gamma_{4,1}\pi-(1/6)\alpha_{1,1}\alpha_{4,1}\beta_{1,1}\pi-(14/9)\beta_{2,1}\beta_{6,1}+4\alpha_{2,1}\beta_{2,1}-(2/9)\alpha_{1,1}\beta_{2,1}+(1/12)\alpha_{4,1}^2\pi+(5/2)\alpha_{2,1}^2\pi+(14/9)\beta_{1,1}\beta_{6,1}+(16/9)\alpha_{4,1}\alpha_{2,1}+(2/3)\alpha_{1,1}^2\pi-(1/4)\alpha_{11,1}\pi-(9/4)\beta_{7,1}\pi-(22/9)\alpha_{1,1}\beta_{1,1}-(2/9)\alpha_{1,1}\alpha_{4,1}-(20/9)\alpha_{4,1}\beta_{4,1}-(14/3)\beta_{4,1}\beta_{2,1}-(8/3)\beta_{1,1}\alpha_{2,1}+(16/45)\alpha_{1,1}^2\alpha_{4,1}+(1/2)\alpha_{1,1}\pi+\beta_{2,1}^2\pi+(9/4)\alpha_{8,1}\pi-(1/8)\beta_{5,1}\gamma_{6,1}\pi+(1/12)\alpha_{4,1}\alpha_{6,1}\pi+(1/12)\beta_{4,1}\beta_{6,1}\pi+(1/6)\alpha_{1,1}\beta_{6,1}\pi+(1/6)\beta_{2,1}\alpha_{6,1}\pi+(4/9)\alpha_{2,1}\alpha_{6,1}-(3/4)\beta_{10,1}\pi)$,

$b_8 = 1/6(16\beta_{2,1} + 8\alpha_{4,1} - 4\beta_{1,1} + 4\alpha_{6,1});$
$c_1 = (1/2)\pi(\gamma_{1,1} + \gamma_{2,1}),$

$c_3 = 1/6(4\gamma_{2,1}\beta_{2,1}\pi + 3\gamma_{6,1}\gamma_{2,1}\pi + (1/3)\gamma_{4,1}\beta_{4,1}\pi + 4\gamma_{1,1}\beta_{2,1}\pi + 2\gamma_{2,1}\beta_{1,1}\pi + 2\gamma_{1,1}\beta_{1,1}\pi + 4\gamma_{8,1} + 2\gamma_{1,1}\alpha_{4,1}\pi + 3\gamma_{6,1}\gamma_{1,1}\pi - (16/3)\gamma_{4,1}\beta_{2,1} - 4\gamma_{5,1}\gamma_{2,1} + (26/3)\gamma_{2,1}\alpha_{2,1} - (1/3)\gamma_{4,1}\alpha_{1,1}\pi - (2/3)\gamma_{2,1}\beta_{4,1} - (1/3)\gamma_{4,1}\alpha_{2,1}\pi + 2\gamma_{2,1}\alpha_{4,1}\pi + 2\gamma_{11,1} - (14/3)\gamma_{1,1}\alpha_{2,1} + 2\gamma_{6,1}\gamma_{4,1} + (2/3)\gamma_{1,1}\alpha_{1,1} - 2\gamma_{5,1}\gamma_{1,1} - (8/3)\gamma_{4,1}\beta_{1,1} + (26/3)\gamma_{2,1}\alpha_{1,1} - (4/3)\gamma_{4,1}\alpha_{4,1} + (10/3)\gamma_{1,1}\beta_{4,1}),$

$c_4 = 1/6(-2\gamma_{11,1} + (2/3)\gamma_{2,1}\beta_{4,1} - 2\gamma_{4,1}\alpha_{4,1} - 2\gamma_{6,1}\gamma_{4,1} - (4/3)\gamma_{2,1}\alpha_{2,1} - 4\gamma_{8,1} + 4\gamma_{1,1}\beta_{2,1}\pi + 4\gamma_{2,1}\beta_{2,1}\pi + 2\gamma_{1,1}\beta_{1,1}\pi + 2\gamma_{2,1}\beta_{1,1}\pi + 2\gamma_{1,1}\alpha_{4,1}\pi + 2\gamma_{2,1}\alpha_{4,1}\pi + 2\gamma_{5,1}\gamma_{1,1} + 4\gamma_{5,1}\gamma_{2,1} - (14/3)\gamma_{2,1}\alpha_{1,1} - (4/3)\gamma_{1,1}\alpha_{2,1} + (2/3)\gamma_{1,1}\beta_{4,1} - (14/3)\gamma_{1,1}\alpha_{1,1}),$

$c_5 = 1/2(-4\gamma_{2,1}\beta_{2,1}\pi - 3\gamma_{6,1}\gamma_{2,1}\pi + (10/9)\alpha_{6,1}\gamma_{4,1} - 4\gamma_{1,1}\beta_{2,1}\pi - (2/3)\alpha_{6,1}\gamma_{2,1}\pi - (2/3)\alpha_{6,1}\gamma_{1,1}\pi - 2\gamma_{2,1}\beta_{1,1}\pi - 2\gamma_{1,1}\beta_{1,1}\pi - 4\gamma_{8,1} - 2\gamma_{1,1}\alpha_{4,1}\pi - 3\gamma_{6,1}\gamma_{1,1}\pi + 8\gamma_{4,1}\beta_{2,1} + 4\gamma_{5,1}\gamma_{2,1} - (82/9)\gamma_{2,1}\alpha_{2,1} + (2/3)\gamma_{2,1}\beta_{4,1} - 2\gamma_{2,1}\alpha_{4,1}\pi - (2/3)\gamma_{6,1}^2 - (10/9)\beta_{6,1}\gamma_{1,1} - 2\gamma_{11,1} + 6\gamma_{1,1}\alpha_{2,1} - 2\gamma_{6,1}\gamma_{4,1} - (2/3)\gamma_{1,1}\alpha_{1,1} + 2\gamma_{5,1}\gamma_{1,1} + (2/9)\beta_{6,1}\gamma_{2,1} + 4\gamma_{4,1}\beta_{1,1} - (26/3)\gamma_{2,1}\alpha_{1,1} + 2\gamma_{4,1}\alpha_{4,1} - (10/3)\gamma_{1,1}\beta_{4,1}),$

$c_6 = 1/2((4/3)\gamma_{2,1}\beta_{2,1}\pi + 2\gamma_{6,1}\gamma_{2,1}\pi - (2/3)\alpha_{6,1}\gamma_{4,1} + (4/3)\gamma_{1,1}\beta_{2,1}\pi + (2/3)\alpha_{6,1}\gamma_{2,1}\pi + (2/3)\alpha_{6,1}\gamma_{1,1}\pi + (2/3)\gamma_{2,1}\beta_{1,1}\pi + (2/3)\gamma_{1,1}\beta_{1,1}\pi + 4\gamma_{8,1} + (2/3)\gamma_{1,1}\alpha_{4,1}\pi + 2\gamma_{6,1}\gamma_{1,1}\pi - (16/3)\gamma_{4,1}\beta_{2,1} - 4\gamma_{5,1}\gamma_{2,1} + (20/3)\gamma_{2,1}\alpha_{2,1} + (2/3)\gamma_{2,1}\beta_{4,1} + (2/3)\gamma_{2,1}\alpha_{4,1}\pi + (4/3)\gamma_{1,1} + (2/3)\gamma_{6,1}^2 + (10/9)\beta_{6,1}\gamma_{1,1} - (2/3)\gamma_{11,1} - 4\gamma_{1,1}\alpha_{2,1} + 2\gamma_{6,1}\gamma_{4,1} + 2\gamma_{1,1}\alpha_{1,1} - 2\gamma_{5,1}\gamma_{1,1} - (2/9)\beta_{6,1}\gamma_{2,1} - (8/3)\gamma_{4,1}\beta_{1,1} + (22/3)\gamma_{2,1}\alpha_{1,1} - (2/3)\gamma_{4,1}\alpha_{4,1} + 2\gamma_{1,1}\beta_{4,1}).$

The coefficients in the formulas of $F_2$, $H_2$ and $G_2$ are similar with this difference that we must replace $-_{*,1}$ with $-_{*,2}$.

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