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RELATIVE C-(n,m)-COTORSION MODULES

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Abstract. Let R be a ring, C a (faithfully) semidualizing module, and $n, m \geq 0$ be integers or potentially $n = \infty$. We introduce and analyze C-(n,m)-cotorsion modules, utilizing the class of R-modules with C-FP_n-flat dimensions at most m as a broad generalization. This framework provides a new classification of C-cotorsion and C-m-weak cotorsion modules, along with their equivalent characterizations. Furthermore, we examine the Foxby equivalences between subclasses of the Auslander and Bass classes with respect to C. Lastly, we explore the properties of strongly C-(n,m)-cotorsion dimensions in the context of almost excellent extensions, considering the class of R-modules with C-FP_n-flat (or injective) dimensions at most m.

Keywords: C-(n,m)-Cotorsion module, C-FP_n-flat module, C-FP_n-injective module, Foxby equivalence, semidualizing module.
2020 MSC: Primary 16D80, 16E05, 16E30, 16E65.

1. Introduction

Throughout this document, the symbol R will represent an associative ring with identity. All R-modules are assumed to be unital left R-modules unless explicitly stated otherwise. Right R-modules are interpreted as left modules over the opposite ring R^{op}. The variables m, n, and t are non-negative integers.

In 1984, Enochs introduced the concept of cotorsion modules as a generalization of cotorsion abelian groups [5]. Since then, the study of these modules has become a highly active area of research. We recommend consulting [5, 11] for foundational background on cotorsion modules. In recent years, new classifications of cotorsion modules have been explored in works such as [1, 10, 15]. In 2007, Mao and Ding introduced the concept of m-cotorsion modules as a new classification based on classes of modules with finite flat dimension [10]. Later, in 2018, Selvaraj and Prabakaran proposed the concept of m-weak cotorsion modules as a further classification using classes of modules with finite weak flat dimension [15]. Prior to this, in 2015, Gao and Wang introduced the notion of weak flat modules [7].

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Semidualizing modules C over general associative rings have been extensively studied, beginning with the work of Holm and White in [9], where they introduced and analyzed the concepts of the Auslander and Bass classes, as well as C-flat and C-injective modules with respect to a semidualizing bimodule $C = {}_SC_R$ over rings R and S. They established the Foxby equivalence between subclasses of the Auslander class and the Bass class. Considering $R = S$, in 2016, X. Chen and J. Chen, in [4], defined C-cotorsion (and C-flat cotorsion) modules by utilizing C-flat (and flat cotorsion) modules over commutative rings. They further examined the Foxby equivalences between subclasses of the Auslander and Bass classes relative to C, as well as the strongly C-cotorsion dimensions of modules in the context of almost excellent extensions, and also, in 2024, Amini et al., in [2], introduc the concept of C-m-weak cotorsion modules and represent a new classification within C-cotorsion modules.

In Section 2, also considering $R = S$, we state some required concepts. For example, the definitions of cotorsion, m-cotorsion, m-weak cotorsion, C-cotorsion, C-m-weak cotorsion (and flat cotorsion, C-flat C-cotorsion) modules provide motivation for the definition of C-(n,m)-cotorsion (and C-FP_n-flat C-(n,m)-cotorsion) modules as common generalizations of these concepts. It is noteworthy that this approach enables us to deal with several important concepts on homological theory comprehensively. So, in Section 3, we introduce C-(n,m)-cotorsion modules, strongly C-(n,m)-cotorsion and C-FP_n-flat C-(n,m)-cotorsion modules. Then we obtain some results of homological relationships between the classes $Cot^{(n,m)}(R)$, $Cot_C^{(n,m)}(R)$, $SCot_C^{(n,m)}(R)$, $\mathcal{P}(R)$, $\mathcal{FF}^{cot^{(n,m)}}(R)$, $\mathcal{FF}_C^{cot^{(n,m)}}(R)$, $\mathcal{A}_C(R)$, $\mathcal{B}_C(R)$, $\mathcal{FF}^n(R)_{\leq m}$ and $\mathcal{FF}_C^n(R)_{\leq m}$, $\mathcal{FF}^{\mathrm{cot}^{(n,m)}}(R)_{\leq t}$ and $\mathcal{FF}_C^{\mathrm{cot}^{(n,m)}}(R)_{\leq t}$, where these classes are the class of (n,m)-cotorsion, C-(n,m)-cotorsion, strongly C-(n,m)-cotorsion, projective, FP_n-flat (n,m)-cotorsion, C-FP_n-flat C-(n,m)-cotorsion R-modules, Auslander class, Bass class, R-modules with FP_n-flat dimension at most m, R-modules with C-FP_n-flat dimension at most m, R-modules with FP_n-flat (n,m)-cotorsion projective dimension at most t and R-modules with C-FP_n-flat C-(n,m)-cotorsion projective dimension at most t, respectively. Then, using these results, we investigate Foxby equivalence relative to theses classes, see Theorems 3.4, 3.6 and 3.14.

Section 4 is dedicated to investigating the (n,m)-cotorsion dimensions relative to semidualizing modules, specifically the strongly C-(n,m)-cotorsion dimensions. Additionally, we examine the transfer properties of the classes C-FP_n-injective dimensions, C-FP_n-flat dimensions, and strongly C-(n,m)-cotorsion dimensions under almost excellent extensions, see Theorem 4.9.

2. Preliminaries

In this section, fundamental concepts and notation are outlined.

Definition 2.1. (1) An R-module U is said to be *finitely n-presented* [17] if there exists an exact sequence

$$F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow U \longrightarrow 0,$$

where each F_i is finitely generated and projective;

(2) A module M is called *FP_n-injective* [3] if $\mathrm{Ext}^1_R(U,M) = 0$ for every finitely n-presented R-module U. Similarly, an R-module M is called *FP_n-flat* [3] if $\mathrm{Tor}^R_1(M,U) = 0$ for every right finitely n-presented R-module U;

(3) An R-module M is said to be *cotorsion* [5] if $\mathrm{Ext}^1_R(L,M) = 0$ for every flat R-module L. An R-module M is called *m-cotorsion* [10] if $\mathrm{Ext}^1_R(L,M) = 0$ for all R-modules L within the class $\mathcal{F}(R)_{\leq m}$, where $\mathcal{F}(R)_{\leq m}$ denotes the collection of all R-modules with flat dimension less than or equal to m.

Definition 2.2. (1) An R-module U is referred to as *super finitely presented* [6] if there exists an exact sequence

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow U \longrightarrow 0,$$

where each F_i is finitely generated and projective;

(2) A module M is termed *weak injective* or *FP_∞-injective* [7] if $\mathrm{Ext}^1_R(U,M) = 0$ for every super finitely presented R-module U. Similarly, an R-module M is called *weak flat* or *FP_∞-flat* [7] if $\mathrm{Tor}^R_1(M,U) = 0$ for every right super finitely presented R-module U;

(3) An R-module M is called *m-weak cotorsion* (see [15]) if $\mathrm{Ext}^1_R(L,M) = 0$ for every R-module L that belongs to $\mathcal{WF}(R)_{\leq m}$. Here, $\mathcal{WF}(R)_{\leq m}$ denotes the class of all R-modules with weak flat dimension less than or equal to m.

Definition 2.3. Let R be a ring.

(1) An R-module C is considered *semidualizing* [9] if it satisfies the following conditions:
(a_1) C possesses a degreewise finite projective resolution;
(a_2) The homothety map $\gamma : R \to \mathrm{Hom}_R(C,C)$ is an isomorphism;
(a_3) $\mathrm{Ext}^i_R(C,C) = 0$ for all $i \geq 1$.

(2) A semidualizing module C is said to be *faithfully semidualizing* [9] if for every R-module M, $\mathrm{Hom}_R(C,M) = 0$ implies that $M = 0$

We define

$$C^{\perp_1} = \{M \mid \mathrm{Ext^1_R(C,M)} = 0,\ \text{where } M \text{ is an } R\text{−module}\}$$

and

$$C^{\perp} = \{M \mid \mathrm{Ext^i_R(C,M)} = 0\ \textit{for any}\ i \geq 1,\ \text{where } M \text{ is an } R\text{−module}\}.$$

Definition 2.4. (1) The *Auslander class* $\mathcal{A}_C(R)$ [9] with respect to C comprises all R-modules M that satisfy the following conditions:

(A_1)] $\mathrm{Tor}_i^R(C, M) = 0$ for all $i \geq 1$;
(A_2) $\mathrm{Ext}_R^i(C, C \otimes_R M) = 0$ for all $i \geq 1$;
(A_3) The natural evaluation homomorphism $\mu_M : M \to \mathrm{Hom}_R(C, C \otimes_R M)$ is an isomorphism of R-modules.

(2) The *Bass class* $\mathcal{B}_C(R)$ [9] with respect to C consists of all R-modules N satisfying the following conditions:
(B_1) $\mathrm{Ext}_R^i(C, N) = 0$ for all integers $i \geq 1$;
(B_2) $\mathrm{Tor}_i^R(C, \mathrm{Hom}_R(C, N)) = 0$ for all integers $i \geq 1$;
(B_3) The natural evaluation homomorphism $\nu_N : C \otimes_R \mathrm{Hom}_R(C, N) \to N$ is an isomorphism of R-modules.

Lemma 2.5. ([9, Proposition 4.1]) *Let C be a semidualizing module. The following categories are equivalent:*

$$\mathcal{A}_C(R) \underset{\mathrm{Hom}_R(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathcal{B}_C(R)\ .$$

Definition 2.6. (1) An R-module is considered C*-injective* [9] if it can be expressed as $\mathrm{Hom}_R(C, I)$ for some injective R-module I. An R-module is termed C*-flat* [9] if it can be written as $C \otimes_R F$ for some flat R-module F. Additionally, an R-module is defined as C*-projective* [9] if it has the form $C \otimes_R P$ for some projective R-module P;
(2) An R-module M is called C*-cotorsion* [4] if the group $\mathrm{Ext}_R^1(F, M)$ vanishes for every C-flat R-module F;
(3) An R-module is termed C*-weakly injective* [8] if it can be expressed as $\mathrm{Hom}_R(C, I)$ for some weakly injective R-module I. Similarly, an R-module is called C*-weakly flat* [8] if it is of the form $C \otimes_R F$ where F is a weakly flat R-module;
(4) An R-module is termed C*-m-weak cotorsion* [2] if, for every module $L \in \mathcal{WF}_C(R)_{\leq m}$, the group $\mathrm{Ext}_R^1(L, M)$ vanishes. Here, $\mathcal{WF}_C(R)_{\leq m}$ denotes the class of all R-modules with C-weak flat dimension less than or equal to m.

Definition 2.7. An R-module is referred to as C*-FP_n-injective* [17] if it is isomorphic to $\mathrm{Hom}_R(C, X)$ for some FP_n-injective R-module X. Similarly, an R-module is considered C*-FP_n-flat* [17] if it is isomorphic to $C \otimes_R Y$ for some FP_n-flat R-module Y.

We define the following classes:

(1) $\mathcal{P}_C(R)_{\leq m}$ as the class of R-modules with C-projective dimension at most m,
(2) $\mathcal{FI}^n(R)_{\leq m}$ as the class of R-modules with FP_n-injective dimension at most m,

(3) $\mathcal{FF}^n(R)_{\leq m}$ as the class of R-modules with FP_n-flat dimension at most m,
(4) $\mathcal{FI}^n_C(R)_{\leq m}$ as the class of R-modules with C-FP_n-injective dimension at most m, and
(5) $\mathcal{FF}^n_C(R)_{\leq m}$ as the class of R-modules with C-FP_n-flat dimension at most m.

When $m = 0$, these classes specialize as follows:

(1) $\mathcal{FI}^n_C(R)_{\leq 0} = \mathcal{FI}^n_C(R)$, which is the class of C-FP_n-injective R-modules,
(2) $\mathcal{FF}^n_C(R)_{\leq 0} = \mathcal{FF}^n_C(R)$, the class of C-FP_n-flat R-modules, and
(3) $\mathcal{P}_C(R)_{\leq 0} = \mathcal{P}_C(R)$, representing the class of C-projective R-modules.

Definition 2.8. An R-module M is referred to as *flat cotorsion* [5] if it is both flat and cotorsion. An R-module M is called *C-flat C-cotorsion* [4] if it is isomorphic to an R-module of the form $C \otimes_R F$, where F is flat and cotorsion.

3. C-(n,m)-cotorsion modules

In this section, we present and examine the concept of C-(n,m)-cotorsion modules. We begin with their formal definition.

Definition 3.1. Let n, m be non-negative integers. Then, an R-module M is called *C-(n,m)-cotorsion* if $\mathrm{Ext}^1_R(N, M) = 0$ for every $N \in \mathcal{FF}^n_C(R)_{\leq m}$. The module M is said to be *strongly C-(n,m)-cotorsion* if $\mathrm{Ext}^i_R(N, M) = 0$ for all $i \geq 1$ and for every $N \in \mathcal{FF}^n_C(R)_{\leq m}$. If $C = R$, then C-(n,m)-cotorsion modules are referred to as *(n,m)-cotorsion*.

The classes of C-(n,m)-cotorsion R-modules, strongly C-(n,m)-cotorsion R-modules, and (n,m)-cotorsion R-modules are denoted by $Cot_C^{(n,m)}(R)$, $SCot_C^{(n,m)}(R)$, and $Cot^{(n,m)}(R)$, respectively.

Remark 3.2. (1) If $n = m = 0$, then the C-(n,m)-cotorsion modules coincide with the C-cotorsion modules;
(2) If $n = \infty$, then the C-(∞, m)-cotorsion modules are precisely the C-m-cotorsion modules;
(3) If $C = R$ and $n = \infty$, then C-(n,m)-cotorsion modules coincide precisely with m-weak cotorsion modules;
(4) If $C = R$ and $n = 0$ or 1, then C-(n,m)-cotorsion modules coincide precisely with m-cotorsion modules;
(5) Every C-(n_2,m)-cotorsion module is also C-(n_1,m)-cotorsion for any $n_2 \geq n_1$, though the converse does not necessarily hold (see Example 3.3(2)). In fact, the following inclusions hold:
$$\cdots \subseteq Cot_C^{(n_3,m)}(R) \subseteq Cot_C^{(n_2,m)}(R) \subseteq Cot_C^{(n_1,m)}(R).$$
(6) $SCot_C^{(n,m)}(R) \subseteq Cot_C^{(n,m)}(R)$.

A ring R is referred to as an $(n,0)$-ring [19] if every finitely n-presented R-module is projective.

Example 3.3. (1) Let K be a field, E a nonzero K-vector space, and $B = K \ltimes E$ a trivial extension of K by E, where E has infinite rank over K. Additionally, let B' be a Noetherian ring with global dimension zero, and define $R = B' \times B$. According to [12, Theorem 3.4], R is a $(2,0)$-ring but not a $(1,0)$-ring. Consequently, every finitely 2-presented R-module U is projective, implying that $\mathrm{Ext}^1_R(U, M) = 0$ for any R-module M. Therefore, M is FP_2-injective (resp. FP_2-flat). Furthermore, there exists an R-module that is not FP_1-injective (resp. FP_1-flat). To see this, note that if every R-module were FP_1-injective (resp. FP_1-flat), then every finitely 1-presented R-module would be projective, which would imply that R is a $(1,0)$-ring, contradicting the prior result. Finally, when $C = R$, it follows that $M \cong \mathrm{Hom}_R(C, M)$ (resp. $M \cong C \otimes_R M$) for any R-module M. Thus, we conclude that every R-module is C-FP_2-injective (resp. C-FP_2-flat), while also recognizing the existence of modules that are not C-FP_1-injective (resp. C-FP_1-flat).

(2) In (1), if $C = R$, then every R-module is C-FP_2-flat, and there exists an R-module that is not C-FP_1-flat. Consequently, not every member of $\mathcal{FF}^2_C(R^{op})_{\leq m}$ belongs to $\mathcal{FF}^1_C(R^{op})_{\leq m}$. Therefore, it follows that there exists a module in $Cot_C^{(1,m)}(R)$ that is not contained in $Cot_C^{(2,m)}(R)$.

(3) Let R be a semisimple ring and $R = C$. Then every module is in $Cot_C^{(n,m)}(R)$ and $SCot_C^{(n,m)}(R)$.

The following result establishes the relationship between the classes $Cot_C^{(n,m)}(R)$ and $Cot^{(n,m)}(R)$. Specifically, we verify the equivalent conditions involving C-(n,m)-cotorsion modules.

Theorem 3.4. *Let M be an R-module. Then M belongs to $Cot_C^{(n,m)}(R)$ if and only if M is in $C^{\perp_1}$ and $\mathrm{Hom}_R(C, M)$ is in $Cot^{(n,m)}(R)$.*

Proof. ($\Rightarrow$). It is evident that $C \cong C \otimes_R R$ is a C-FP_n-flat R-module, and consequently, $C \in \mathcal{FF}^n_C(R)_{\leq m}$. Therefore, we have $\mathrm{Ext}^1_R(C, M) = 0$ for any R-module M in $\mathrm{Cot}_C^{(n,m)}(R)$, i.e., $M \in C^{\perp_1}$. Let $N \in \mathcal{FF}^n(R)_{\leq m}$. Then there exists a finite exact sequence of R-modules:

$$0 \longrightarrow F_m \longrightarrow F_{m-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow N \longrightarrow 0,$$

where each $F_i \in \mathcal{FF}^n(R)$ for $0 \leq i \leq m$. Consider the short exact sequences

$$0 \longrightarrow K_0 \longrightarrow F_0 \longrightarrow N \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow K_i \longrightarrow F_i \longrightarrow K_{i-1} \longrightarrow 0,$$

where $K_0 = \ker(F_0 \to N)$ and $K_i = \ker(F_i \to F_{i-1})$. According to [17, Corollary 3.4], both N and K_i is an element of $\mathcal{A}_C(R)$. Hence, the following exact sequence exists

$$0 \longrightarrow C \otimes_R F_m \longrightarrow C \otimes_R F_{m-1} \longrightarrow \cdots \longrightarrow C \otimes_R F_1 \longrightarrow C \otimes_R F_0 \longrightarrow C \otimes_R N \longrightarrow 0,$$

with each $C\otimes_R F_i \in \mathcal{FF}^n_C(R)$. From this, it follows that $C\otimes_R N \in \mathcal{FF}^n_C(R)_{\leq m}$. Consequently, $\operatorname{Ext}^1_R(C\otimes_R N, M) = 0$ because M lies in $\operatorname{Cot}^{(n,m)}_C(R)$. Furthermore, there exists a short exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow N \longrightarrow 0,$$

where F is a projective R-module. Applying the functor $C\otimes_R -$ yields the exact sequence

$$0 \longrightarrow C\otimes_R K \longrightarrow C\otimes_R F \longrightarrow C\otimes_R N \longrightarrow 0,$$

and since $C\otimes_R F$ is in $\mathcal{FF}^n_C(R)_{\leq m}$, we conclude that $\operatorname{Ext}^1_R(C\otimes_R F, M) = 0$.

According to [14, Theorem 2.75], one can readily obtain the following commutative diagram with exact rows:

$$\operatorname{Hom}_R(F, \operatorname{Hom}_R(C,M)) \longrightarrow \operatorname{Hom}_R(K, \operatorname{Hom}_R(C,M)) \longrightarrow \operatorname{Ext}^1_R(N, \operatorname{Hom}_R(C,M)) \longrightarrow 0.$$

By the Five Lemma, it follows that $\operatorname{Ext}^1_R(C\otimes_R N, M) \cong \operatorname{Ext}^1_R(N, \operatorname{Hom}_R(C,M))$. Therefore, since $\operatorname{Ext}^1_R(N, \operatorname{Hom}_R(C,M)) = 0$, it follows that $\operatorname{Hom}_R(C,M)$ belongs to the class $Cot^{(n,m)}(R)$.

($\Leftarrow$). There exists a short exact sequence $0 \to M \to E \to D \to 0$, where E is an injective module. Consequently, the sequence

$$0 \to \operatorname{Hom}_R(C,M) \to \operatorname{Hom}_R(C,E) \to \operatorname{Hom}_R(C,D) \to 0$$

is exact, given that $M \in C^{\perp_1}$.

If $N \in \mathcal{FF}^n_C(R)_{\leq m}$, then, according to [17, Proposition 4.2 and Theorem 4.4], we have $N \in \mathcal{B}_C(R)$, which implies that $\operatorname{Hom}_R(C,N) \in \mathcal{A}_C(R)$. Additionally, $\operatorname{Hom}_R(C,E) \in \mathcal{A}_C(R)$ as established by [9, Lemma 4.1] and [17, Lemma 3.9]. Furthermore, by applying [9, Theorem 6.4] and [9, Lemma 4.1], it follows that: $\operatorname{Ext}^1_R(\operatorname{Hom}_R(C,N), \operatorname{Hom}_R(C,E)) \cong \operatorname{Ext}^1_R(C\otimes_R \operatorname{Hom}_R(C,N), C\otimes_R \operatorname{Hom}_R(C,E)) \cong \operatorname{Ext}^1_R(N,E) = 0$. Consequently, the following exact commutative diagram can be derived:

$$\begin{tikzcd}
{} \arrow[r] \arrow[d, "\cong"] & \operatorname{Hom}_R(\operatorname{Hom}_R(C,N), \operatorname{Hom}_R(C,D)) \arrow[r] \arrow[d, "\cong"] & \operatorname{Ext}^1_R(\operatorname{Hom}_R(C,N), \operatorname{Hom}_R(C,M)) \arrow[r] \arrow[d] & 0 \\
{} \arrow[r] \arrow[d, "\cong"] & \operatorname{Hom}_R(C\otimes_R \operatorname{Hom}_R(C,N), D) \arrow[r] \arrow[d, "\cong"] & \operatorname{Ext}^1_R(C\otimes_R \operatorname{Hom}_R(C,N), M) \arrow[d] & \\
{} \arrow[r] & \operatorname{Hom}_R(N,D) \arrow[r] & \operatorname{Ext}^1_R(N,M) \arrow[r] & 0
\end{tikzcd}$$

Therefore, by the Five Lemma, we have an isomorphism $\operatorname{Ext}^1_R(\operatorname{Hom}_R(C,N), \operatorname{Hom}_R(C,M)) \cong \operatorname{Ext}^1_R(N,M)$. Furthermore, from [17, Proposition 4.3], we know that $\operatorname{Hom}_R(C,N) \in \mathcal{FF}^n(R)_{\leq m}$. Under our assumptions, it follows that $\operatorname{Hom}_R(C,M) \in Cot^{(n,m)}(R)$. Consequently, this implies that $\operatorname{Ext}^1_R(\operatorname{Hom}_R(C,N), \operatorname{Hom}_R(C,M)) = 0$. Hence, we deduce that $\operatorname{Ext}^1_R(N,M) = 0$, which implies that M belongs to $Cot^{(n,m)}_C(R)$. $\square$

Additionally, for strongly C-(n,m)-cotorsion modules, the following result holds:

Proposition 3.5. *Let M be an R-module. Then M belongs to $SCot_C^{(n,m)}(R)$ if and only if M is in $C^{\perp}$ and $\mathrm{Hom}_R(C,M)$ is an element of $Cot^{(n,m)}(R)$.*

Proof. ($\Rightarrow$). By Definition 3.1, we have that $M \in C^{\perp}$, since $C \cong C \otimes_R R$ is in $\mathcal{FF}_C^n(R)$, and $C \in \mathcal{FF}_C^n(R)_{\leq m}$. Furthermore, $\mathrm{Hom}_R(C,M)$ belongs to $Cot^{(n,m)}(R)$ by Remark 3.2(5) and Theorem 3.4.

Let $N \in \mathcal{FF}_C^n(R)_{\leq m}$. Then, by [17, Proposition 4.2 and Theorem 4.4], we have $N \in \mathcal{B}_C(R)$, and consequently, $\mathrm{Hom}_R(C,N) \in \mathcal{FF}^n(R)_{\leq m}$ as established in [17, Proposition 4.3]. Since $M \in C^{\perp}$, it follows that $\mathrm{Ext}_R^{i\geq 1}(C,M) = 0$. Therefore, by [9, Theorem 6.4] and the given hypothesis, we obtain $\mathrm{Ext}_R^{i\geq 1}(N,M) \cong \mathrm{Ext}_R^{i\geq 1}(\mathrm{Hom}_R(C,N), \mathrm{Hom}_R(C,M)) = 0$. Hence, M belongs to $SCot_C^{(n,m)}(R)$. $\square$

The following result characterizes the conditions under which each module in $\mathcal{FF}^n(R)_{\leq m}$ is projective. The class of projective R-modules is denoted by $\mathcal{P}(R)$.

Theorem 3.6. *An element $N \in \mathcal{FF}^n(R)_{\leq m}$ belongs to $\mathcal{P}(R)$ if and only if every $M \in C^{\perp_1}$ is contained in $Cot_C^{(n,m)}(R)$.*

Proof. ($\Rightarrow$). Let $N \in \mathcal{FF}^n(R)_{\leq m}$. Then $\mathrm{Ext}_R^1(N, \mathrm{Hom}_R(C,M)) = 0$, which implies that $\mathrm{Hom}_R(C,M)$ is (n,m)-cotorsion. Since $M \in C^{\perp_1}$, Theorem 3.4 indicates that M belongs to the class $\mathrm{Cot}_C^{(n,m)}(R)$.

($\Leftarrow$). Let $N \in \mathcal{FF}^n(R)_{\leq m}$. Then there exists an exact sequence

$$0 \longrightarrow F_m \longrightarrow F_{m-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow N \longrightarrow 0$$

of R-modules, where each $F_i \in \mathcal{FF}^n(R)$ for all $0 \leq i \leq m$. Define $K_0 = \ker(F_0 \to N)$ and $K_i = \ker(F_i \to F_{i-1})$. According to [17, Corollary 3.4], N and each K_i (for $0 \leq i \leq m$) belong to $\mathcal{A}_C(R)$. Consequently, the following exact sequence exists:

$$0 \longrightarrow C \otimes_R F_m \longrightarrow C \otimes_R F_{m-1} \longrightarrow \cdots \longrightarrow C \otimes_R F_1 \longrightarrow C \otimes_R F_0 \longrightarrow C \otimes_R N \longrightarrow 0,$$

where each $C \otimes_R F_i \in \mathcal{FF}_C^n(R)$. It follows that $C \otimes_R N \in \mathcal{FF}_C^n(R)_{\leq m}$. Furthermore, consider the short exact sequence of R-modules:

$$0 \longrightarrow K \longrightarrow P \longrightarrow N \longrightarrow 0,$$

where $P \in \mathcal{P}(R)$. By [17, Corollary 3.4], we have $N, K, P \in \mathcal{A}_C(R)$. Thus, the sequence

$$0 \longrightarrow C \otimes_R K \longrightarrow C \otimes_R P \longrightarrow C \otimes_R N \longrightarrow 0$$

is exact, with each term in $\mathcal{B}_C(R)$. Applying the functor $\mathrm{Hom}_R(C, \cdot)$, we obtain the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(C, C \otimes_R K) \longrightarrow \mathrm{Hom}_R(C, C \otimes_R P) \longrightarrow \mathrm{Hom}_R(C, C \otimes_R N) \longrightarrow 0.$$

This implies that $\mathrm{Ext}_R^1(C, C \otimes_R K) = 0$. Therefore, $C \otimes_R K \in C^{\perp_1}$, and by hypothesis, $C \otimes_R K \in \mathrm{Cot}_C^{(n,m)}(R)$. Consequently, $\mathrm{Ext}_R^1(C \otimes_R N, C \otimes_R K) = 0$, which ensures the splitting of the exact sequence

$$0 \longrightarrow C \otimes_R K \longrightarrow C \otimes_R P \longrightarrow C \otimes_R N \longrightarrow 0.$$

Hence, $C \otimes_R N$ belongs to $\mathcal{P}_C(R)$. Applying [9, Lemma 5.1(b)], we see that $\mathrm{Hom}_R(C, C \otimes_R N) \in \mathcal{P}(R)$. Since $N \cong \mathrm{Hom}_R(C, C \otimes_R N)$ (because $N \in \mathcal{A}_C(R)$), it follows that $N \in \mathcal{P}(R)$. $\square$

Lemma 3.7. *Let $M \in \mathcal{B}_C(R)$. The following statements are equivalent:*

(1) *M belongs to $SCot_C^{(n,m)}(R)$;*
(2) *M belongs to $Cot_C^{(n,m)}(R)$;*
(3) *The module $\mathrm{Hom}_R(C, M)$ belongs to $Cot^{(n,m)}(R)$.*

Proof. (1)⇔(3). Since $M \in \mathcal{B}_C(R)$, it follows that $\mathrm{Ext}_R^{i \geq 1}(C, M) = 0$, implying that $M \in C^{\perp}$. Therefore, by Proposition 3.5, we conclude that $\mathrm{Hom}_R(C, M)$ belongs to $Cot^{(n,m)}(R)$ if and only if M is in $SCot_C^{(n,m)}(R)$.

(2)⇔(3). Assuming the hypothesis that $\mathrm{Ext}_R^1(C, M) = 0$, it follows that $M \in C^{\perp_1}$. Consequently, Theorem 3.4 establishes that $\mathrm{Hom}_R(C, M)$ belongs to $Cot^{(n,m)}(R)$ if and only if M belongs to $Cot_C^{(n,m)}(R)$. $\square$

Lemma 3.8. *Let $M \in \mathcal{A}_C(R)$. Then the following statements are equivalent:*

(1) *$C \otimes_R M$ belongs to $SCot_C^{(n,m)}(R)$;*
(2) *$C \otimes_R M$ belongs to $Cot_C^{(n,m)}(R)$;*
(3) *M belongs to $Cot^{(n,m)}(R)$.*

Proof. (1)⇔(3). Let $M \in \mathcal{A}_C(R)$. Then, by Lemma 2.5, we have $C \otimes_R M \in \mathcal{B}_C(R)$. Consequently, $\mathrm{Ext}_R^i(C, C \otimes_R M) = 0$ for all $i \geq 1$, which implies that $C \otimes_R M$ belongs to $C^{\perp}$. Therefore, by Proposition 3.5, we conclude that $\mathrm{Hom}_R(C, C \otimes_R M) \cong M$ is in $\mathrm{Cot}^{(n,m)}(R)$ if and only if $C \otimes_R M$ is in $SCot_C^{(n,m)}(R)$.

(2)⇔(3). Based on the hypothesis and Lemma 2.5, we have that $\mathrm{Ext}_R^1(C, C \otimes_R M) = 0$, which implies $C \otimes_R M \in C^{\perp_1}$. Consequently, according to Theorem 3.4, the isomorphism $\mathrm{Hom}_R(C, C \otimes_R M) \cong M$ holds, and M belongs to $Cot^{(n,m)}(R)$ if and only if $C \otimes_R M$ belongs to $SCot_C^{(n,m)}(R)$. $\square$

Definition 3.9. Let n, m be non-negative integers. An R-module M is defined as *C-FP_n-flat C-(n,m)-cotorsion* if it is isomorphic to an R-module of the form $C \otimes_R N$, where N belongs to the class $\mathcal{FF}^n(R)_{\leq m}$ and to $Cot^{(n,m)}(R)$.

It is evident that when $n = m = 0$, the C-FP_n-flat C-(n,m)-cotorsion R-modules coincide with the C-flat C-cotorsion R-modules.

Example 3.10. Let R be a hereditary ring and $R = C$. Then by [13, Theorem 4.23], we deduce that every R-module is C-FP_n-flat C-(n,m)-cotorsion.

Proposition 3.11. *The following conditions are equivalent:*

(1) *M belongs to $\mathcal{FF}_C^{cot^{(n,m)}}(R)$;*

(2) *M is an element of $\mathcal{FF}_C^n(R)_{\leq m}$ and belongs to $Cot_C^{(n,m)}(R)$;*
(3) *M is an element of $\mathcal{FF}_C^n(R)_{\leq m}$ and belongs to $SCot_C^{(n,m)}(R)$;*
(4) *M is in $\mathcal{B}_C(R)$ and $\mathrm{Hom}_R(C,M)$ is in $\mathcal{FF}^{cot^{(n,m)}}(R)$;*
(5) *$\mathrm{Hom}_R(C,M)$ is in $\mathcal{FF}^{cot^{(n,m)}}(R)$.*

Proof. (1)$\Leftrightarrow$(2) According to [17, Corollary 3.4], the modules N and N' are elements of $\mathcal{A}_C(R)$ whenever $N, N' \in \mathcal{FF}^n(R)_{\leq m}$. Consequently, it follows that $\mathrm{Tor}_i^R(C,N') = 0$ for all $i \geq 1$. Let $M = C \otimes_R N$, with $N \in \mathcal{FF}^n(R)_{\leq m}$. Then, by [9, Theorem 6.4], there is an isomorphism:

$$\mathrm{Ext}_R^{i\geq 0}(C \otimes_R N', C \otimes_R N) \cong \mathrm{Ext}_R^{i\geq 0}(N', N).$$

(1)$\Rightarrow$(4). Let M be an element of $\mathcal{FF}_C^{cot^{(n,m)}}(R)$. Then M is isomorphic to $C \otimes_R N$, where N belongs to both $\mathcal{FF}^n(R)_{\leq m}$ and $Cot^{(n,m)}(R)$; that is, $N \in \mathcal{FF}^{cot^{(n,m)}}(R)$. According to [17, Corollary 3.4], we have $N \in \mathcal{A}_C(R)$. Consequently, we obtain

$$\mathrm{Hom}_R(C,M) \cong \mathrm{Hom}_R(C, C \otimes_R N) \cong N \in \mathcal{FF}^{cot^{(n,m)}}(R).$$

Furthermore, it is evident that $M \in \mathcal{FF}_C^{cot^{(n,m)}}(R) \subseteq \mathcal{FF}_C^n(R)_{\leq m}$. By [17, Proposition 4.2], it follows that $\mathcal{FF}_C^n(R)_{\leq m} \subseteq \mathcal{B}_C(R)$, and thus $M \in \mathcal{B}_C(R)$.
(4)$\Rightarrow$(1). Let $M \in \mathcal{B}_C(R)$ and assume that $\mathrm{Hom}_R(C,M) \in \mathcal{FF}^{\mathrm{cot}^{(n,m)}}(R)$. Then it follows that $M \cong C \otimes_R \mathrm{Hom}_R(C,M) \in \mathcal{FF}_C^{\mathrm{cot}^{(n,m)}}(R)$.

(4)$\Rightarrow$(5). It is clear.

(5)$\Rightarrow$(4). Let $\mathrm{Hom}_R(C,M) \in \mathcal{FF}^{\mathrm{cot}^{(n,m)}}(R) \subseteq \mathcal{FF}^n(R)_{\leq m}$. Then, by [17, Corollary 3.4], it follows that $\mathrm{Hom}_R(C,M) \in \mathcal{A}_C(R)$. Consequently, applying Lemma 2.5, we deduce that $M \in \mathcal{B}_C(R)$.

(3)$\Rightarrow$(4). Let $M \in \mathcal{FF}_C^n(R)_{\leq m}$. Then, there exists an exact sequence

$$0 \longrightarrow N_m \longrightarrow N_{m-1} \longrightarrow \cdots \longrightarrow N_1 \longrightarrow N_0 \longrightarrow M \longrightarrow 0,$$

where each $N_i \in \mathcal{FF}_C^n(R)$ for all $0 \leq i \leq m$. According to [17, Theorem 4.4], both M and the N_i are elements of $\mathcal{B}_C(R)$. Consequently, by [9, Theorem 6.3], there exists an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(C,N_m) \longrightarrow \mathrm{Hom}_R(C,N_{m-1}) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_R(C,N_0) \longrightarrow \mathrm{Hom}_R(C,M) \longrightarrow 0,$$

of R-modules. Furthermore, by [17, Theorem 4.4], each $\mathrm{Hom}_R(C,N_i)$ belongs to $\mathcal{FF}^n(R)$ for all $0 \leq i \leq m$. As a result, $\mathrm{Hom}_R(C,M)$ is an element of $\mathcal{FF}^n(R)_{\leq m}$. Additionally, by Lemma 3.7, we have $\mathrm{Hom}_R(C,M) \in \mathrm{Cot}^{(n,m)}(R)$. Therefore, it follows that $\mathrm{Hom}_R(C,M)$ belongs to $\mathcal{FF}^{\mathrm{cot}^{(n,m)}}(R)$.

(4)$\Rightarrow$(3). By condition (4), the module $\mathrm{Hom}_R(C,M)$ belongs to the class $\mathcal{FF}^n(R)_{\leq m}$, and M is contained in $\mathcal{B}_C(R)$. According to [17, Theorem 4.4], this yields that M is in $\mathcal{FF}_C^n(R)_{\leq m}$. Additionally, we observe that $M \in C^{\perp}$ and $\mathrm{Hom}_R(C,M)$ lies in $\mathcal{FF}^{cot(n,m)}(R)$. Proposition 3.5 then ensures that M belongs to the class $SCot_C^{(n,m)}(R)$. $\square$

In the following, we explore Foxby equivalence concerning the classes $\mathcal{FF}^{cot^{(n,m)}}(R)$ and $\mathcal{FF}_C^{cot^{(n,m)}}(R)$, extending the concept of Foxby equivalence beyond the classes $\mathcal{F}^{cot}(R)$ and $\mathcal{F}_C^{cot}(R)$ as discussed in [4,16].

Proposition 3.12. *The following categories are equivalent:*

(1) $\mathcal{FF}^{cot^{(n,m)}}(R) \underset{\mathrm{Hom}_R(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathcal{FF}_C^{cot^{(n,m)}}(R)$;

(2) $\mathcal{FF}^{cot^{(n,m)}}(R)_{\leq t} \underset{\mathrm{Hom}_R(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathcal{FF}_C^{cot^{(n,m)}}(R)_{\leq t}$.

Proof. (a). If $M \in \mathcal{FF}^{cot^{(n,m)}}(R)$, then $M \in \mathcal{FF}^{cot^{(n,m)}}(R) \subseteq \mathcal{FF}^n(R)_{\leq m}$, and consequently, $M \in \mathcal{A}_C(R)$ as established in [17, Corollary 3.4]. By Lemma 3.8, it follows that $C\otimes_R M \in Cot_C^{(n,m)}(R)$. Additionally, $C\otimes_R M \in \mathcal{FF}_C^n(R)_{\leq m}$ according to [17, Theorem 4.4]. The tensor functor $C\otimes_R -$ thus maps objects in $\mathcal{FF}^{cot^{(n,m)}}(R)$ to objects in $\mathcal{FF}_C^{cot^{(n,m)}}(R)$. Conversely, this is clear by Proposition 3.11. Moreover, if $N \in \mathcal{FF}_C^{cot^{(n,m)}}(R)$, then $N \in \mathcal{FF}_C^n(R)_{\leq m}$, which implies $N \in \mathcal{B}_C(R)$ by [17, Theorem 4.4]. Consequently, there exist natural isomorphisms: $M \cong \mathrm{Hom}_R(C, C\otimes_R M)$ and $N \cong C\otimes_R \mathrm{Hom}_R(C,N)$. Therefore, the assertion holds.

(b). The case $t = 0$ is straightforward by part (a). For $t \geq 1$, assume $M \in \mathcal{FF}^{\mathrm{cot}^{(n,m)}}(R)_{\leq t}$. Then there exists an exact sequence of R-modules:

$$0 \longrightarrow N_t \longrightarrow N_{t-1} \longrightarrow \cdots \longrightarrow N_1 \longrightarrow N_0 \longrightarrow M \longrightarrow 0,$$

where each $N_i \in \mathcal{FF}^{\mathrm{cot}^{(n,m)}}(R) \subseteq \mathcal{A}_C(R)$ for $0 \leq i \leq t$. According to [9, Corollary 6.3], both M and $\mathrm{coker}(N_i \to N_{i-1})$ are contained in $\mathcal{A}_C(R)$. Applying the functor $C\otimes_R -$ to this exact sequence yields:

$$0 \longrightarrow C\otimes_R N_t \longrightarrow C\otimes_R N_{t-1} \longrightarrow \cdots \longrightarrow C\otimes_R N_1 \longrightarrow C\otimes_R N_0 \longrightarrow C\otimes_R M \longrightarrow 0,$$

which is an exact sequence of R-modules where each $C\otimes_R N_i \in \mathcal{FF}_C^{\mathrm{cot}^{(n,m)}}(R)$. Consequently, it follows that $C\otimes_R M \in \mathcal{FF}_C^{\mathrm{cot}^{(n,m)}}(R)_{\leq t}$, establishing the result. Conversely, suppose $M \in \mathcal{FF}_C^{\mathrm{cot}^{(n,m)}}(R)_{\leq t}$. Then there exists an exact sequence:

$$0 \longrightarrow C\otimes_R N_t \longrightarrow C\otimes_R N_{t-1} \longrightarrow \cdots \longrightarrow C\otimes_R N_1 \longrightarrow C\otimes_R N_0 \longrightarrow M \longrightarrow 0,$$

where each $N_i \in \mathcal{FF}^{\mathrm{cot}^{(n,m)}}(R) \subseteq \mathcal{FF}_C^n(R)_{\leq m} \subseteq \mathcal{A}_C(R)$. Using Lemma 2.5, we find that each $C\otimes_R N_i \in \mathcal{B}_C(R)$. Applying [9, Corollary 6.3] to this sequence produces an exact sequence:

$$0 \longrightarrow \mathrm{Hom}_R(C, C\otimes_R N_t) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_R(C, C\otimes_R N_0) \longrightarrow \mathrm{Hom}_R(C,M) \longrightarrow 0,$$

with each $N_i \cong \mathrm{Hom}_R(C, C\otimes_R N_i)$. This leads to the exact sequence:

$$0 \longrightarrow N_t \longrightarrow N_{t-1} \longrightarrow \cdots \longrightarrow N_1 \longrightarrow N_0 \longrightarrow \mathrm{Hom}_R(C,M) \longrightarrow 0,$$

which confirms that $\mathrm{Hom}_R(C,M) \in \mathcal{FF}^{\mathrm{cot}^{(n,m)}}(R)_{\leq t}$. $\square$

Proposition 3.13. *The following category equivalences are established:*

$$Cot^{(n,m)}(R)\bigcap\mathcal{A}_C(R)\ \underset{\mathrm{Hom}_R(C,-)}{\overset{C\otimes_R-}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}}\ Cot_C^{(n,m)}(R)\bigcap\mathcal{B}_C(R)\ .$$

Proof. It is concluded from Lemmas 2.5, 3.7, and 3.8. □

The following result demonstrates that the class of (n,m)-cotorsion R-modules within the Auslander class $\mathcal{A}_C(R)$ and the class of C-(n,m)-cotorsion R-modules within the Bass class $\mathcal{B}_C(R)$ are equivalent under Foxby equivalence.

Theorem 3.14. (Foxby Equivalence) *Let C be a semidualizing module. There are equivalences of categories:*

$$\begin{tikzcd}
\mathcal{FF}^{cot^{(n,m)}}(R) \arrow[r, shift left, "C\otimes_R-"] \arrow[d, hook] & \mathcal{FF}_C^{cot^{(n,m)}}(R) \arrow[l, shift left, "\mathrm{Hom}_R(C,-)"] \arrow[d, hook] \\
\mathcal{FF}^{cot^{(n,m)}}(R)_{\leq t} \arrow[r, shift left, "C\otimes_R-"] \arrow[d, hook] & \mathcal{FF}_C^{cot^{(n,m)}}(R)_{\leq t} \arrow[l, shift left, "\mathrm{Hom}_R(C,-)"] \arrow[d, hook] \\
\mathcal{A}_C(R) \arrow[r, shift left, "C\otimes_R-"] & \mathcal{B}_C(R) \arrow[l, shift left, "\mathrm{Hom}_R(C,-)"] \\
Cot^{(n,m)}(R)\bigcap\mathcal{A}_C(R) \arrow[r, shift left, "C\otimes_R-"] \arrow[u, hook] & Cot_C^{(n,m)}(R)\bigcap\mathcal{B}_C(R) \arrow[l, shift left, "\mathrm{Hom}_R(C,-)"] \arrow[u, hook]
\end{tikzcd}$$

Proof. This derives from Propositions 3.12 and 3.13. □

Corollary 3.15. *An R-module in $\mathcal{FF}_C^{cot^{(n,m)}}(R)$ belongs to $\mathcal{P}_C(R)_{\leq m}$ if and only if every R-module in $\mathcal{FF}^{cot^{(n,m)}}(R)$ belongs to $\mathcal{P}(R)_{\leq m}$.*

Proof. ($\Rightarrow$). Let $M \in \mathcal{FF}^{cot^{(n,m)}}(R)$. By Theorem 3.14, the module $C\otimes_R M$ belongs to $\mathcal{FF}_C^{cot^{(n,m)}}(R)$, which implies that $C\otimes_R M \in \mathcal{P}_C(R)_{\leq m}$. Consequently, there exists an exact sequence of R-modules:

$$0 \longrightarrow C\otimes_R P_m \longrightarrow C\otimes_R P_{m-1} \longrightarrow \cdots \longrightarrow C\otimes_R P_1 \longrightarrow C\otimes_R P_0 \longrightarrow C\otimes_R M \longrightarrow 0,$$

where each $P_i \in \mathcal{P}(R)$ for $0 \leq i \leq m$. Since $M \in \mathcal{FF}^n(R)_{\leq m}$, it follows from [17, Corollary 3.4] that $M \in \mathcal{A}_C(R)$, and by Theorem 3.14, we have an isomorphism $M \cong \mathrm{Hom}_R(C, C\otimes_R M)$. Furthermore, by [9, Lemma 4.1], each P_i also belongs to $\mathcal{A}_C(R)$, leading to the isomorphism $P_i \cong \mathrm{Hom}_R(C, C\otimes_R P_i)$. Additionally, both $C\otimes_R M$ and $C\otimes_R P_i$ lie in $\mathcal{B}_C(R)$. Therefore, applying [17, Corollary 6.3], we obtain a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(C, C\otimes_R P_m) \arrow[r] \arrow[d, "\cong"] & \cdots \arrow[r] & \mathrm{Hom}_R(C, C\otimes_R P_0) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_R(C, C\otimes_R M) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & P_m \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

From this, it follows that $M \in \mathcal{P}(R)_{\leq m}$.

($\Leftarrow$). Let $M \in \mathcal{FF}_C^{\mathrm{cot}^{(n,m)}}(R)$. Then, by Theorem 3.14, $\mathrm{Hom}_R(C, M) \in \mathcal{FF}^{\mathrm{cot}^{(n,m)}}(R)$, which implies that $\mathrm{Hom}_R(C, M)$ is in $\mathcal{P}(R)_{\leq m}$. Consequently, there exists an exact sequence of R-modules:

$$0 \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathrm{Hom}_R(C, M) \longrightarrow 0,$$

where each P_i belongs to $\mathcal{P}(R)$ for all $0 \leq i \leq m$. By Lemma 3.7 and Theorem 3.14, it follows that $\mathrm{Hom}_R(C, M) \in \mathcal{A}_C(R)$. Additionally, referencing [9, Lemma 4.1], each P_i also belongs to $\mathcal{A}_C(R)$. Moreover, by Theorem 3.14, we have the isomorphism $M \cong C \otimes_R \mathrm{Hom}_R(C, M)$. Using [17, Corollary 6.3], this leads to the construction of the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & C\otimes_R P_m \arrow[r] \arrow[d, "\cong"] & \cdots \arrow[r] & C\otimes_R P_0 \arrow[r] \arrow[d, "\cong"] & C\otimes_R \mathrm{Hom}_R(C, M) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & C\otimes_R P_m \arrow[r] & \cdots \arrow[r] & C\otimes_R P_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

From this, we conclude that M belongs to $\mathcal{P}_C(R)_{\leq m}$. $\square$

Proposition 3.16. *Let M be an R-module. The following statements are equivalent:*

(1) *M belongs to $\mathrm{Cot}_C^{(n,m)}(R)$;*
(2) *M is injective with respect to every short exact sequence of R-modules $0 \to K \to N \to F \to 0$, where F is an element of $\mathcal{FF}_C^n(R)_{\leq m}$.*

Proof. It is clear. $\square$

Example 3.17. Let R be a von Neumann ring and $R = C$. Then every R-module N is flat, and so by [17, Remark 3.2], N is in $\mathcal{FF}_C^n(R)_{\leq 0}$. Also, every injective R-module is in $Cot_C^{(n,0)}(R)$. So if the class $Cot_C^{(n,0)}(R)$ closed under direct sums, then the direct sum of injective R-modules is injective, which is a contradiction.

Proposition 3.18. *Let M be an R-module. Then, the following conditions are equivalent:*

(1) *M belongs to $Cot_C^{(n,m)}(R)$;*
(2) *For any $D \in \mathcal{P}(R)$, the module $\mathrm{Hom}_R(D, M)$ is in $Cot_C^{(n,m)}(R)$;*

Furthermore, if the class $Cot_C^{(n,m)}(R)$ is closed under direct sums, then these conditions are also equivalent to:

(1) *For any $D \in \mathcal{P}(R)$, the module $D \otimes_R M$ is in $Cot_C^{(n,m)}(R)$.*

Proof. (1) $\Rightarrow$ (2). Let $M \in \mathrm{Cot}_C^{(n,m)}(R)$ and $F \in \mathcal{FF}_C^n(R)_{\leq m}$. We aim to demonstrate that $\mathrm{Ext}_R^1(F, \mathrm{Hom}_R(D, M)) = 0$. Consider an exact sequence: $0 \to K \to P \to F \to 0$, where P belongs to $\mathcal{P}(R)$. From this, we obtain the following exact sequence:

$$\mathrm{Hom}_R(P, \mathrm{Hom}_R(D, M)) \to \mathrm{Hom}_R(K, \mathrm{Hom}_R(D, M)) \to \mathrm{Ext}_R^1(F, \mathrm{Hom}_R(D, M)) \to 0.$$

Furthermore, the sequence $0 \to K \otimes_R D \to P \otimes_R D \to F \otimes_R D \to 0$ is exact. This leads to another exact sequence:

$$\mathrm{Hom}_R(P \otimes_R D, M) \to \mathrm{Hom}_R(K \otimes_R D, M) \to \mathrm{Ext}_R^1(F \otimes_R D, M) \to 0.$$

These sequences induce the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_R(P, \mathrm{Hom}_R(D, M)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_R(K, \mathrm{Hom}_R(D, M)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Ext}_R^1(F, \mathrm{Hom}_R(D, M)) \arrow[r] \arrow[d] & 0 \\
\mathrm{Hom}_R(P \otimes_R D, M) \arrow[r] & \mathrm{Hom}_R(K \otimes_R D, M) \arrow[r] & \mathrm{Ext}_R^1(F \otimes_R D, M) \arrow[r] & 0
\end{tikzcd}$$

Consequently, we deduce that: $\mathrm{Ext}_R^1(F, \mathrm{Hom}_R(D, M)) \cong \mathrm{Ext}_R^1(F \otimes_R D, M)$. Since $F \in \mathcal{FF}_C^n(R)_{\leq m}$ and $D \in \mathcal{P}(R)$, there exists an exact sequence:

$$0 \to C \otimes_R F_m \otimes_R D \to C \otimes_R F_{m-1} \otimes_R D \to \cdots \to C \otimes_R F_0 \otimes_R D \to F \otimes_R D \to 0,$$

which implies that $F \otimes_R D$ belongs to $\mathcal{FF}_C^n(R)_{\leq m}$. Therefore, $\mathrm{Ext}_R^1(F \otimes_R D, M) = 0$. As a result, $\mathrm{Ext}_R^1(F, \mathrm{Hom}_R(D, M)) = 0$, which shows that $\mathrm{Hom}_R(D, M) \in \mathrm{Cot}_C^{(n,m)}(R)$.

The implications (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) are straightforward.

(1) $\Rightarrow$ (3). We have an isomorphism $F \cong K \oplus D$, where F is a free R-module. Consequently, we obtain $M \otimes_R F \cong \bigoplus_{i \in I} M \cong M \otimes_R K \oplus M \otimes_R D$. Since the direct sum $\bigoplus_{i \in I} M$ belongs to $Cot_C^{(n,m)}(R)$, it follows that $M \otimes_R D$ also belongs to $Cot_C^{(n,m)}(R)$. $\square$

4. Strongly C-(n, m)-Cotorsion Dimensions

Let t be a non-negative integer. For convenience, we define $SCot_C^{(n,m)}(R)_{\leq t}$ as the class of R-modules with strongly C-(n, m)-Cotorsion dimension at most t. Similarly, we denote $SCot_C^{(n,m)} - \mathrm{id}_R(M) \leq t$ as indicating that an R-module M has strongly C-(n, m)-Cotorsion dimension at most t.

Proposition 4.1. *Let M be an R-module and t a non-negative integer. The following conditions are equivalent:*

(1) $SCot_C^{(n,m)}\text{-}\mathrm{id}_R(M) \leq t$;
(2) $\mathrm{Ext}_R^{t+i}(F, M) = 0$ *for every* $F \in \mathcal{FF}_C^n(R)_{\leq m}$ *and all integers* $i \geq 1$;
(3) *For any exact sequence* $0 \longrightarrow M \longrightarrow Y_0 \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_{t-1} \longrightarrow Y_t \longrightarrow 0$ *where each* $Y_i \in SCot_C^{(n,m)}(R)$ *for* $0 \leq i \leq t-1$, *the module* Y_t *is also in* $SCot_C^{(n,m)}(R)$.

Proof. (1) ⇒ (2). Assume that $\mathrm{SCot}_C^{(n,m)}$-$\mathrm{id}_R(M) \leq t$. Then there exists an exact sequence

$$0 \longrightarrow M \longrightarrow Y_0 \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_{t-1} \longrightarrow Y_t \longrightarrow 0,$$

where each Y_i belongs to $\mathrm{SCot}_C^{(n,m)}(R)$ for all $0 \leq i \leq t$. Applying dimension shifting techniques, it follows that for any $F \in \mathcal{FF}_C^n(R)_{\leq m}$ and any $i \geq 1$, $\mathrm{Ext}_R^{t+i}(F, M) \cong \mathrm{Ext}_R^i(F, Y_t) = 0$.

(2) ⇒ (3). Using dimension shifting again, we observe that $0 = \mathrm{Ext}_R^{t+i}(F, M) \cong \mathrm{Ext}_R^i(F, Y_t)$. (3) ⇒ (1). This implication is straightforward. □

Proposition 4.2. *The class $SCot_C^{(n,m)}(R)_{<\infty}$ is closed under extensions, kernels of epimorphisms, and cokernels of monomorphisms.*

Proof. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules. If $SCot_C^{(n,m)}$-$\mathrm{id}_R(M') \leq SCot_C^{(n,m)}$-$\mathrm{id}_R(M'') \leq t < \infty$, then, by Proposition 4.1, for any $F \in \mathcal{FF}_C^n(R)_{\leq m}$, we have

$$0 = \mathrm{Ext}_R^{t+i}(F, M') \longrightarrow \mathrm{Ext}_R^{t+i}(F, M) \longrightarrow \mathrm{Ext}_R^{t+i}(F, M'') = 0.$$

Hence, $\mathrm{Ext}_R^{t+i}(F, M) = 0$, which implies that $SCot_C^{(n,m)}$-$\mathrm{id}_R(M) \leq t$. Similarly, it follows that the class $SCot_C^{(n,m)}(R)_{<\infty}$ is closed under kernels of epimorphisms and cokernels of monomorphisms. □

In this section, we consider the scenario where $S^{'}$ is a unitary ring extension of R with the assumption $S^{'} \geq R$. The ring $S^{'}$ is defined as being *right R-projective*, as described in [18, 19], if for any right $S^{'}$-module $M_{S'}$ and any $S^{'}$-module $N_{S'}$, the condition $N_R \mid M_R$ implies $N_{S'} \mid M_{S'}$. Here, $N \mid M$ indicates that N is a direct summand of M. A finite normalizing extension $S^{'} \geq R$ is one for which there exist elements $a_1, \dots, a_n \in S^{'}$ with $a_1 = 1$ such that $S^{'} = Ra_1 + Ra_2 + \cdots + Ra_n$. If, in addition, the extension $S^{'} \geq R$ is such that ${}_RS^{'}$ is flat, $S^{'}_R$ is projective, and $S^{'}$ is right R-projective, then $S^{'} \geq R$ is called an *almost excellent extension*. Furthermore, an almost excellent extension $S^{'} \geq R$ is termed an *excellent extension* if both ${}_RS^{'}$ and $S^{'}_R$ are free modules with a common basis $\{a_1, \dots, a_n\}$.

In this section, we examine the modules of the classes $SCot_C^{(n,m)}(R)$ and $SCot_C^{(n,m)}(R)_{<\infty}$ within the context of an almost excellent extension of rings, where C is a faithfully semidualizing R-module.

Lemma 4.3. *Let $S^{'} \geq R$ be an almost excellent extension, and let C be a (faithfully) semidualizing R-module. Then, the module $C \otimes_R S^{'}$ is a (faithfully) semidualizing $S^{'}$-module.*

Proof. Let C be a faithfully semidualizing R-module. Then, according to [4, Lemma 3.4], the module $C \otimes_R S^{'}$ is a semidualizing $S^{'}$-module. Suppose $\mathrm{Hom}_{S'}(C \otimes_R S^{'}, N) = 0$ for some $S^{'}$-module N. Then, we have:

$$0 = \mathrm{Hom}_{S'}(C \otimes_R S^{'}, N) \cong \mathrm{Hom}_R(C, \mathrm{Hom}_{S'}(C, N)) \cong \mathrm{Hom}_R(C, N).$$

Since C is faithfully semidualizing, it follows that $N = 0$. $\square$

We investigate modules with C-FP_n-injective dimension at most m and modules with C-FP_n-flat dimension at most m within the context of an almost excellent extension of rings, where C is a faithfully semidualizing R-module.

Lemma 4.4. *Let $S^{'} \geq R$ be an almost excellent extension. The following statements are valid:*

(1) *If M belongs to $\mathcal{FI}^n(R)_{\leq m}$, then $\mathrm{Hom}_R(S^{'}, M)$ belongs to $\mathcal{FI}^n(S^{'})_{\leq m}$;*

(2) *If M belongs to $\mathcal{FF}^n(R)_{\leq m}$, then $S^{'} \otimes_R M$ belongs to $\mathcal{FF}^n(S^{'})_{\leq m}$.*

Proof. (1). Let U be a finitely n-presented $S^{'}$-module. According to [18, Theorem 5], U is also a finitely n-presented R-module. There exists an exact sequence $0 \longrightarrow K_0 \longrightarrow F_0 \longrightarrow U \longrightarrow 0$, where F_0 is a free $S^{'}$-module. Utilizing [14, Theorem 2.76], we obtain the following commutative diagram:

$$\begin{tikzcd}
Hom_{S'}(F_0, \mathrm{Hom}_R(S^{'}, M)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{S'}(K_0, \mathrm{Hom}_R(S^{'}, M)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Ext}^1_{S'}(U, \mathrm{Hom}_R(S^{'}, M)) \arrow[r] \arrow[d] & 0 \\
\mathrm{Hom}_R(F_0, M) \arrow[r] & \mathrm{Hom}_R(K_0, M) \arrow[r] & \mathrm{Ext}^1_R(U, M) \arrow[r] & 0,
\end{tikzcd}$$

From this diagram, it follows that $\mathrm{Ext}^1_{S'}(U, \mathrm{Hom}_R(S^{'}, M)) \cong \mathrm{Ext}^1_R(U, M)$. Given the hypothesis that $\mathrm{Ext}^1_R(U, M) = 0$, we conclude that $\mathrm{Ext}^1_{S'}(U, \mathrm{Hom}_R(S^{'}, M)) = 0$. Consequently, $\mathrm{Hom}_R(S^{'}, M)$ belongs to the class $\mathcal{FI}^n(S^{'})$.

Let M be an element of $\mathcal{FI}^n(R)_{\leq m}$. Then there exists an exact sequence

$$0 \longrightarrow M \longrightarrow M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_m \longrightarrow 0,$$

of R-modules, where each M_i belongs to $\mathcal{FI}^n(R)$ for all $0 \leq i \leq m$. Since $S^{'}_R$ is projective, applying the functor $\mathrm{Hom}_R(S^{'}, -)$ yields an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(S^{'}, M) \longrightarrow \mathrm{Hom}_R(S^{'}, M_0) \longrightarrow \mathrm{Hom}_R(S^{'}, M_1) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_R(S^{'}, M_m) \longrightarrow 0,$$

of $S^{'}$-modules, with each $\mathrm{Hom}_R(S^{'}, M_i)$ belongs to $\mathcal{FI}^n(S^{'})$ for all $0 \leq i \leq m$. Consequently, $\mathrm{Hom}_R(S^{'}, M)$ is an element of $\mathcal{FI}^n(S^{'})_{\leq m}$.

(2). According to [17, Proposition 3.13], it follows that for an R-module N, N belongs to $\mathcal{FI}^n(R)_{\leq m}$ if and only if its dual N^* belongs to $\mathcal{FF}^n(R^{op})_{\leq m}$, and similarly, N is in $\mathcal{FF}^n(R)_{\leq m}$ if and only if N^* is in $\mathcal{FI}^n(R^{op})_{\leq m}$. Consequently, if M is in $\mathcal{FF}^n(R)_{\leq m}$, then M^* is in $\mathcal{FI}^n(R^{op})_{\leq m}$. Using this, along with (1) and [14, Proposition 2.56 and Theorem 2.76], we observe that $(S^{'} \otimes_R M)^* \cong \mathrm{Hom}_R(S^{'}, M^*)$ is in $\mathcal{FI}^n(S^{'op})_{\leq m}$. As a result, $S^{'} \otimes_R M$ belongs to $\mathcal{FF}^n(S^{'})_{\leq m}$. $\square$

Proposition 4.5. *Let $S^{'} \geq R$ be an almost excellent extension. Then the following statements are valid:*

(1) *If M belongs to $\mathcal{FI}^n_C(R)_{\leq m}$, then $\mathrm{Hom}_R(S^{'}, M)$ belongs to $\mathcal{FI}^n_{(C\otimes_R S')}(S^{'})_{\leq m}$;*

(2) *If M belongs to $\mathcal{FF}^n_C(R)_{\leq m}$, then $S^{'} \otimes_R M$ belongs to $\mathcal{FF}^n_{(C\otimes_R S')}(S^{'})_{\leq m}$.*

Proof. (1). If $m = 0$, then $M = \mathrm{Hom}_R(C, X)$ for some $X \in \mathcal{FI}^n(R)$. We observe the following isomorphism:

$$\begin{aligned}\mathrm{Hom}_R(S^{'}, M) &\cong \mathrm{Hom}_R(S^{'}, \mathrm{Hom}_R(C, X))\\ &\cong \mathrm{Hom}_R(C \otimes_R S^{'}, X)\\ &\cong \mathrm{Hom}_R(C \otimes_R S^{'} \otimes_{S'} S^{'}, X)\\ &\cong \mathrm{Hom}_{S'}(C \otimes_R S^{'}, \mathrm{Hom}(S^{'}, X)).\end{aligned}$$

According to Lemma 4.4, $\mathrm{Hom}_R(S^{'}, X) \in \mathcal{FI}^n(S^{'})$, and by Lemma 4.3, $C \otimes_R S^{'}$ is a semidualizing $S^{'}$-module. Therefore, we conclude that $\mathrm{Hom}_{S'}(C \otimes_R S^{'}, \mathrm{Hom}(S^{'}, X)) \in \mathcal{FI}^n_{(C\otimes_R S')}(S^{'})$. Consequently, $\mathrm{Hom}_R(S^{'}, M) \in \mathcal{FI}^n_{(C\otimes_R S')}(S^{'})$. Furthermore, if $M \in \mathcal{FI}^n_C(R)_{\leq m}$, then there exists an exact sequence of R-modules:

$$0 \longrightarrow M \longrightarrow X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_m \longrightarrow 0,$$

where each $X_i \in \mathcal{FI}^n_C(R)$ for all $0 \leq i \leq m$. Since $S^{'}_R$ is a projective R-module, applying the functor $\mathrm{Hom}_R(S^{'}, -)$ yields an exact sequence of $S^{'}$-modules:

$$0 \longrightarrow \mathrm{Hom}_R(S^{'}, M) \longrightarrow \mathrm{Hom}_R(S^{'}, X_0) \longrightarrow \mathrm{Hom}_R(S^{'}, X_1) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_R(S^{'}, X_m) \longrightarrow 0,$$

with each $\mathrm{Hom}_R(S^{'}, X_i)$ belonging to $\mathcal{FI}^n_{(C\otimes_R S')}(S^{'})$ for all $0 \leq i \leq m$. Thus, $\mathrm{Hom}_R(S^{'}, M)$ resides in $\mathcal{FI}^n_{(C\otimes_R S')}(S^{'})_{\leq m}$.

(2). The argument follows similarly to that of (1). □

In the following, we present equivalent conditions concerning modules with C-FP_n-injective dimension at most m, as well as modules with C-FP_n-flat dimension at most m, in the context of almost excellent ring extensions.

Proposition 4.6. *Let $S^{'} \geq R$ be an almost excellent extension, and let M be an $S^{'}$-module. Then the following statements are equivalent:*

(1) $M \in \mathcal{FI}^n_C(R)_{\leq m}$;
(2) $\mathrm{Hom}_R(S^{'}, M)$ *belongs to* $\mathcal{FI}^n_{(C\otimes_R S')}(S^{'})_{\leq m}$;
(3) M *belongs to* $\mathcal{FI}^n_{(C\otimes_R S')}(S^{'})_{\leq m}$.

Proof. (1)⇒(2). Let M be an element of $\mathcal{FI}^n_C(R)_{\leq m}$. Then, by Proposition 4.5(1), the module $\mathrm{Hom}_R(S^{'}, M)$ belongs to $\mathcal{FI}^n_{(C\otimes_R S')}(S^{'})_{\leq m}$.

(2)⇒(3). According to [18, Lemma 1.1], ${}_{S'}M$ is isomorphic to a direct summand of the $S^{'}$-module $\mathrm{Hom}_R(S^{'}, M)$. Subsequently, by (2) and [17, Proposition 3.12], it follows that M belongs to $\mathcal{FI}^n_{(C\otimes_R S')}(S^{'})_{\leq m}$. (3)⇒(1). Let $m = 0$. Suppose $M \in \mathcal{FI}^n_{(C\otimes_R S')}(S^{'})$. Then there exists some $X \in \mathcal{FI}^n(S^{'})$ such that $M \cong \mathrm{Hom}_{S'}(C \otimes_R S^{'}, X)$. We observe that

$$M \cong \mathrm{Hom}_{S'}(C \otimes_R S^{'}, X) \cong \mathrm{Hom}_R(C, \mathrm{Hom}_{S'}(S^{'}, X)) \cong \mathrm{Hom}_R(C, X).$$

Our goal is to demonstrate that $X \in \mathcal{FI}^n(R)$. Consider a finitely n-presented R-module U. There exists an exact sequence

$$0 \longrightarrow K_0 \longrightarrow F_0 \longrightarrow U \longrightarrow 0,$$

where F_0 is a free R-module. Since $S^{'}$ is a flat R-module, tensoring with $S^{'}$ yields the exact sequence

$$0 \longrightarrow K_0 \otimes_R S^{'} \longrightarrow F_0 \otimes_R S^{'} \longrightarrow U \otimes_R S^{'} \longrightarrow 0.$$

By [18, Lemma 4], $U \otimes_R S^{'}$ is finitely n-presented over $S^{'}$. Utilizing [14, Theorem 2.76], we obtain a commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{S'}(F_0 \otimes_R S^{'}, X) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{S'}(K_0 \otimes_R S^{'}, X) \arrow[r] \arrow[d, "\cong"] & \mathrm{Ext}^1_{S'}(U \otimes_R S^{'}, X)) \arrow[r] \arrow[d] & 0 \\
\mathrm{Hom}_R(F_0, X) \arrow[r] & \mathrm{Hom}_R(K_0, X) \arrow[r] & \mathrm{Ext}^1_R(U, X) \arrow[r] & 0,
\end{tikzcd}$$

from which it follows that $\mathrm{Ext}^1_{S'}(U \otimes_R S^{'}, X) \cong \mathrm{Ext}^1_R(U, X)$. Under the hypothesis, $\mathrm{Ext}^1_{S'}(U \otimes_R S^{'}, X) = 0$, hence $\mathrm{Ext}^1_{S'}(U, X) = 0$. Consequently, $X \in \mathcal{FI}^n(R)$. Since $M \cong \mathrm{Hom}_R(C, X)$, this implies $M \in \mathcal{FI}^n_C(R)$. Furthermore, if $M \in \mathcal{FI}^n_{(C\otimes_R S')}(S^{'})_{\leq m}$, it directly follows that $M \in \mathcal{FI}^n_C(R)_{\leq m}$. $\square$

Proposition 4.7. *Let $S^{'} \geq R$ be an almost excellent extension, and let M be an $S^{'}$-module. The following statements are equivalent:*

(1) *M belongs to $\mathcal{FF}^n_C(R)_{\leq m}$;*
(2) *The tensor product $S^{'} \otimes_R M$ belongs to $\mathcal{FF}^n_{(C\otimes_R S')}(S^{'})_{\leq m}$;*
(3) *M belongs to $\mathcal{FF}^n_{(C\otimes_R S')}(S^{'})_{\leq m}$.*

Proof. Based on Proposition 4.6, Proposition 2.56 and Theorem 2.76 from [14], and Proposition 3.13 from [17], we conclude the following:

A module M belongs to $\mathcal{FF}^n_C(R)_{\leq m}$ if and only if its dual M^* belongs to $\mathcal{FI}^n_C(R^{op})_{\leq m}$. This is equivalent to $\mathrm{Hom}_R(S^{'}, M^*)$ lying in $\mathcal{FI}^n_{(C\otimes_R S')}(S^{'op})_{\leq m}$, and also equivalent to $(S^{'} \otimes_R M)^*$ being in $\mathcal{FI}^n_{(C\otimes_R S')}(S^{'op})_{\leq m}$. Furthermore, this is equivalent to $S^{'} \otimes_R M$ belongs to $\mathcal{FF}^n_{(C\otimes_R S')}(S^{'})_{\leq m}$.

Additionally, M is in $\mathcal{FF}^n_C(R)_{\leq m}$ if and only if M^* is in $\mathcal{FI}^n_C(R^{op})_{\leq m}$. This is equivalent to M^* being in $\mathcal{FI}^n_{(C\otimes_R S')}(S^{'op})_{\leq m}$, which in turn is equivalent to M being in $\mathcal{FF}^n_{(C\otimes_R S')}(S^{'})_{\leq m}$. $\square$

Proposition 4.8. *Let $S^{'} \geq R$ be an almost excellent extension, and let M be an $S^{'}$-module. Then the following statements are equivalent:*

(1) *M belongs to $SCot^{(n,m)}_{(C\otimes_R S')}(S^{'})$;*
(2) *M belongs to $SCot^{(n,m)}_C(R)$;*
(3) $\mathrm{Hom}_R(S^{'}, M)$ *belongs to $SCot^{(n,m)}_{(C\otimes_R S')}(S^{'})$.*

Proof. (1)⇒(2). Let M be an object in $SCot^{(n,m)}_{(C\otimes_R S')}(S^{'})$, and suppose $F \in \mathcal{FF}^n_C(R)_{\leq m}$. By [13, Theorem 11.65], we have an isomorphism $\mathrm{Ext}^n_R(F, M) \cong$

$\mathrm{Ext}^n_{S'}(S^{'} \otimes_R F, M)$, for all $n \geq 1$, since $S^{'}$ is a projective R-module. Furthermore, by Proposition 4.7, the module $S^{'} \otimes_R F$ belongs to $\mathcal{FF}^n_{(C\otimes_R S')}(S^{'})_{\leq m}$. Consequently, $\mathrm{Ext}^n_{S'}(S^{'} \otimes_R F, M) = 0$, which implies that $\mathrm{Ext}^n_R(F, M) = 0$. Therefore, M lies in $SCot^{(n,m)}_C(R)$.

(2)⇒(3). Assume that $M \in SCot^{(n,m)}_C(R)$, and take $F \in \mathcal{FF}^n_{(C\otimes_R S')}(S^{'})_{\leq m}$. By Proposition 4.7, $F \in \mathcal{FF}^n_C(R)_{\leq m}$. Since $S^{'}$ is a projective R-module, by [13, Exercise 9.21], there are isomorphisms $\mathrm{Ext}^n_{S'}(F, \mathrm{Hom}_R(S^{'}, M)) \cong \mathrm{Ext}^n_R(S^{'} \otimes_R F, M) \cong \mathrm{Ext}^n_R(F, M) = 0$. Thus, $\mathrm{Ext}^n_{S'}(F, \mathrm{Hom}_R(S^{'}, M)) = 0$. It follows that $\mathrm{Hom}_R(S^{'}, M)$ belongs to $SCot^{(n,m)}_{(C\otimes_R S')}(S^{'})$.

(3)⇒(1). By [18, Lemma 1.1], the $S^{'}$-module ${}_{S'}M$ is isomorphic to a direct summand of $\mathrm{Hom}_R(S^{'}, M)$. Using the assumption in (3), we conclude that

$$M \in SCot^{(n,m)}_{(C\otimes_R S')}(S^{'})$$

. □

Theorem 4.9. *Let $S^{'} \geq R$ be an almost excellent extension and M an $S^{'}$-module. Then, the $SCot^{(n,m)}_C$-injective dimension of M over R is equal to the $SCot^{(n,m)}_{(C\otimes_R S')}$-injective dimension of M over $S^{'}$, which is also equal to the $SCot^{(n,m)}_{(C\otimes_R S')}$-injective dimension of* $\mathrm{Hom}_R(S^{'}, M)$.

Proof. According to [18, Lemma 1.1], the module ${}_{S'}M$ is isomorphic to a direct summand of the $S^{'}$-module $\mathrm{Hom}_R(S^{'}, M)$. Consequently, from Proposition 4.1, we deduce that $SCot^{(n,m)}_C\text{-id}_R(M) \leq SCot^{(n,m)}_{(C\otimes_R S')}\text{-id}_{S'}(\mathrm{Hom}_R(S^{'}, M))$.

We demonstrate that $SCot^{(n,m)}_C\text{-id}_R(M) \leq SCot^{(n,m)}_{(C\otimes_R S')}\text{-id}_{S'}(M)$.

Let $SCot^{(n,m)}_{(C\otimes_R S')}\text{-id}_{S'}(M) = t$. Then, there exists an exact sequence of $S^{'}$-modules:

$$0 \longrightarrow M \longrightarrow Y_0 \longrightarrow \cdots \longrightarrow Y_{t-1} \longrightarrow Y_t \longrightarrow 0,$$

where each Y_i belongs to $SCot^{(n,m)}_{(C\otimes_R S')}(S^{'})$ for $0 \leq i \leq t$. By Proposition 4.8, each Y_i is contained within $SCot^{(n,m)}_C(R)$. Consequently, it follows that:

$$SCot^{(n,m)}_C\text{-id}_R(M) \leq t.$$

If we demonstrate that $SCot^{(n,m)}_{(C\otimes_R S')}\text{-id}_{S'}(\mathrm{Hom}_R(S^{'}, M)) \leq SCot^{(n,m)}_C\text{-id}_R(M)$, then the proof is complete. Let us define $SCot^{(n,m)}_C\text{-id}_R(M) = t$. By this definition, there exists an exact sequence of R-modules:

$$0 \longrightarrow M \longrightarrow Y_0 \longrightarrow \cdots \longrightarrow Y_{t-1} \longrightarrow Y_t \longrightarrow 0$$

where each Y_i is within $SCot^{(n,m)}_C\text{-id}_R(M)$ for all $0 \leq i \leq t$. Given the projectivity of the R-module $S^{'}$, it follows that the application of the functor

$\mathrm{Hom}_R(S^{'},-)$ produces the exact sequence:
$0 \longrightarrow \mathrm{Hom}_R(S^{'},M) \longrightarrow \mathrm{Hom}_R(S^{'},Y_0) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_R(S^{'},Y_{t-1}) \longrightarrow \mathrm{Hom}_R(S^{'},Y_t) \longrightarrow 0$.
Furthermore, by Proposition 4.8, each $\mathrm{Hom}_R(S^{'},Y_i)$ belongs to $SCot^{(n,m)}_{(C\otimes_R S^{'})}(S^{'})$ for all $0 \leq i \leq t$. Therefore, it follows that $SCot^{(n,m)}_{(C\otimes_R S^{'})}$-$\mathrm{id}_{S^{'}}(\mathrm{Hom}_R(S^{'},M)) \leq t$. □

5. Conclusion

Employing relative homological techniques, we have expanded several fundamental homological concepts to incorporate findings related to semidualizing modules. This paper focuses on the (n,m)-cotorsion, strongly (n,m)-cotorsion and FP_n-flat (n,m)-cotorsion modules under a (faithfully) semidualizing module C. So, we introduce and analyze the C-(n,m)-cotorsion, strongly C-(n,m)-cotorsion and C-FP_n-flat C-(n,m)-cotorsion modules. Our investigation includes a study of Foxby equivalence concerning the classes $\mathcal{FF}^{cot^{(n,m)}}(R)_{\leq t}$ and $\mathcal{FF}_C^{cot^{(n,m)}}(R)_{\leq t}$, and also the study of the strongly C-(n,m)-cotorsion dimensions under the change of rings. Given the importance of these topics in the field of homological algebra, in the continuation of future research work, we suggest further investigation into the Gorenstein properties of these modules.
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