

CONNECTIVITY OF 2-DISTANCE GRAPHS

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ABSTRACT. For a simple graph G , the 2-distance graph, $D_2(G)$, is a graph with the vertex set $V(G)$ and two vertices are adjacent if their distance is 2 in the graph G . In this paper, we characterize all graphs with connected 2-distance graphs. For graphs with diameter 2, we prove that $D_2(G)$ is connected if and only if G has no spanning complete bipartite subgraphs. For graphs whose diameter is greater than 2, we define a maximal fine set, and by contracting G with respect to these subsets, we obtain a new graph \hat{G} such that $D_2(G)$ is connected if and only if $D_2(\hat{G})$ is connected. In particular, $D_2(G)$ is disconnected if and only if \hat{G} is bipartite.

Keywords: 2-distance graph, power graph, diameter, connectivity.
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1. Introduction

In this paper, we consider only the finite, simple, undirected graphs $G = (V(G), E(G))$.

A graph $H = (V(H), E(H))$ is called a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $M \subseteq V(G)$, then the subgraph $H = (V(H), E(H))$ is called the induced subgraph by M , and denoted by $G[M]$, if $V(H) = M$ and any two vertices in M are adjacent in $G[M]$ if and only if they are adjacent in G .

The complement graph of G , denoted by \overline{G} , is a graph with vertex set $V(G)$, and two vertices u and v are adjacent if and only if u and v are not adjacent in G . The distance between two vertices of $x, y \in V(G)$, $d_G(x, y)$, is the length of a shortest path between them. The diameter of G , $diam(G)$, is the maximum distance between vertices of G . The graph G is a bipartite graph if $V(G)$ is the union of two disjoint non-empty sets A and B where every edge connects a vertex in A to a vertex in B . It is well known that: A graph G is bipartite if and only if G has no odd cycle. Let A and B be two distinct sets of vertices. The symbol $[A, B]$ denotes the complete bipartite graph constructed with partite sets A and B .

The contracted graph corresponding to the partition $P = \{A_1, A_2, \dots, A_t\}$ of $V(G)$ is the graph G_P with vertex set $V(G_P) = \{A_1, \dots, A_t\}$ and two vertices A_i and A_j are adjacent if there is an edge $xy \in E(G)$ such that $x \in A_i$ and $y \in A_j$.

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For a positive integer k , the k -power of G , denoted by G^k , is the graph with the vertex set $V(G^k) = V(G)$ and two vertices x and y are adjacent if and only if $d_G(x, y) \leq k$. The 2-distance graph of a graph G , $D_2(G)$, is a graph with the vertex set $V(G)$ and two vertices are adjacent if and only if their distance is 2 in the graph G . One can see that for any graph G , $E(D_2(G)) = E(G^2) \setminus E(G)$.

The idea of 2-distance graphs is a particular case of a general notion of k -distance graphs, which was first studied by Harary et al. [6]. They investigated the connectivity of 2-distance graphs of some graphs. In his book, Prisner [9, pp. 157-159], explored the dynamics of the k -distance operator. Furthermore, Boland et al. [3] extended the k -distance operator to a graph invariant called the distance- n domination number. To solve a problem posed by Prisner [9, Open Problem 29], Zelinka [11] constructed a graph that is r -periodic under the 2-distance operator for each positive even integer r . Prisner's problem was completely solved by Leonor in her master's thesis [8], where she used a graph construction different from Zelinka's. Azimi and Farrokhi [2] studied all graphs whose 2-distance graphs have maximum degree 2. They also solved the problem of finding all graphs whose 2-distance graphs are paths or cycles. The same authors [1] determined all finite simple graphs with no square, diamond, or triangles with a common vertex that are self 2-distance graphs (a graph G is called self 2-distance if $D_2(G) \cong G$). They further showed the nonexistence of cubic self 2-distance graphs. In [4], Ching gave some characterizations of 2-distance graphs and found all graphs X such that $D_2(X) \cong kP_2$ or $K_m \cup K_n$, where $k \geq 2$ is an integer, P_2 is the path of order 2, and K_m is the complete graph of order $m \geq 1$. Also, Rajkumar and Celine solved some equations of the form $D_2(X) \cong G_1 \cup G_2$, where G_1 and G_2 are classes of graphs [10]. In [7], Khormali gave some conditions for connectivity $D_k(G)$. For $D_2(G)$, he proved that if G has no odd cycle, then $D_2(G)$ is disconnected. Finally, Gaar and Krenn considered graphs with k -regular 2-distance graph, and they derived a complete characterization for each $k = 0, 1, 2$ [5].

In this paper, we characterize all graphs with connected 2-distance graphs. For graphs with diameter 2, we prove that $D_2(G)$ is connected if and only if G has no spanning complete bipartite subgraphs. For a graph G with diameter greater than 2, we define maximal fine sets and by contracting G on these subsets, we get a new graph \widehat{G} . Then we show that $D_2(G)$ is connected if and only if $D_2(\widehat{G})$ is connected. Especially, $D_2(G)$ is disconnected if and only if \widehat{G} has no odd cycle.

2. Connectivity of $D_2(G)$

Let G be a graph with $diam(G) = 2$, then one can see that $D_2(G) = \overline{G}$. Therefore, the connectivity of $D_2(G)$ reduces to that of \overline{G} . Thus, we have the following theorem, which characterizes all graphs with diameter 2 and connected 2-distance graphs.

Theorem 2.1. *Let G be a graph with $\text{diam}(G) = 2$, then $D_2(G)$ is disconnected if and only if G has a spanning complete bipartite subgraph.*

Proof. Let $D_2(G)$ be disconnected, G_1 be a connectivity component of $D_2(G)$, $A = V(G_1)$, and $B = V(G) \setminus A$. Since $D_2(G) = \overline{G}$, then $[A, B]$ is a spanning complete bipartite subgraph of G .

Conversely, let $H = [A, B]$ be a spanning complete bipartite subgraph of G . Then, there exists no edge from any vertex in A to any vertex in B , as required. \square

Based on Theorem 2.1, we now focus on connected graphs with diameter at least 3. First, we introduce a family of graphs whose 2-distance graphs are connected and play a central role in our proofs.

Notation 2.1. The family \mathcal{H} of graphs is the set of all cycle graphs of odd order greater than 4 and three special graphs depicted in Figure 1, in other words, $\mathcal{H} = \{C_{2n+1} | n = 2, 3, \dots\} \cup \{F_1, F_2, F_3\}$.

We immediately obtain the following results.

Lemma 2.2. (1) *If $G \in \mathcal{H}$, then $D_2(G)$ is connected.*

(2) *If G is a connected graph and $2 \leq |V(G)| \leq 4$, then $D_2(G)$ is disconnected.*

(3) *Let G be a connected graph and $|V(G)| = 5$. Then $D_2(G)$ is connected if and only if $G \in \mathcal{H}$.*

Proof. The parts (1) and (2) are elementary.

Let G be a graph with 5 vertices, and $D_2(G)$ be connected. If G has a vertex u of degree 4, then u has no neighbor in $D_2(G)$, a contradiction. Then G has no vertex of degree 4. If G has no vertex of degree 3, then $G = C_5$ or $G = P_5$. But $D_2(C_5)$ is connected and $D_2(P_5)$ is disconnected. Now, assume that G has at least one vertex of degree 3. If G has exactly four vertices of degree 3, then G is isomorphic to the graph L_1 in Figure 2, where $D_2(G)$ is disconnected. If G has exactly three vertices of degree 3, then G is isomorphic to the graph L_2 in Figure 2, where $D_2(G)$ is disconnected. If G has exactly two vertices of degree 3, then either $G \in \mathcal{H}$ or G is isomorphic to the graph L_3 in Figure 2. In the latter case, $D_2(G)$ is disconnected. If G has exactly one vertex of degree 3, then G is isomorphic to one of the graphs in Figure 3, where $D_2(G)$ is disconnected. \square

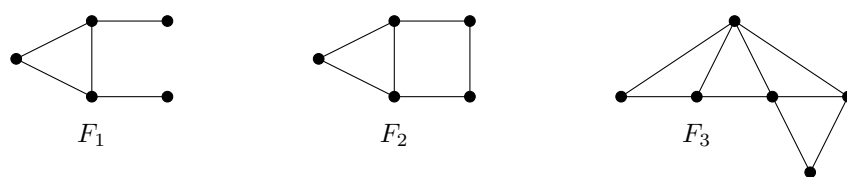


FIGURE 1. Three special graphs of the family \mathcal{H}

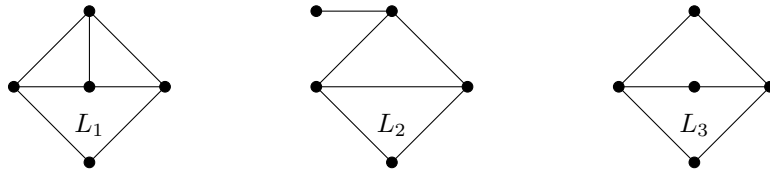


FIGURE 2. Graphs with exactly 2, 3 or 4 vertices of degree 3

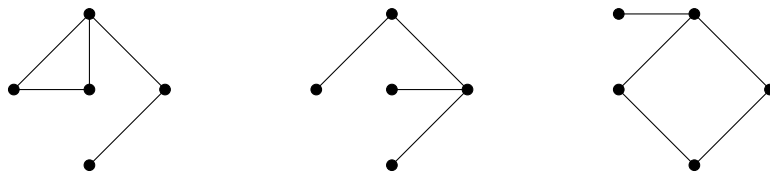


FIGURE 3. Graphs with exactly one vertex of degree 3

Let H_1 be a subgraph of G_1 . In general, $D_2(H_1)$ does not need to be a subgraph of $D_2(G_1)$. For example, if $H_1 = C_5$ and $G_1 = H_1 + e$, where e is an edge between two vertices of H_1 with distance 2, then $D_2(H_1) = C_5$ but $D_2(G_1) = P_5$. (See Figure 4.) Also, for an induced subgraph H_2 of G_2 , $D_2(H_2)$ does not need to be an induced subgraph of $D_2(G_2)$. (See Figure 5.)

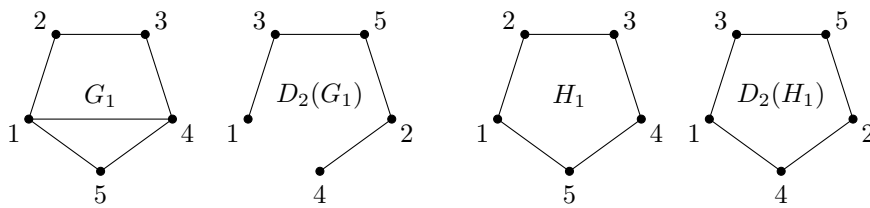


FIGURE 4. H_1 is a subgraph of G_1 while $D_2(H_1)$ is not a subgraph of $D_2(G_1)$.

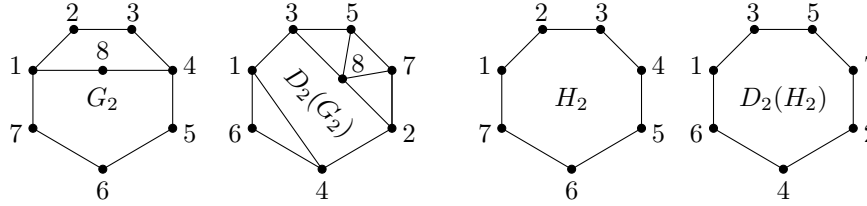


FIGURE 5. H_2 is an induced subgraph of G_2 while $D_2(H_2)$ is not an induced subgraph of $D_2(G_2)$.

Lemma 2.3. *Let G be a connected graph. If $H \in \mathcal{H}$ is an induced subgraph of G such that $V(H) \not\subseteq N(x)$ for any vertex $x \in V(G)$, then $D_2(G)$ is connected.*

Proof. Since $D_2(H)$ is connected and H is an induced subgraph, then for any two non-adjacent vertices $x, y \in V(H)$, we have $d_G(x, y) \geq 2$, and thus $D_2(H)$ is a subgraph of $D_2(G)$. Let $u \in V(G) \setminus V(H)$, $d_G(u, V(H)) = t$, and $u = u_0u_1 \cdots u_t = v$ be a path with $v \in V(H)$. Since $V(H) \not\subseteq N(u_{t-1})$, then there exists $w \in V(H)$ such that $d_G(u_{t-1}, w) = 2$. Consequently, there exists a path from u to v or w in $D_2(G)$ according to the parity of t , as required. \square

Lemma 2.4. *Let G be a graph such that $D_2(G)$ is disconnected. If H is a connected induced subgraph of G , and $D_2(H)$ is connected, then there is a subgraph K of G such that $H \leq K$, $D_2(K)$ is connected, and $[V(K), N_G(V(K)) \setminus V(K)]$ is a subgraph of G .*

Proof. If $[V(H), N_G(V(H)) \setminus V(H)]$ is a subgraph of G , then $K = H$. Otherwise, there is $x \in N_G(V(H)) \setminus V(H)$ such that $V(H) \not\subseteq N_G(x)$. Similar to the proof of Lemma 2.3, $D_2(G[V(H) \cup \{x\}])$ is connected. By setting $H_1 = G[V(H) \cup \{x\}]$, we can continue this process by replacing H_1 with H . Since G is finite, there is H_t such that $H_t \neq G$ and for any $x \in N_G(V(H_t)) \setminus V(H_t)$, $V(H_t) \subseteq N_G(x)$. By setting $K = H_t$, $[V(K), N_G(V(K)) \setminus V(K)]$ is a subgraph of G , which completes the proof. \square

Due to the previous lemma, we introduce the following definition:

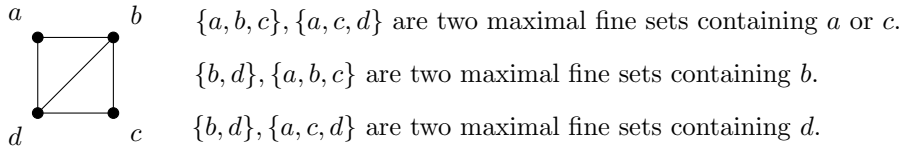
Definition 2.5. Let G be a graph and $A \subseteq V(G)$. The set A is called **fine** if $N(A) \setminus A \neq \emptyset$ and $[A, N_G(A) \setminus A]$ is a subgraph of G , or equivalently, for any $x \in A$, $N(x) \setminus A = N(A) \setminus A \neq \emptyset$.

Lemma 2.6. *Assume that G is a graph with $\text{diam}(G) \geq 3$. Then, for each $x \in V(G)$ there exists a unique maximal fine set A such that $x \in A$.*

Proof. For $x \in V(G)$, $\{x\}$ is fine. Since G is finite, there are some maximal fine sets containing x . Let A and B be two maximal fine sets containing x . We claim that $A \cup B$ is fine. By the hypothesis, $N(x) \setminus A = N(A) \setminus A$ and $N(x) \setminus B = N(B) \setminus B$. First, we prove that $N(A \cup B) \setminus (A \cup B) \neq \emptyset$. If

$N(A \cup B) \subseteq (A \cup B)$, then since G is connected, $A \cup B = V(G)$ and $N(B) \setminus B \subseteq A$ and $N(A) \setminus A \subseteq B$. Let $y \in N(B) \setminus B$, then $y \in A$, and since B is fine, then $B \subseteq N(y) \subseteq N(A)$. Since A is fine and $y \in A$, $B \setminus A \subseteq N(A) \setminus A$ and therefore $B \setminus A = N(A) \setminus A$. Therefore, $\text{diam}(G) = 2$ because A is fine, a contradiction. Since $x \in A \cup B$ then it is trivial that $N(x) \setminus (A \cup B) \subseteq N(A \cup B) \setminus (A \cup B)$. Now let $y \in N(A \cup B) \setminus (A \cup B)$ then $y \notin A \cup B$ and $y \in N(A \cup B)$. So $y \in N(A)$ or $y \in N(B)$. Without loss of generality, let $y \in N(A)$. Then $y \in N(A) \setminus A = N(x) \setminus A$ and so $y \in N(x) \setminus (A \cup B)$. Therefore, $N(x) \setminus (A \cup B) = N(A \cup B) \setminus (A \cup B)$. On the other hand, for any $a \in A$, $N(a) \setminus A = N(A) \setminus A = N(x) \setminus N(A)$ and so $N(a) \setminus (A \cup B) = N(x) \setminus (A \cup B) = N(A \cup B) \setminus (A \cup B)$. Similarly, for any $b \in B$, $N(b) \setminus (A \cup B) = N(A \cup B) \setminus (A \cup B)$. Consequently, $A \cup B$ is a fine set, and by the maximality of A and B , $A = A \cup B = B$. \square

Remark 2.7. A connected graph with diameter 2 may have more than one maximal fine set containing a given vertex. For example, in the following graph, the maximal fine set containing a vertex is not unique.



Remark 2.8. One may think that a non-maximal fine set can always be extended by adding a vertex, but this is not true. For example, in the graph shown in Figure 6, $\{a, b\}$ is a fine set, but it is not maximal, because $\{a, b, c, x, y\}$ is a maximal fine set containing both a and b . However, any 3-element set containing $\{a, b\}$ is not a fine set.

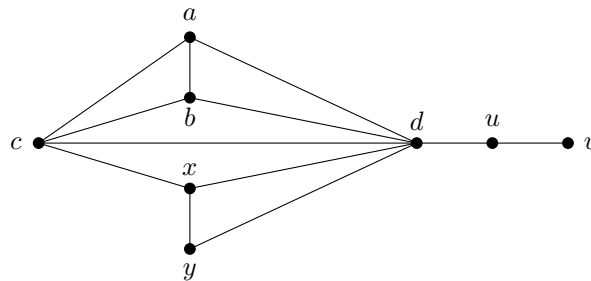


FIGURE 6. A special graph

By Lemma 2.6, we have the following immediate corollary:

Corollary 2.9. *The maximal fine sets partition the vertex set of a graph with diameter greater than 2.*

Definition 2.10. Let G be a graph with $\text{diam}(G) \geq 3$ and \widehat{G} be the graph with vertex set

$$V(\widehat{G}) = \{\hat{x} : \hat{x} \text{ is the maximal fine set of } G \text{ containing } x\}$$

and two distinct vertices $\hat{x}, \hat{y} \in V(\widehat{G})$ are adjacent if and only if x and y are adjacent in G .

Lemma 2.11. Let G be a graph with $\text{diam}(G) \geq 3$ and $\hat{x}, \hat{y} \in V(\widehat{G})$ be distinct vertices, where $x, y \in V(G)$. Then $d_G(x, y) = d_{\widehat{G}}(\hat{x}, \hat{y})$.

Proof. Let $\hat{x} = \hat{x}_0\hat{x}_1\hat{x}_2 \cdots \hat{x}_t = \hat{y}$ be an $\hat{x}\hat{y}$ -path. By the properties of fine sets, $x = x_0x_1x_2 \cdots x_t = y$ is an xy -path. Then $d_G(x, y) \leq d_{\widehat{G}}(\hat{x}, \hat{y})$. On the other hand, if $x = x_0x_1x_2 \cdots x_t = y$ is an xy -path, then $\hat{x} = \hat{x}_0\hat{x}_1\hat{x}_2 \cdots \hat{x}_t = \hat{y}$ is an $\hat{x}\hat{y}$ -walk and then contains an $\hat{x}\hat{y}$ -path such that its length is less than t . Consequently $d_{\widehat{G}}(\hat{x}, \hat{y}) \leq d_G(x, y)$ and then $d_{\widehat{G}}(\hat{x}, \hat{y}) = d_G(x, y)$. \square

One can see that if $x, y \in V(G)$ and $\hat{x} = \hat{y}$, then $d_G(x, y) \leq 2$ and the following holds.

Corollary 2.12. If G is a graph with $\text{diam}(G) \geq 3$, then $\text{diam}(G) = \text{diam}(\widehat{G})$.

Remark 2.13. Let G be a graph with $\text{diam}(G) \geq 3$. Then $\text{diam}(\widehat{G}) \geq 3$. Thus, we can consider $\widehat{\widehat{G}}$. However, one can see that if $S \subseteq \widehat{G}$ is a fine set, then $S_1 = \{a : \hat{a} \in S\}$ is a fine set of G , which means that S has a unique element, and therefore $\widehat{\widehat{G}} = \widehat{G}$.

Theorem 2.14. Let G be a graph with $\text{diam}(G) \geq 3$. Then $D_2(G)$ is connected if and only if $D_2(\widehat{G})$ is connected.

Proof. First, let $D_2(\widehat{G})$ be connected, and $x, y \in V(G)$. We consider the following two cases:

Case 1. $\hat{x} \neq \hat{y}$. There is a $\hat{x}\hat{y}$ -path, $\hat{x} = \hat{x}_1\hat{x}_2 \cdots \hat{x}_t = \hat{y}$ in $D_2(\widehat{G})$. Since $\hat{x}_i \neq \hat{x}_{i+1}$, by Lemma 2.11, $d_G(x_i, x_{i+1}) = d_{\widehat{G}}(\hat{x}_i, \hat{x}_{i+1}) = 2$, and then $x = x_1x_2 \cdots x_t = y$ is a xy -path in $D_2(G)$.

Case 2. $\hat{x} = \hat{y}$. Since $\text{diam}(G) \geq 3$, there is $z \in V(G)$ such that $\hat{x} \neq \hat{z}$. Then, $\hat{y} \neq \hat{z}$, and by Case 1, there are some paths between x and y to z in $D_2(G)$. Consequently, $D_2(G)$ is connected.

Now, assume that $D_2(G)$ is connected, $\hat{x} \neq \hat{y} \in V(\widehat{G})$, and $x = x_1x_2 \cdots x_t = y$ is an xy -path in $D_2(G)$. Since $\hat{x}_i = \hat{x}_{i+1}$ or $d_{\widehat{G}}(\hat{x}_i, \hat{x}_{i+1}) = 2$, then $\hat{x} = \hat{x}_1\hat{x}_2 \cdots \hat{x}_t = \hat{y}$ is an $\hat{x}\hat{y}$ -walk and contains an $\hat{x}\hat{y}$ -path in $D_2(\widehat{G})$. Thus, $D_2(\widehat{G})$ is connected. \square

By a similar argument, one can deduce the following immediate corollary.

Corollary 2.15. Let G be a graph with $\text{diam}(G) \geq 3$. Then, $\text{diam}(D_2(G)) = \text{diam}(D_2(\widehat{G}))$.

Now, we are ready to state and prove the main result of the paper.

Theorem 2.16. *If G is a connected graph and $\text{diam}(G) \geq 3$, then $D_2(G)$ is disconnected if and only if \widehat{G} has no odd cycle.*

Proof. If \widehat{G} has no odd cycle, then it is trivial that $D_2(\widehat{G})$ is disconnected, and by Theorem 2.14, $D_2(G)$ is disconnected.

On the contrary, let $D_2(G)$ be disconnected, but \widehat{G} contains an odd cycle. Let $\widehat{H} \leq \widehat{G}$ where \widehat{H} is a non-trivial induced connected subgraph with $V(\widehat{H}) = \{\hat{x}_1, \hat{x}_2, \dots, \hat{x}_t\}$ and $D_2(\widehat{H})$ is connected. Obviously, $H = G[\{x_1, x_2, \dots, x_t\}]$ is a non-trivial connected induced subgraph of G , and $D_2(H)$ is connected too. According to the proof of Lemma 2.4, there is an induced subgraph K which contains H , and $[V(K), N_G(V(K)) \setminus V(K)]$ is a subgraph of G . Therefore, $V(K)$ is fine and then there is $x \in V(G)$ such that $V(H) \subseteq \hat{x}$, a contradiction. Thus, \widehat{G} has no non-trivial connected induced subgraph \widehat{H} with the connected 2-distance graph. Consequently, by Lemma 2.2, \widehat{G} has no induced subgraph belonging to \mathcal{H} , in particular, \widehat{G} has no induced odd cycle of length greater than 3.

Let \widehat{G} have a triangle.

Claim 1. For any triangle $\hat{a}\hat{b}\hat{c}\hat{a}$ in \widehat{G} , $\widehat{G}[N(\hat{a})]$ is connected.

Proof: On the contrary, let $\widehat{G}[N(\hat{a})]$ be disconnected and L be the connected component of $\widehat{G}[N(\hat{a})]$ containing the edge $\hat{b}\hat{c}$. Let $\hat{y} \in N(\hat{a}) \setminus V(L)$. If $N(V(L)) \subseteq N[\hat{a}]$, then $N(V(L)) \setminus V(L) = \{\hat{a}\}$. Therefore, $V(L)$ is fine and contains \hat{b} and \hat{c} . By Remark 2.13, we have $\hat{b} = \hat{c}$, a contradiction. Then, we assume that $\hat{u} \in N(V(L)) \setminus N(\hat{a})$. If $V(L) \not\subseteq N(\hat{u})$ then since L is connected, there are $\hat{v}, \hat{w} \in V(L)$ such that $\hat{v}\hat{w} \in E(L)$, $\hat{v} \in N(\hat{u})$ and $\hat{w} \notin N(\hat{u})$. Consequently, one can see that $\widehat{G}[\{\hat{y}, \hat{a}, \hat{v}, \hat{w}, \hat{u}\}] \in \mathcal{H}$, a contradiction. Thus, $V(L) \subseteq N(\hat{u})$ for all $\hat{u} \in N(V(L)) \setminus N(\hat{a})$, and again $V(L)$ is fine, which contradicts Remark 2.13.

Claim 2. Any vertex of \widehat{G} is on a triangle.

Proof: On the contrary, there is a vertex \hat{a} that lies on no triangle. Since \widehat{G} is connected and \widehat{G} has at least a triangle, then there is a vertex \hat{b} and a triangle $\hat{x}\hat{y}\hat{z}\hat{x}$ such that $\hat{b} \in N(\hat{x})$ and \widehat{G} has no triangle containing \hat{b} , that is, $N(\hat{b}) \cap N(\hat{x}) = \emptyset$. Now, $\widehat{G}[N(\hat{x})]$ is disconnected, a contradiction by Claim 1.

Claim 3. \widehat{G} has a subgraph \widehat{H} with vertex set $V(\widehat{H}) = \{\hat{a}, \hat{b}, \hat{x}, \hat{c}, \hat{d}\}$ and $E(\widehat{H}) = \{\hat{a}\hat{b}, \hat{a}\hat{x}, \hat{b}\hat{x}, \hat{c}\hat{x}, \hat{b}\hat{c}, \hat{c}\hat{d}\}$ where $d_{\widehat{G}}(\hat{a}, \hat{d}) = 3$ (Figure 7).

Proof: Since $\text{diam}(\widehat{G}) \geq 3$, there is a path $\hat{a}\hat{b}\hat{c}\hat{d}$ with $d_{\widehat{G}}(\hat{a}, \hat{d}) = 3$. By Claim 2, \hat{b} lies on a triangle, and by Claim 1, $\widehat{G}[N(\hat{b})]$ is connected. Then there is $\hat{x} \in N(\hat{b}) \cap N(\hat{c})$. If $\hat{x} \notin N(\hat{a}) \cup N(\hat{d})$, then $\widehat{G}[\{\hat{a}, \hat{b}, \hat{x}, \hat{c}, \hat{d}\}] \in \mathcal{H}$, a contradiction. Therefore, $\hat{x} \in N(\hat{a}) \cup N(\hat{d})$, and by symmetry, we can assume $\hat{x} \in N(\hat{a})$ and $\hat{x} \notin N(\hat{d})$, which completes the proof.

We now have all the necessary information to proceed with the argument.

Let H be the subgraph constructed in Claim 3. Let K be a connected component of $\widehat{G}[N(\hat{a}) \cap N(\hat{c})]$ containing the edge $\hat{b}\hat{x}$. We claim that $V(K)$ is fine.

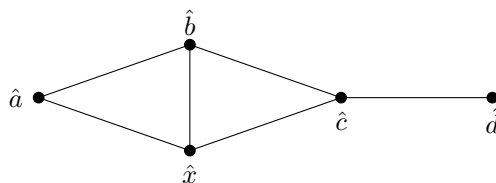


FIGURE 7. The graph constructed in Claim 3.

Since K is connected, if $\hat{y} \in N(V(K)) \setminus N(\hat{a})$ and $V(K) \not\subseteq N(\hat{y})$, then there is $\hat{u}, \hat{v} \in V(K)$ such that $\hat{u}\hat{v}, \hat{u}\hat{y} \in E(\hat{G})$ but $\hat{v} \notin N(\hat{y})$.

The subgraph $\hat{G}[\{\hat{a}, \hat{c}, \hat{d}, \hat{u}, \hat{v}, \hat{y}\}]$ can be one of the four graphs shown in Figure 8.

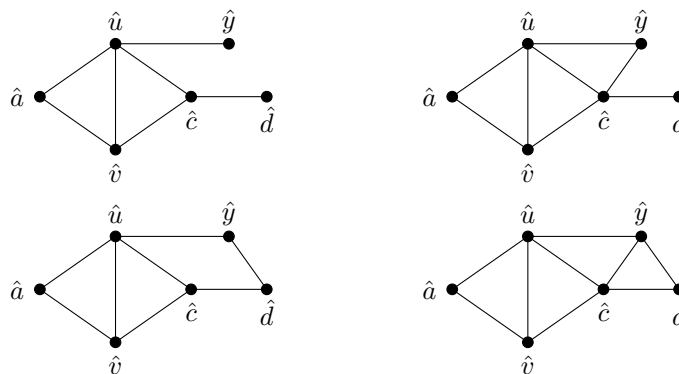


FIGURE 8

One can see that in each case, $\hat{G}[\{\hat{a}, \hat{c}, \hat{d}, \hat{u}, \hat{v}, \hat{y}\}]$ contains an induced subgraph of \hat{G} which belongs to \mathcal{H} , a contradiction.

If $\hat{y} \in N(\hat{a}) \cap N(V(K)) \setminus N(\hat{c})$ and $V(K) \not\subseteq N(\hat{y})$, then there are $\hat{u}\hat{v} \in E(K)$ such that $\hat{u}\hat{y}, \hat{u}\hat{v} \in E(\hat{G})$ but $\hat{v} \notin N(\hat{y})$. Then, $\hat{G}[\{\hat{y}, \hat{u}, \hat{v}, \hat{c}, \hat{d}\}] \in \mathcal{H}$, a contradiction. Thus, $V(K)$ is fine, again a contradiction. Therefore, \hat{G} has no triangle that completes the proof. \square

Declarations

Conflict of interest: No potential conflict of interest was reported by the authors.

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