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# WEAK NONCOMPACTNESS IN FRACTIONAL DIFFERENTIAL BOUNDARY PROBLEMS: EXISTENCE RESULTS

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Abstract. This paper addresses the existence of solutions for a class of fractional differential boundary value problems. The main novelty of this paper lies in providing a new analytical framework to prove solution existence for this class of equations. Our approach is grounded in a combination of a Krasnoselskii-type fixed point theorem with the concept of the measure of weak noncompactness (MWNC) in the Banach space $L^1(I)$. This technique allows us to prove existence under more general growth conditions than in previous works that relied on complete continuity. By establishing a set of sufficient conditions, we demonstrate that the boundary value problem under consideration admits at least one solution. Several illustrative examples are presented to highlight the applicability and effectiveness of the obtained results.

*Keywords*: Liouville-Caputo fractional derivative, Integral boundary condition, Measure of weak noncompactness.
*2020 MSC*: 34A08, 47H08, 47H10.

## 1. Introduction

The field of fractional calculus, owing to its capability of handling derivatives and integrals of arbitrary (non-integer) order, has seen remarkable advancements across numerous scientific and engineering areas. It is particularly effective in describing phenomena with memory and hereditary properties.

For instance, it finds wide application in physics and dynamics to describe anomalous diffusion and fractional dynamical systems (see, e.g., [15, 24, 26]), in control engineering and electrical systems (e.g., [5, 17]), as well as in the development of its theoretical and analytical foundations (e.g., [10, 16, 20]). This breadth of application motivates the deeper study of fractional differential equations. When a partial differential equation is used to describe a phenomenon, failing to consider whether a solution exists may lead to incomplete mathematical modeling. The problem of identifying whether solutions exist for nonlinear

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boundary conditions of fractional type has been the subject of investigation by numerous researchers through various methodologies (see [18, 21, 28, 29, 31]).

Additionally, from a mathematical standpoint, the analysis of whether the fractional analogs of differential equations with integer orders possess solutions has drawn the interest of numerous researchers; notable examples appear in works such as [1, 22, 23, 27, 30] and the references cited therein.

Inspired by [33], this manuscript investigates the solvability of the following differential equation involving fractional boundary terms.

$$(1)\qquad \begin{cases} {}^{\mathfrak{C}}\mathfrak{D}^{\vartheta}_{\varsigma_0}\left[\dfrac{\mathfrak{u}(\varsigma)-\rho(\varsigma,\mathfrak{u}(\varsigma))}{\nu(\varsigma,\mathfrak{u}(\varsigma))}\right]+\tau(\varsigma,\mathfrak{u}(\varsigma))=0, \quad \text{for } \ \varsigma\in[\varsigma_0,\varsigma_1], \\ \mathfrak{u}(\varsigma_0)=\rho(\varsigma_0,\mathfrak{u}(\varsigma_0)), \\ \mathfrak{u}(\varsigma_1)=\rho(\varsigma_1,\mathfrak{u}(\varsigma_1)); \end{cases}$$

where $\vartheta \in (1,2]$, $\tau, \rho : [\varsigma_0,\varsigma_1]\times\mathbb{R}\to\mathbb{R}$ and $\nu : [\varsigma_0,\varsigma_1]\times\mathbb{R}\to\mathbb{R}\setminus\{0\}$ are Caratheodoty functions, and ${}^{\mathfrak{C}}\mathfrak{D}^{\vartheta}_{\varsigma_0}$ denotes the Liouville-Caputo derivative of fractional order $\vartheta$. Model (1) represents a significant generalization of standard fractional boundary value problems. The composite nonlinear term $\left[\dfrac{\mathfrak{u}(\varsigma)-\rho(\varsigma,\mathfrak{u}(\varsigma))}{\nu(\varsigma,\mathfrak{u}(\varsigma))}\right]$ is specifically designed to model phenomena where the system's dynamics depend not only on the current state $\mathfrak{u}(\varsigma)$ but also on a state-dependent state-transformation or modulation [7].

Note that this model also assumes the boundary conditions at $\varsigma_0$ and $\varsigma_1$ are governed by the same nonlinear function $\rho$ that appears in the differential operator. This symmetrical condition is a feature of the model, implying that the state's relationship with its equilibrium $\rho$ at the boundaries is consistent with its relationship within the fractional operator itself.

For instance, in modeling nonlinear anomalous diffusion [19] $\rho$ could represent an equilibrium state (analogous to an ambient temperature) and $\nu$ could be a diffusion coefficient or thermal conductivity that depends on the state $\mathfrak{u}$. This form allows for the study of systems where the driving force is coupled with the system's state in a complex manner, which provides the primary motivation for studying this particular structure.

The novelty of this investigation lies in a combined methodological approach to prove the existence of solutions for the complex model (1). Unlike many existing works that rely on $C(I)$ spaces and standard fixed point theorems (like Schauder's), our method is centered in the $L^1(I)$ space. We integrate the Krasnosel'skii fixed point theorem with functional analytic tools based on the measure of weak noncompactness (MWNC). This combination permits us to handle the corresponding integral operators under linear growth assumptions (D1)-(D3) without requiring the often restrictive assumption of complete continuity. This paper demonstrates how this powerful technique can be applied to guarantee at least one solution for this class of nonlinear fractional boundary value problems.

The structure of this article is arranged as follows: In Section 2, we provide the essential background information on fractional calculus, the measure of weak noncompactness, and a fixed point theorem of Krasnosel'skii type. Section 3 reformulates the existence problem into a fixed point problem by transforming the differential equation into its corresponding integral form, leading us to a discussion of existence results. In Section 4, we present two illustrative examples to support our findings. Finally, we employ a numerical method to derive approximate solutions for the examples mentioned.

## 2. Preliminaries

Let $I := [\varsigma_0, \varsigma_1]$ and recall the Banach space $L^1(I)$, which is comprised of all real-valued functions that are Lebesgue integrable over the interval $I$. This space is equipped with the standard norm, i.e.,

$$\|\nu\|_1 = \int_{\varsigma_0}^{\varsigma_1} |\nu(\varsigma)| d\varsigma.$$

### 2.1. Fractional Calculus.

**Definition 2.1.** [14] Let $\Theta \in L^1(I)$. The Riemann-Liouville fractional integral of the function $\Theta$ with order $\vartheta > 0$ is expressed as

$$\mathfrak{I}^{\vartheta}_{\varsigma_0}\Theta(\varsigma) = \frac{1}{\Gamma(\vartheta)} \int_{\varsigma_0}^{\varsigma} (\varsigma - s)^{\vartheta-1}\Theta(s)ds, \quad \varsigma > 0,$$

in which $\Gamma(\cdot)$ represents the gamma function.

**Definition 2.2.** [14] Let the function $\Theta \in L^1(I)$ and $\vartheta > 0$. The Liouville-Caputo fractional derivative of $\Theta$ of order $\vartheta$ is formulated as

$$^{\mathfrak{C}}\mathfrak{D}^{\vartheta}_{\varsigma_0}\Theta(\varsigma) = \frac{1}{\Gamma(n-\vartheta)} \int_{\varsigma_0}^{\varsigma} (\varsigma - s)^{n-\vartheta-1}\Theta^{(n)}(s)ds,$$

with $n = [\vartheta] + 1$, in which $[\vartheta]$ represents the integer part of $\vartheta$.

**Lemma 2.3.** *[34] Let $\vartheta > 0$, then*

$$\mathfrak{I}^{\vartheta}_{\varsigma_0}\,{}^{\mathfrak{C}}\mathfrak{D}^{\vartheta}_{\varsigma_0}\Theta(\varsigma) = \Theta(\varsigma) + \iota_0 + \iota_1(\varsigma - \varsigma_0) + \iota_2(\varsigma - \varsigma_0)^2 + \cdots + \iota_{n-1}(\varsigma - \varsigma_0)^{n-1},$$

*where $n = [\vartheta] + 1$ and for $i \in \{0, 1, 2, \cdots, n-1\}$, $\iota_i \in \mathbb{R}$.*

### 2.2. On the Measure of Weak Noncompactness and a Fixed Point Theorem of Krasnosel'skii Type.

The following notations will be used:

- $\mathfrak{M}(X)$ : the collection of all nonempty and bounded subsets of the Banach space $X$.
- $\mathcal{W}(X)$ : a sub collection of $\mathfrak{M}(X)$ consisting of all relatively weakly compact subsets within the Banach space $X$.
- $\overline{Y^{\omega}}$ : the weak closure of $Y$.
- $co(Y)$ : the closed convex hull of $Y$.

**Definition 2.4.** [6] A function $\omega : \mathfrak{M}(X) \to \mathbb{R}^+$ is called a measure of weak noncompactness (MWNC), if it fulfills the subsequent properties for any $Y, Y_1, Y_2 \in \mathfrak{M}(X)$:

($\omega_1$) The set $ker(\omega) := \{Y \in \mathfrak{M}(X) : \omega(Y) = 0\}$ forms a nonempty subcollection of $\mathcal{W}(X)$;
($\omega_2$) $Y_1 \subseteq Y_2 \Rightarrow \omega(Y_1) \leq \omega(Y_2)$;
($\omega_3$) $\omega(\overline{Y^{\omega}}) = \omega(Y)$;
($\omega_4$) $\omega(\lambda Y) = |\lambda|\omega(Y)$ for all $\lambda \in \mathbb{R}$;
($\omega_5$) $\omega(co(Y)) = \omega(Y)$;
($\omega_6$) For any decreasing sequence $(Y_n)_{n=1}^{\infty}$ of nonempty, bounded, and weakly closed subsets of $X$ that satisfies $\lim_{n\to\infty} \omega(Y_n) = 0$, the intersection $Y_\infty := \bigcap_{n=1}^{\infty} Y_n$ is nonempty.

**Theorem 2.5.** *[2] For each $\varepsilon > 0$ and $Y \subset L^1(I)$, let*

$$\omega_\varepsilon(Y) := \sup_{\mathfrak{u}\in Y} \left\{ \int_D |\mathfrak{u}(\varsigma)| d\varsigma : D \subseteq I, \mathrm{m}(D) \leq \varepsilon \right\}, \tag{2}$$

*in which* $\mathrm{m}(\cdot)$ *represents the standard Lebesgue measure. Then* $\omega_0(Y) := \limsup_{\varepsilon\to 0} \omega_\varepsilon(Y)$ *defines an MWNC on* $L^1(I)$.

**Theorem 2.6.** *[9] A bounded subset $Y \subset L^1(I)$ has a weakly compact closure in $L^1(I)$ precisely when $Y$ is uniformly integrable. Formally, this means:*

$$\forall \varepsilon > 0, \exists \delta > 0 \quad \text{such that} \quad \int_D |\mathfrak{u}(\varsigma)| d\varsigma \leq \varepsilon, \quad \forall \mathfrak{u} \in Y,$$

*whenever $D \subseteq I$ is a measurable set satisfying* $\mathrm{m}(D) \leq \delta$.

**Theorem 2.7.** *[8]( Dunford-Pettis Theorem ) For a bounded sequence $\{\mathfrak{u}_n\}$ in $L^1(I)$, the subsequent statements are equivalent:*

(i) *The sequence $\{\mathfrak{u}_n\}_{n=1}^{\infty}$ possesses the property of uniform integrability on $I$.*
(ii) *From any subsequence of $\{\mathfrak{u}_n\}_{n=1}^{\infty}$, one can extract a further subsequence that is weakly convergent in $L^1(I)$.*

**Definition 2.8.** Consider a non-empty bounded set $Y \subset L^1(I)$ for which $\omega_0(Y) = 0$. From Theorem 2.6, it follows that $Y$ must be relatively compact, a condition equivalent to $Y$ being uniformly integrable in $L^1(I)$. Therefore, as stated in Theorem 2.7, every sequence $\{\mathfrak{u}_n\}_{n=1}^{\infty} \subset Y$ has a subsequence that converges weakly.

**Definition 2.9.** [11, 12] Let $X_1$ and $X_2$ be two Banach spaces, and let $Y$ be a nonempty subset of $X_1$. Then

An operator $\Upsilon : Y \to X_2$ is defined as ws-compact if, for every sequence $(\mathfrak{u}_n)_{n=1}^{\infty} \subseteq Y$ that converges weakly, the sequence $(\Upsilon\mathfrak{u}_n)_{n=1}^{\infty}$ contains a subsequence that converges strongly.

An operator $\Upsilon : Y \to X_2$ is defined as ww-compact, for every sequence $(\mathfrak{u}_n)_{n=1}^{\infty} \subseteq Y$ that converges weakly, the sequence $(\Upsilon\mathfrak{u}_n)_{n=1}^{\infty}$ contains a subsequence that converges weakly.

**Definition 2.10.** Let $Y$ be a nonempty subset within a Banach space $X$.

For $\beta \in [0,1)$, an operator $\Upsilon : Y \to X$ is called a contraction with constant $\beta$ if $\|\Upsilon\mathfrak{u}_1 - \Upsilon\mathfrak{u}_2\| \le \beta\|\mathfrak{u}_1 - \mathfrak{u}_2\|$ for every $\mathfrak{u}_1, \mathfrak{u}_2 \in Y$.
For $\eta \in [0,1)$ and any given MWNC $\omega$ on $X$, the operator $\Upsilon : Y \to X$ is termed an $\omega$-contraction with constant $\eta$ provided that it transforms bounded sets into bounded sets, and satisfies $\omega(\Upsilon(S)) \le \eta\omega(S)$ for every bounded subset $S \subseteq Y$.

Now, we state a Krasnosel'skii-type fixed point result which will be used later.

**Theorem 2.11.** *[32] Assume that $M$ is a nonempty, bounded, closed, and convex subset of the Banach space $X$. Suppose the mappings $\Upsilon_1 : X \to X$ and $\Upsilon_2 : M \to X$ satisfy:*

(I) *$\Upsilon_1$ is a ww-compact contraction with parameter $\beta$.*
(II) *$\Upsilon_2$ is a ws-compact and $\omega$-contractive mapping with constant $\eta$.*
(III) *If the equation $\xi = \Upsilon_1\xi + \Upsilon_2\zeta$ holds for some $\zeta \in M$, then $\xi \in M$.*

*Then there exists $\mathfrak{u} \in M$ such that $\mathfrak{u} = \Upsilon_2\mathfrak{u} + \Upsilon_1\mathfrak{u}$ whenever $\eta + \beta < 1$.*

## 3. Main results

This section forms the core of the paper. Our objective is to establish the existence of at least one solution to the fractional boundary value problem (1) in the Banach space $L^1(I)$.

Our proof strategy is as follows:

1. First, in Section 3.1, we transform the BVP (1) into an equivalent fixed point problem for an integral operator, $\Upsilon$, using the associated Green's function.
2. Then, in Section 3.2, we decompose the operator $\Upsilon$ into a sum of two operators, $\Upsilon_1$ and $\Upsilon_2$.
3. Finally, using the concepts introduced in the preliminaries (notably the MWNC and Theorem 2.11), we will show that $\Upsilon_1$ is a ww-compact contraction and $\Upsilon_2$ is a ws-compact and $\omega$-contractive mapping.

By satisfying these conditions, the Krasnosel'skii-type fixed point theorem will guarantee the existence of a fixed point for $\Upsilon$, and thus a solution to the original problem.

### 3.1. From the existence problem to a fixed point formulation.

**Lemma 3.1.** *Assume that $\vartheta \in (1,2]$. A function $\mathfrak{u}(\cdot) \in L^1(I)$ is a solution to the following fractional boundary value problem:*

$$
\begin{cases} {}^{\mathfrak{C}}\mathfrak{D}^{\vartheta}_{\varsigma_o}\left[\dfrac{\mathfrak{u}(\varsigma)-\rho(\varsigma,\mathfrak{u}(\varsigma))}{\nu(\varsigma,\mathfrak{u}(\varsigma))}\right]+\tau(\varsigma,\mathfrak{u}(\varsigma))=0, \quad \textit{for} \;\; \varsigma\in[\varsigma_0,\varsigma_1] \\ \mathfrak{u}(\varsigma_0)=\rho(\varsigma_0,\mathfrak{u}(\varsigma_0)) \\ \mathfrak{u}(\varsigma_1)=\rho(\varsigma_1,\mathfrak{u}(\varsigma_1)), \end{cases} \tag{3}
$$

*if and only if* $\mathfrak{u}(\cdot)$ *satisfies the corresponding fractional integral formulation:*

$$
\mathfrak{u}(\varsigma)=\rho(\varsigma,\mathfrak{u}(\varsigma))+\nu(\varsigma,\mathfrak{u}(\varsigma))\int_{\varsigma_0}^{\varsigma_1}\mathfrak{G}(\varsigma,s)\tau(s,\mathfrak{u}(s))\,ds, \tag{4}
$$

*where* $\mathfrak{G}:[\varsigma_0,\varsigma_1]^2\to\mathbb{R}$ *denotes the associated Green's function, defined by*

$$
\mathfrak{G}(\varsigma,s)=\frac{1}{\Gamma(\vartheta)}\begin{cases} -(\varsigma-s)^{\vartheta-1}+\dfrac{\varsigma-\varsigma_0}{\varsigma_1-\varsigma_0}(\varsigma_1-s)^{\vartheta-1}, & \varsigma_0\le s\le\varsigma\le\varsigma_1 \\ \dfrac{\varsigma-\varsigma_0}{\varsigma_1-\varsigma_0}(\varsigma_1-s)^{\vartheta-1}, & \varsigma_0\le\varsigma\le s\le\varsigma_1. \end{cases} \tag{5}
$$

*Proof.* Let us consider a function $\mathfrak{u}(\cdot)$ that fulfills the requirements of equation (3). According to Lemma 2.3, if we apply the fractional integral $\mathfrak{I}^{\vartheta}_{\varsigma_0}$ to equation (3) while taking into account the boundary conditions, we deduce that

$$
\frac{\mathfrak{u}(\varsigma)-\rho(\varsigma,\mathfrak{u}(\varsigma))}{\nu(\varsigma,\mathfrak{u}(\varsigma))}+\iota_0+\iota_1(\varsigma-\varsigma_0)+\mathfrak{I}^{\vartheta}_{\varsigma_0}\tau(\varsigma,\mathfrak{u}(\varsigma))=0,
$$

where

$$
\iota_0=0 \quad \text{and} \quad \iota_1=\frac{-1}{(\varsigma_1-\varsigma_0)\Gamma(\vartheta)}\int_{\varsigma_0}^{\varsigma_1}(\varsigma_1-s)^{\vartheta-1}\tau(s,\mathfrak{u}(s))ds.
$$

Thus

$$
\begin{aligned}
\mathfrak{u}(\varsigma)&=\rho(\varsigma,\mathfrak{u}(\varsigma))+\nu(\varsigma,\mathfrak{u}(\varsigma))\left[\left(\frac{1}{(\varsigma_1-\varsigma_0)\Gamma(\vartheta)}\int_{\varsigma_0}^{\varsigma_1}(\varsigma_1-s)^{\vartheta-1}\tau(s,\mathfrak{u}(s))ds\right)(\varsigma-\varsigma_0)\right.\\
&\quad\left.-\frac{1}{\Gamma(\vartheta)}\int_{\varsigma_0}^{\varsigma}(\varsigma-s)^{\vartheta-1}\tau(s,\mathfrak{u}(s))ds\right]\\
&=\rho(\varsigma,\mathfrak{u}(\varsigma))+\nu(\varsigma,\mathfrak{u}(\varsigma))\frac{1}{\Gamma(\vartheta)}\left[\int_{\varsigma_0}^{\varsigma}\left(\frac{(\varsigma_1-s)^{\vartheta-1}}{(\varsigma_1-\varsigma_0)}(\varsigma-\varsigma_0)-(\varsigma-s)^{\vartheta-1}\right)\tau(s,\mathfrak{u}(s))ds\right.\\
&\quad\left.+\int_{\varsigma}^{\varsigma_1}\frac{(\varsigma_1-s)^{\vartheta-1}}{(\varsigma_1-\varsigma_0)}(\varsigma-\varsigma_0)\tau(s,\mathfrak{u}(s))ds\right]\\
&=\rho(\varsigma,\mathfrak{u}(\varsigma))+\nu(\varsigma,\mathfrak{u}(\varsigma))\int_{\varsigma_0}^{\varsigma_1}\mathfrak{G}(\varsigma,s)\tau(s,\mathfrak{u}(s))ds.
\end{aligned}
$$

On the other hand, it is evident that if $\mathfrak{u}(\cdot)$ meets the conditions stated in (4), then (3) will also be satisfied. Additionally, given that $\mathfrak{G}(\varsigma_0,s)=\mathfrak{G}(\varsigma_1,s)=0$, the boundary conditions are adequately met as well. □

*Remark* 3.2. The Green's function $\mathfrak{G}(\varsigma, s)$ remains continuous throughout the interval $[\varsigma_0, \varsigma_1]$. It follows that for each fixed $\varsigma$, if $s \leq \varsigma$ then $\frac{\partial \mathfrak{G}}{\partial s}(\varsigma, s) > 0$, whereas for $\varsigma \leq s$ one has $\frac{\partial \mathfrak{G}}{\partial s}(\varsigma, s) < 0$. Consequently,

$$\max_{\varsigma, s \in [\varsigma_0, \varsigma_1]} \mathfrak{G}(\varsigma, s) = \max_{\varsigma \in [\varsigma_0, \varsigma_1]} \mathfrak{G}(\varsigma, \varsigma) = \frac{(\varsigma_1 - \varsigma_0)^{\vartheta - 1}}{2^{\vartheta} \Gamma(\vartheta)}.$$

As a result, the linear operator $\mathbb{S}$, mapping $L^1(I)$ into $L^\infty(I)$, is introduced by

$$\mathbb{S}\mathfrak{u}(\varsigma) := \int_{\varsigma_0}^{\varsigma_1} \mathfrak{G}(\varsigma, s)\mathfrak{u}(s) ds, \quad \forall \mathfrak{u} \in L^1(I), \tag{6}$$

is bounded. Indeed, we have

$$\|\mathbb{S}\mathfrak{u}\|_\infty \leq \frac{(\varsigma_1 - \varsigma_0)^{\vartheta - 1}}{2^{\vartheta} \Gamma(\vartheta)} \|\mathfrak{u}\|_1. \tag{7}$$

We now define the operator $\Upsilon : L^1([\varsigma_0, \varsigma_1]) \to L^1([\varsigma_0, \varsigma_1])$, such that for any function $\mathfrak{u} \in L^1([\varsigma_0, \varsigma_1])$, the action of $\Upsilon$ is described by

$$\Upsilon\mathfrak{u}(\varsigma) = \rho(\varsigma, \mathfrak{u}(\varsigma)) + \nu(\varsigma, \mathfrak{u}(\varsigma)) \int_{\varsigma_0}^{\varsigma_1} \mathfrak{G}(\varsigma, s)\tau(s, \mathfrak{u}(s))\, ds, \tag{8}$$

where the function $\mathfrak{G}(\varsigma, s)$ has been introduced previously in Lemma 3.1.

**Corollary 3.3.** *A function* $\mathfrak{u}(\cdot)$ *solves the fractional boundary value problem* (1) *if and only if it is a fixed point of the operator* $\Upsilon$.

3.2. **Existence Results.** This part is devoted to studying the fractional boundary value problem (1) under a specific set of assumptions. By employing an appropriate fixed point principle, we deduce that the operator $\Upsilon$ admits at least one fixed point.

To keep the presentation clear, we write $(\Upsilon\mathfrak{u})(\varsigma) = (\Upsilon_1\mathfrak{u})(\varsigma) + (\Upsilon_2\mathfrak{u})(\varsigma)$, where

$$(\Upsilon_1\mathfrak{u})(\varsigma) = \mathcal{N}_\rho\mathfrak{u}, \tag{9}$$

and

$$(\Upsilon_2\mathfrak{u})(\varsigma) = (\mathcal{N}_\nu\mathfrak{u}) \cdot (\mathbb{S}\mathcal{N}_\tau\mathfrak{u}), \tag{10}$$

in which $\mathbb{S}$ introduced by (6) and $\mathcal{N}_\tau$ is known Nemytskii operator, associated with $\mathfrak{u}$ where $\mathcal{N}_\tau\mathfrak{u}(\varsigma) := \tau(\varsigma, \mathfrak{u}(\varsigma))$.

Let us consider the following assumptions regarding the fractional BVP (1). We denote $\tau, \rho, \nu : I \times \mathbb{R} \to \mathbb{R}$ as Carathéodory functions. Furthermore, we have non-negative functions $\tau_0, \rho_0, \nu_0 \in L^1(I)$ along with a set of positive constants

$l_\tau, l_\rho, l_\nu$. For every $\mathfrak{u} \in \mathbb{R}$ and almost every $\varsigma \in I$, the following conditions hold true:

**(D1)** $|\tau(\varsigma, \mathfrak{u})| \le \tau_0(\varsigma) + l_\tau |\mathfrak{u}|$,
**(D2)** $|\rho(\varsigma, \mathfrak{u})| \le \rho_0(\varsigma) + l_\rho |\mathfrak{u}|$,
**(D3)** $|\nu(\varsigma, \mathfrak{u})| \le \nu_0(\varsigma) + l_\nu |\mathfrak{u}|$.

*Remark* 3.4. Under the stated assumptions, the mappings $\mathcal{N}_\tau$, $\mathcal{N}_\rho$, and $\mathcal{N}_\nu$ are all continuous and bounded operators acting on $L^1(I)$. (Refer to [3] for further discussion.)

*Remark* 3.5. It is known that any finite set of integrable functions satisfies uniform integrability (see [25], Proposition 24). As a consequence, we obtain $\omega_0(\{\tau_0, \nu_0, \rho_0\}) = 0$. Accordingly, using the assumed properties, we deduce that for every nonempty and bounded set $M \subset L^1(I)$, the following inequalities hold:

$$\omega_0(\mathcal{N}_\tau(M)) \le l_\tau \omega_0(M), \quad \omega_0(\mathcal{N}_\nu(M)) \le l_\nu \omega_0(M), \quad \omega_0(\mathcal{N}_\rho(M)) \le l_\rho \omega_0(M).$$

To simplify the presentation of our main conditions, let us denote the constant $\mathcal{K}$ as

$$\mathcal{K} := \frac{(\varsigma_1 - \varsigma_0)^{\vartheta - 1}}{2^\vartheta \Gamma(\vartheta)}.$$

**Lemma 3.6.** *If **(D1)**–**(D3)** are satisfied, then there exists a radius $r_0 > 0$ such that $\Upsilon$ maps the ball $B_{r_0}$ onto itself, provided that*

$$(11) \quad 1 - \Big( l_\nu \mathcal{K} \|\tau_0\|_1 + l_\tau \mathcal{K} \|\nu_0\|_1 + l_\rho \Big) > 2\sqrt{(l_\tau l_\nu \mathcal{K})\left(\|\rho_0\|_1 + \|\nu_0\|_1 \|\tau_0\|_1 \mathcal{K}\right)}.$$

*Remark* 3.7. From (11) we can deduce the following quadratic function

$$\begin{aligned} P(r) &= \mathcal{K} l_\nu l_\tau r^2 \\ &\quad + \left(\mathcal{K}(l_\nu \|\tau_0\|_1 + l_\tau \|\nu_0\|_1) - (1 - l_\rho)\right) r \\ &\quad + \mathcal{K} \|\nu_0\|_1 \|\tau_0\|_1 + \|\rho_0\|_1, \end{aligned}$$

possesses two nonnegative roots, denoted as $r_1^0$ and $r_2^0$, where $r_1^0 < r_2^0$; and for all $r \in (r_1^0, r_2^0)$, $P(r) < 0$. Moreover, $\min_r P(r) = P(r_m)$, where

$$r_m = \frac{1 - (l_\rho + \mathcal{K}(l_\nu \|\tau_0\|_1 + l_\tau \|\nu_0\|_1))}{2\mathcal{K} l_\tau l_\nu} > 0;$$

thus, we can choose $0 < r_0 < r_m$, such that $P(r_0) < 0$.

*Proof.* of Lemma 3.6: Given any $\mathfrak{u} \in L^1(I)$, it holds that

$$\|\Upsilon_1(\mathfrak{u})\|_1 \le \|\rho_0\|_1 + l_\rho \|\mathfrak{u}\|_1.$$

Moreover,

$$\|\Upsilon_2(\mathfrak{u})\|_1 \le \mathcal{K}(\|\tau_0\|_1 + l_\tau \|\mathfrak{u}\|_1)(\|\nu_0\|_1 + l_\nu \|\mathfrak{u}\|_1).$$

Thus for every $\mathfrak{u} \in B_{r_0}$,

$$\begin{aligned}\|\Upsilon(\mathfrak{u})\|_1 &\le \|\rho_0\|_1 + l_\rho r_0 + \mathcal{K}(\|\tau_0\|_1 + l_\tau r_0)(\|\nu_0\|_1 + l_\nu r_0)\\ &= P(r_0) + r_0\\ &\le r_0.\end{aligned}$$

Hence $\Upsilon$ is a self map on $B_{r_0}$. □

**Theorem 3.8.** *Suppose **(D1)**, **(D2)**, **(D3)** are fulfilled. Then, for some $r_0 > 0$, the fractional BVP (1) admits a solution $\mathfrak{u} \in B_{r_0} \subset L^1(I)$, provided (11) is satisfied.*

*Proof.* We aim to verify that the operators $\Upsilon_1$ and $\Upsilon_2$ fulfill the requirements stated in Theorem 2.11. According to assumption **(D3)** and in light of Remark 3.4, the operator $\Upsilon_1$ is ww-compact and satisfies the contractivity condition with constant $\beta := l_\rho$.

Next, we proceed to demonstrate that $\Upsilon_2$ is an $\omega$-contractive mapping. For this purpose, let $D$ be a measurable subset of the interval $I$.

Based on the conditions **(D1)** and **(D3)**, we can conclude the following for every $\mathfrak{u}$ within the ball $B_{r_0}$:

$$\begin{aligned}\int_D |\Upsilon_2\mathfrak{u}(\varsigma)|d\varsigma &\le \|\mathbb{S}\mathcal{N}_\tau\mathfrak{u}\|_\infty \int_D |\mathcal{N}_\nu\mathfrak{u}(\varsigma)|d\varsigma\\ &\le \mathcal{K}(l_\tau r_0 + \|\tau_0\|_1)\int_D (l_\nu|\mathfrak{u}(\varsigma)| + \nu_0(\varsigma))d\varsigma.\end{aligned}$$

Given that $\nu_0(\cdot) \in L^1(I)$ and based on Remark 3.5, the weak measure of noncompactness for the individual element $\nu_0(\cdot)$ is determined to be zero. Consequently, we can conclude that for any nonempty bounded subset $Y$ of $L^1(I)$,

$$\omega_0(\Upsilon_2(Y)) \le \mathcal{K}(\|\tau_0\|_1 + l_\tau r_0)l_\nu\omega_0(Y) := \eta\omega_0(Y), \tag{12}$$

where $\eta := \mathcal{K}(\|\tau_0\|_1 + l_\tau r_0)l_\nu$.

Moreover, $\Upsilon_2$ is ws-compact map. Indeed, by applying the triangle inequality, the definition of the operators, and the bounds $\|\mathbb{S}\mathfrak{u}\|_\infty \le \mathcal{K}\|\mathfrak{u}\|_1$ from (7) and $\|\mathbb{S}\mathfrak{u}\|_1 \le \mathcal{K}(\varsigma_1 - \varsigma_0)\|\mathfrak{u}\|_1$, we get:

$$\begin{aligned}\|\Upsilon_2\mathfrak{u}_1 - \Upsilon_2\mathfrak{u}_2\|_1 &\le \|\mathcal{N}_\nu\mathfrak{u}_1 \cdot (\mathbb{S}\mathcal{N}_\tau\mathfrak{u}_1 - \mathbb{S}\mathcal{N}_\tau\mathfrak{u}_2)\|_1 + \|(\mathcal{N}_\nu\mathfrak{u}_1 - \mathcal{N}_\nu\mathfrak{u}_2)\cdot\mathbb{S}\mathcal{N}_\tau\mathfrak{u}_2\|_1\\ &\le \|\mathbb{S}(\mathcal{N}_\tau\mathfrak{u}_1 - \mathcal{N}_\tau\mathfrak{u}_2)\|_\infty\|\mathcal{N}_\nu\mathfrak{u}_1\|_1 + \|\mathbb{S}\mathcal{N}_\tau\mathfrak{u}_2\|_\infty\|\mathcal{N}_\nu\mathfrak{u}_1 - \mathcal{N}_\nu\mathfrak{u}_2\|_1\\ &\le \mathcal{K}\|\mathcal{N}_\tau\mathfrak{u}_1 - \mathcal{N}_\tau\mathfrak{u}_2\|_1(\|\nu_0\|_1 + l_\nu\|\mathfrak{u}_1\|_1)\\ &\quad + \mathcal{K}\|\mathcal{N}_\tau\mathfrak{u}_2\|_1\|\mathcal{N}_\nu\mathfrak{u}_1 - \mathcal{N}_\nu\mathfrak{u}_2\|_1\\ &\le \mathcal{K}\Big[(\|\nu_0\|_1 + l_\nu\|\mathfrak{u}_1\|_1)\|\mathcal{N}_\tau\mathfrak{u}_1 - \mathcal{N}_\tau\mathfrak{u}_2\|_1\\ &\quad + (\|\tau_0\|_1 + l_\tau\|\mathfrak{u}_2\|_1)\|\mathcal{N}_\nu\mathfrak{u}_1 - \mathcal{N}_\nu\mathfrak{u}_2\|_1\Big].\end{aligned}$$

As a result of the continuity of the operators $\mathcal{N}_\tau$ and $\mathcal{N}_\nu$, it follows that $\Upsilon_2$ is continuous over the ball $B_r$ for each $r > 0$. Let us now consider a sequence

$(\mathfrak{u}_n)_{n\in\mathbb{N}}$ which converges weakly in $B_r$. Referring to equation (12), we deduce that the collection $\{\Upsilon_2\mathfrak{u}_n\}_{n\in\mathbb{N}}$ exhibits uniform integrability in $L^1(I)$. Then, by Theorem 2.6, for every $\varepsilon > 0$, there exists some $\delta > 0$ so that

$$\int_{D_\delta} |\Upsilon_2\mathfrak{u}_n(\varsigma)|\, d\varsigma \leq \frac{\varepsilon}{3}, \quad \forall n, \tag{13}$$

holds for all measurable subsets $D_\delta \subseteq I$ with $\mathrm{m}(D) \leq \delta$.

For each $\varsigma \in [\varsigma_0, \varsigma_1]$, define the mapping $\mathbb{S}_\varsigma : L^1([\varsigma_0, \varsigma_1]) \to \mathbb{R}$ by the equation $\mathbb{S}_\varsigma(y) = \mathbb{S}y(\varsigma)$. $\mathbb{S}_\varsigma$ acts as a linear continuous functional on $L^1([\varsigma_0, \varsigma_1])$. Consequently, the sequence $\{\mathbb{S}_\varsigma(\mathcal{N}_\tau\mathfrak{u}_n)\}$ converges in $\mathbb{R}$, implying that $\{\mathbb{S}(\mathcal{N}_\tau\mathfrak{u}_n)\}$ converges pointwise in $L^1([\varsigma_0, \varsigma_1])$. By applying Egoroff's theorem, we can conclude that there exists a measurable subset $I_0 \subseteq I$ such that the measure of the set difference $\mathrm{m}(I \setminus I_0) \leq \delta$. Within this subset $I_0$, the sequence $\{\mathbb{S}(\mathcal{N}_\tau\mathfrak{u}_n)\}_{n\in\mathbb{N}}$ converges uniformly. Consequently, the sequence $\{\mathcal{N}_\nu\mathfrak{u}_n \cdot \mathbb{S}(\mathcal{N}_\tau\mathfrak{u}_n)\}_{n\in\mathbb{N}}$ converges strongly in $L^1(I_0)$. Therefore, by referring to equation (13) and noting that $\mathrm{m}(I \setminus I_0) \leq \delta$, along with the fact that $\{\Upsilon_2\mathfrak{u}_n\}_{n\in\mathbb{N}}$ satisfies the Cauchy criterion in $L^1(I_0)$, it follows that there exists a natural number $k_0$ such that for all $i, j \geq k_0$,

$$\begin{aligned}
\int_I \left|\Upsilon_2\mathfrak{u}_j(\varsigma) - \Upsilon_2\mathfrak{u}_i(\varsigma)\right| d\varsigma &\leq \int_{I_0} \left|\Upsilon_2\mathfrak{u}_j(\varsigma) - \Upsilon_2\mathfrak{u}_i(\varsigma)\right| d\varsigma + \int_{I\setminus I_0} \left|\Upsilon_2\mathfrak{u}_j(\varsigma) - \Upsilon_2\mathfrak{u}_i(\varsigma)\right| d\varsigma \\
&\leq \int_{I_0} \left|\Upsilon_2\mathfrak{u}_j(\varsigma) - \Upsilon_2\mathfrak{u}_i(\varsigma)\right| d\varsigma + \int_{I\setminus I_0} \left|\Upsilon_2\mathfrak{u}_j(\varsigma)\right| d\varsigma + \int_{I\setminus I_0} \left|\Upsilon_2\mathfrak{u}_i(\varsigma)\right| d\varsigma \\
&\leq \varepsilon.
\end{aligned}$$

Hence, the sequence $\{\Upsilon_2\mathfrak{u}_n\}_{n=1}^{\infty}$ meets the Cauchy criterion over the interval $I$, which ensures convergence in the space $L^1(I)$. To complete the proof, we need to confirm that condition (III) in Theorem 2.11 is fulfilled by selecting the set $M = B_{r_0}$, with $r_0$ as specified in Remark 3.7. For this purpose, assume

$$\xi = \Upsilon_1\xi + \Upsilon_2\zeta, \quad \zeta \in M,$$

which leads to

$$\begin{aligned}
|\xi(\varsigma)| &\leq |\mathcal{N}_\rho\xi(\varsigma)| + |\mathcal{N}_\nu\zeta(\varsigma)||\mathbb{S}\mathcal{N}_\tau\zeta(\varsigma)| \\
&\leq l_\rho|\xi(\varsigma)| + |\rho_0(\varsigma)| + \mathcal{K}(l_\nu|\zeta(\varsigma)| + |\nu_0(\varsigma)|)(\|\tau_0\|_1 + l_\tau\|\zeta\|_1).
\end{aligned}$$

Integrating over $I$ and noting $\|\zeta\|_1 \leq r_0$:

$$\begin{aligned}(1-l_\rho)\|\xi\|_1 &\leq \int_I |\rho_0(\varsigma)|d\varsigma + \int_I \mathcal{K}(l_\nu|\zeta(\varsigma)| + |\nu_0(\varsigma)|)(\|\tau_0\|_1 + l_\tau\|\zeta\|_1)d\varsigma \\ &\leq \|\rho_0\|_1 + \mathcal{K}(\|\tau_0\|_1 + l_\tau r_0)\int_I (l_\nu|\zeta(\varsigma)| + |\nu_0(\varsigma)|)d\varsigma \\ &\leq \|\rho_0\|_1 + \mathcal{K}(\|\tau_0\|_1 + l_\tau r_0)(l_\nu\|\zeta\|_1 + \|\nu_0\|_1) \\ &\leq \|\rho_0\|_1 + \mathcal{K}(\|\tau_0\|_1 + l_\tau r_0)(l_\nu r_0 + \|\nu_0\|_1) \\ &= \mathcal{K}l_\nu l_\tau r_0^2 + \mathcal{K}(l_\nu\|\tau_0\|_1 + l_\tau\|\nu_0\|_1)r_0 + \mathcal{K}\|\nu_0\|_1\|\tau_0\|_1 + \|\rho_0\|_1 \\ &= P(r_0) + (1-l_\rho)r_0 \\ &< (1-l_\rho)r_0.\end{aligned}$$

This indicates that $\xi$ belongs to the set $M$. The final inequality is derived from Remark 3.7, which states that $P(r_0) < 0$.

Furthermore, as detailed in Remark 3.7, we can choose $r_0$ such that $r_0 < r_m$, which leads to:

$$\begin{aligned}\eta + \beta &= \mathcal{K}(\|\tau_0\|_1 + l_\tau r_0)l_\nu + l_\rho \\ &< \mathcal{K}\left(\|\tau_0\|_1 + l_\tau\left[\frac{1-l_\rho-\mathcal{K}l_\nu\|\tau_0\|_1}{\mathcal{K}l_\tau l_\nu}\right]\right)l_\nu + l_\rho \\ &= \mathcal{K}\|\tau_0\|_1 l_\nu + (1-l_\rho-\mathcal{K}l_\nu\|\tau_0\|_1) + l_\rho \\ &= 1.\end{aligned}$$

Hence, the proof concludes successfully.

□

## 4. **Examples**

It is important to note that the following examples are not merely illustrative; they have been specifically constructed to demonstrate the utility of our MWNC framework in $L^1(I)$. The nonlinearities in these examples do not satisfy standard, restrictive conditions (such as a global Lipschitz condition or a compactness-based growth condition) that are often required by classical fixed point theorems like Schauder's in $C(I)$ spaces. By showing that our conditions (D1)-(D3) are satisfied and that a solution exists (with a computable bound $r_m$), we demonstrate that our theoretical approach is capable of handling a broader and more general class of problems than many existing methods.

**Example 4.1.** *Consider*

$$\begin{cases} {}^{\mathfrak{C}}\mathfrak{D}_{0}^{\frac{7}{4}}\left[\dfrac{\mathfrak{u}(\varsigma) - \frac{(\varsigma-\varsigma^2)\arctan(\mathfrak{u})}{15}}{\frac{|\mathfrak{u}(\varsigma)|}{20(1+|\mathfrak{u}(\varsigma)|)}}\right] + \frac{\sqrt{e^{\varsigma}+\mathfrak{u}^2(\varsigma)}}{2} = 0, \quad \textit{for} \quad \varsigma \in [0,1], \\ \mathfrak{u}(0) = \mathfrak{u}(1) = 0. \end{cases} \tag{14}$$

*In this example, specifically, we are considering the BVP (1) defined on the interval* $[0,1]$ *with* $\vartheta = \frac{7}{4}$, *and*

$$\rho(\varsigma, \mathfrak{u}) = \frac{(\varsigma - \varsigma^2)\arctan(\mathfrak{u})}{15}, \quad \nu(\varsigma, \mathfrak{u}) = \frac{|\mathfrak{u}|}{20(1+|\mathfrak{u}|)}, \quad \tau(\varsigma, \mathfrak{u}) = \frac{\sqrt{e^{\varsigma} + \mathfrak{u}^2}}{2}.$$

*It is evident that because of the concave nature of* $\sqrt{\varsigma}$, *for all* $\mathfrak{u} \in \mathbb{R}$ *and* $\varsigma \in [0,1]$, *the inequality below is valid:*

$$|\tau(\varsigma, \mathfrak{u})| \leq \frac{1}{2}\sqrt{e^{\varsigma}} + \frac{1}{2}|\mathfrak{u}|.$$

*Hence, condition* (**D1**) *holds with* $\tau_0(\varsigma) = \frac{1}{2}\sqrt{e^{\varsigma}}$ *and* $l_\tau = \frac{1}{2}$. *Additionally, for each* $\mathfrak{u}_1, \mathfrak{u}_2 \in \mathbb{R}$ *and* $\varsigma \in [0,1]$, *we have*

$$|\rho(\varsigma, \mathfrak{u})| \leq \frac{1}{60}|\mathfrak{u}| \quad \textit{and} \quad |\nu(\varsigma, \mathfrak{u})| \leq \frac{1}{20}|\mathfrak{u}|.$$

*Therefore, conditions (D2) and (D3) hold with*

$$l_\rho = \frac{1}{60}, \quad \rho_0(\varsigma) = 0, \quad l_\nu = \frac{1}{20}, \quad \nu_0(\varsigma) = 0.$$

*Using MATLAB computations, it was found that the inequality referenced in (11) is satisfied and that* $r_m \approx 60.14$. *Thus, applying Theorem 3.8, we deduce the existence of a solution* $\mathfrak{u}^* \in L^1([0,1])$ *such that*

$$\int_0^1 |\mathfrak{u}^*(\varsigma)|\, d\varsigma < 60.14.$$

**Example 4.2.** *Consider*

(15)
$$\begin{cases} {}^{\mathfrak{C}}\mathfrak{D}_1^{\frac{16}{5}} \left[ \dfrac{\mathfrak{u}(\varsigma) - \frac{1}{15}\mathfrak{u}(\arctan(\mathfrak{u}) - \sin(\varsigma)) + \frac{1}{30}\ln(1+\mathfrak{u}^2)}{\frac{1}{10}\sqrt{\mathfrak{u}}\varsigma^2} \right] \\ \qquad + (2\sin(\varsigma) - 3\cos(\varsigma))\ln(1+|\mathfrak{u}|) + \dfrac{e^{1-|\mathfrak{u}|}}{\sqrt{\varsigma - 1}} = 0, \quad \textit{for } \varsigma \in [1,2], \\ \mathfrak{u}(1) = \mathfrak{u}(2) = 0. \end{cases}$$

*This problem is an example of the fractional BVP (1), where the interval* $[\varsigma_0, \varsigma_1] = [1,2]$, *with* $\vartheta = \frac{16}{5}$. *The functions are defined as follows:*

$$\rho(\varsigma, \mathfrak{u}) := \frac{1}{15}\mathfrak{u}(\arctan(\mathfrak{u}) - \sin(\varsigma)) + \frac{1}{30}\ln(1+\mathfrak{u}^2),$$

$$\nu(\varsigma, \mathfrak{u}) := \frac{1}{10}\sqrt{\mathfrak{u}}\varsigma^2,$$

*and*

$$\tau(\varsigma, \mathfrak{u}) := (2\sin(\varsigma) - 3\cos(\varsigma))\ln(1+|\mathfrak{u}|) + \frac{e^{1-|\mathfrak{u}|}}{\sqrt{\varsigma - 1}}.$$

*Using basic calculus, we can express that* $|\tau(\varsigma, \mathfrak{u})| \leq \sqrt{13}|\mathfrak{u}| + \frac{e}{\sqrt{\varsigma-1}}$. *Therefore, condition* (***D1***) *is fulfilled with* $\tau_0(\varsigma) = \frac{e}{\sqrt{\varsigma-1}}$ *and* $l_\tau = \sqrt{13}$. *Moreover, for any* $\mathfrak{u}, y \in \mathbb{R}$ *and* $\varsigma \in [1,2]$, *we can assert that*

$$|\nu(\varsigma, \mathfrak{u})| = \frac{1}{10}|\sqrt{\mathfrak{u}}\varsigma^2| \leq \frac{4}{10}(|\mathfrak{u}|+1),$$

*and also*

$$|\rho(\varsigma, \mathfrak{u}) - \rho(\varsigma, y)| \leq \frac{1}{15}\big|\mathfrak{u}\arctan(\mathfrak{u}) - \frac{1}{2}\ln(1+\mathfrak{u}^2) - y\arctan(y) + \frac{1}{2}\ln(1+y^2)\big| + \frac{1}{15}|\mathfrak{u}-y||\sin(\varsigma)|.$$

*Since*

$$\Big|\frac{d\big(\mathfrak{u}\arctan(\mathfrak{u}) - \frac{1}{2}\ln(1+\mathfrak{u}^2)\big)}{d\mathfrak{u}}\Big| = |\arctan(\mathfrak{u})| \leq \frac{\pi}{2},$$

*we insert that*

$$|\rho(\varsigma, \mathfrak{u}) - \rho(\varsigma, y)| \leq \frac{\pi}{30}|\mathfrak{u}-y| + \frac{1}{15}|\mathfrak{u}-y|.$$

*Consequently, conditions (D2) and (D3) are fulfilled with* $l_\rho = \frac{2+\pi}{30}, \rho_0(\varsigma) = 0$ *and* $l_\nu = \frac{4}{10}, \nu_0(\varsigma) = \frac{4}{10}$. *Utilizing MATLAB, we determined that the inequality in (11) is satisfied, yielding* $r_m \approx 5.14$. *Furthermore, by implementing Theorem 3.8, we conclude that a solution* $\mathfrak{u}^*$ *exists within* $L^1([1,2])$, *satisfying the condition* $\int_1^2 |\mathfrak{u}^*(\varsigma)|d\varsigma < 5.14$.

## 5. Numerical results

In the preceding sections, we theoretically established the existence of at least one solution in the $L^1(I)$ space. The purpose of this section is twofold: first, to provide a numerical method to approximate these solutions, and second, to numerically validate the theoretical results (specifically the condition (11) and the radius $r_m$) for our given examples.

We employ the Finite Element Method (FEM) with linear basis functions. It is important to note that the objective of this numerical study is illustrative and validating, rather than comparative. Our aim is to demonstrate that solutions for our examples do exist and that they respect our theoretical bounds, not to propose the FEM as a computationally superior algorithm.

A detailed performance comparison with other established numerical schemes (such as finite difference or spectral methods), which would constitute a separate computational study, is respectfully considered beyond the primary scope of this theoretical-existence paper. The FEM was chosen here for its robustness and flexibility in handling the complex nonlinear integral operators present in our formulation.

This approach is recognized as one of the most efficient techniques for the numerical treatment of differential equations commonly found in engineering disciplines. The finite element method provides various advantages, such as accurately modeling localized effects, facilitating the representation of complex solutions, adapting to heterogeneous material characteristics, and precisely describing intricate geometrical configurations.

Moreover, this method has a significant advantage that we have highlighted in this paper. Specifically, the basis functions employed in this approach can be selected based on nodal properties, facilitating the resolution of nonlinear equations. For further insights into this methodology, please refer to [4].

For numerical analysis of BVP (1), we define a standard grid by dividing the interval $I = [\varsigma_0, \varsigma_1]$ into $N$ subintervals $[\varsigma_{j-1}, \varsigma_j]$ for $j = 1, \ldots, N$. The grid points are defined as $\varsigma_j = \varsigma_0 + j(d\varsigma)$ for $j = 0, 1, \ldots, N$, where the step size is $(d\varsigma) = (\varsigma_1 - \varsigma_0)/N$. We use the standard piecewise linear (triangular) basis functions, $\text{tri}_j(\varsigma)$, associated with each node $\varsigma_j$ for $j = 0, \ldots, N$. These functions are defined as:

$$\text{tri}_j(\varsigma) = \begin{cases} (\varsigma - \varsigma_{j-1})/(d\varsigma), & \varsigma \in [\varsigma_{j-1}, \varsigma_j], \quad \text{for } j = 1, \ldots, N, \\ (\varsigma_{j+1} - \varsigma)/(d\varsigma), & \varsigma \in [\varsigma_j, \varsigma_{j+1}], \quad \text{for } j = 0, \ldots, N-1, \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

The function $\text{tri}_j$ is defined as a piecewise linear function, which takes on the value of one at $\varsigma_j$ and equals zero at all other points within the partition. This characteristic endows it with the nodal property, meaning that $\text{tri}_j(\varsigma_n) = \delta_{n,j}$, where $\delta_{n,j}$ represents the Kronecker delta for any integer $n = 0, 1, \ldots, N$. We represent the approximation of $\mathfrak{u}(\varsigma)$ as $\mathfrak{u}_N(\varsigma)$:

$$\mathfrak{u}_N(\varsigma) = \sum_{k=0}^{N} c_k \ \text{tri}_k(\varsigma), \ \text{such that } c_k = \mathfrak{u}_N(\varsigma_k) \text{ for } k = 0, 1, \cdots, N, \tag{17}$$

where $c_j$ are the nodal values of the finite element solution $\mathfrak{u}_N$. According to Lemma 3.1, the solution of the BVP (1) satisfies

$$\mathfrak{u}(\varsigma) = \rho(\varsigma, \mathfrak{u}(\varsigma)) + \nu(\varsigma, \mathfrak{u}(\varsigma)) \int_{\varsigma_0}^{\varsigma_1} \mathfrak{G}(\varsigma, s)\tau(s, \mathfrak{u}(s))ds. \tag{18}$$

Inserting $\mathfrak{u}_N$ into (18), using the nodal property of $\text{tri}_j$ and taking $\varsigma = \varsigma_j$ yield

$$\begin{aligned} \mathfrak{u}_N(\varsigma_j) = &\sum_{k=0}^{N} \rho(\varsigma_j, c_k) \ \text{tri}_k(\varsigma_j) + \left( \sum_{k=0}^{N} \nu(\varsigma_j, c_k) \ \text{tri}_k(\varsigma_j) \right) \\ &\times \sum_{k=0}^{N} \int_{\varsigma_0}^{\varsigma_1} \mathfrak{G}(\varsigma_j, s) \, \text{tri}_k(s) \, \tau(s, c_k) ds. \end{aligned} \tag{19}$$

Employing the composite trapezoidal rule [13], the integral in (19) can be approximated as

(20)
$$\int_{\varsigma_0}^{\varsigma_1} \mathfrak{G}(\varsigma_j, s)\ \mathrm{tri}_k(s)\, \tau(s, c_k)\, ds \approx \frac{d\varsigma}{2}\Big[\mathfrak{G}(\varsigma_j, \varsigma_0)\, \mathrm{tri}_k(\varsigma_0)\, \tau(\varsigma_0, c_k) + \mathfrak{G}(\varsigma_j, \varsigma_1)\, \mathrm{tri}_k(\varsigma_1)\, \tau(\varsigma_1, c_k)\Big] + d\varsigma \sum_{l=1}^{N-1} \mathfrak{G}(\varsigma_j, \varsigma_l)\, \mathrm{tri}_k(\varsigma_l)\, \tau(\varsigma_l, c_k).$$

By substituting (20) into (19) and applying the nodal property $\mathrm{tri}_k(\varsigma_j) = \delta_{j,k}$, we deduce

$$\begin{aligned} c_j &= \rho(\varsigma_j, c_j) + \frac{d\varsigma}{2}\, \nu(\varsigma_j, c_j)\ \big(\mathfrak{G}(\varsigma_j, \varsigma_0)\, \tau(\varsigma_0, c_0) + \mathfrak{G}(\varsigma_j, \varsigma_1)\, \tau(\varsigma_1, c_N)\big) \\ &\quad + d\varsigma\, \nu(\varsigma_j, c_j) \sum_{l=1}^{N-1} \mathfrak{G}(\varsigma_j, \varsigma_l)\, \tau(\varsigma_l, c_l). \end{aligned} \tag{21}$$

Taking $j = 0, N$ we obtain

$$c_0 = \rho(\varsigma_0, \mathfrak{u}(\varsigma_0)) = \mathfrak{u}(\varsigma_0), \text{ and } c_N = \rho(\varsigma_1, \mathfrak{u}(\varsigma_1)) = \mathfrak{u}(\varsigma_1). \tag{22}$$

Here, we used that $\mathfrak{G}(\varsigma_0, s) = \mathfrak{G}(\varsigma_1, s) = 0$ comes from (5).

Also, for $j = 1, 2, \cdots, N-1$, the coefficients $c_j$ are determined by solving the nonlinear system of $N-1$ equations given by (21). To solve this system, we used the built-in `fsolve` function in MATLAB (Version R2025b), which employs a trust-region-dogleg algorithm.

The theoretical contractive properties of the operator $\Upsilon_1$ (which forms a core part of the full operator $\Upsilon$) are what fundamentally guarantee that the underlying problem is well-posed and locally unique. This well-posedness ensures that a robust numerical solver like `fsolve` can converge to a stable solution. The rapid convergence observed in the relative errors (see Table 1) numerically supports the stability of the solution guaranteed by our theoretical analysis.

Using this algorithm, the solutions of the BVP (14) and BVP (15) are approximated. We plot the approximate solutions of both problems for $N = 10$ in the following figures (Figures 1 and 2). Also, Table shows the $L^\infty-$ relative errors of the approximate solutions by increasing $N$.

As a direct verification of our theoretical findings, we also computed the $L^1$-norm of the approximate solutions for both examples.

- For Example 5.1, using the numerical data from the FEM for $N = 15$ and applying the composite trapezoidal rule, we find that $\|\mathfrak{u}_N\|_1 \approx 58.23$. This computed norm is clearly less than the theoretical bound $r_m \approx 60.14$ calculated in the example.
- Similarly, for Example 5.2 (with $N = 15$), the computed norm is $\|\mathfrak{u}_N\|_1 \approx 4.91$, which is well below its theoretical bound $r_m \approx 5.14$.

This provides strong numerical support for our theoretical results and demonstrates the consistency between our analysis and the computational output.

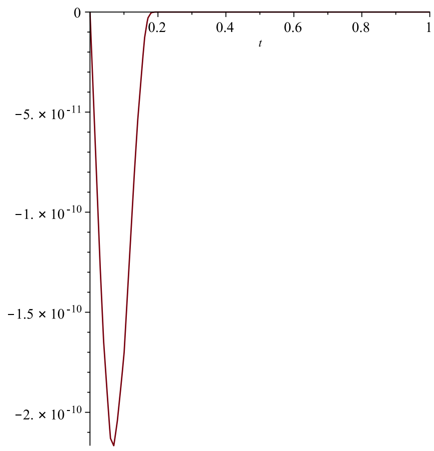


Figure 1. Finite element approximation of the solution in Example 5.1

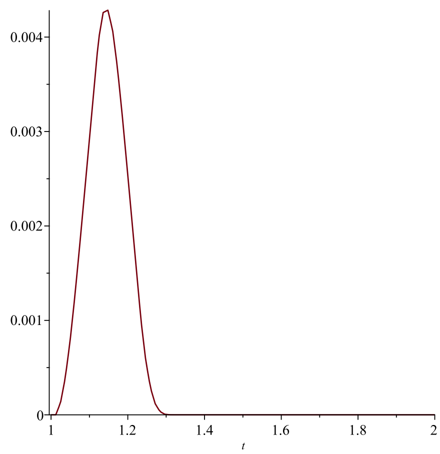


Figure 2. Finite element approximation of the solution in Example 5.2

## 6. **Conclusion**

In this paper, we have investigated the existence of solutions for a generalized class of fractional differential boundary value problems in the Banach space $L^1(I)$. Our methodology was based on a fixed point theorem of Krasnosel'skii type, combined with the concept of the measure of weak noncompactness. We established a set of sufficient conditions, based on linear growth assumptions,

| N | $L^\infty$–relative error for Example 5.1 | $L^\infty$–relative error for Example 5.2 |
|---|---|---|
| 5 | $2.41E-1$ | $1.06E-0$ |
| 10 | $5.28E-2$ | $6.12E-1$ |
| 15 | $8.70E-3$ | $9.48E-2$ |

Table 1. Convergence of the approximate solution in Example 5.1 and 5.2 with increasing $N$.

that guarantee the existence of at least one solution. The theoretical results were then illustrated through two detailed examples, for which numerical solutions were also approximated using the Finite Element Method, validating our abstract findings. This theoretical framework could be particularly useful for models in anomalous diffusion or control theory where the system's dynamics are state-dependent in the manner described by Eq. (1).

For future work, several interesting directions remain. One potential direction is to extend this analysis to systems of fractional differential equations using similar MWNC techniques. Another research direction could be to investigate the uniqueness of the solution, which was not the focus of the present study. Finally, applying this method to other classes of fractional operators, such as $\psi$-Hilfer or Caputo-Hadamard derivatives, would be a valuable contribution to the field.

## 7. Author Contributions

All authors participated in drafting this manuscript. Detailed designs of the problem were performed by S. S. Nezhad and M. Sangi. All authors read and approved the final manuscript.

## 8. Data Availability Statement

Due to the nature of this research, there is no relevant data.

## 9. Aknowledgement

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## 10. Ethical considerations

This study is completely theoretical and does not involve any experiments on humans or animals. The authors declare that this manuscript is original, has not been published before, and is not currently being considered for publication elsewhere.

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## 12. Conflict of interest

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