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ON THE DUALITY OF FUNCTION VALUED FRAMES IN $L^2(0,\infty)$

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Abstract. By the notion of F_a-factorable operators, we establish a new version of the Riesz Representation Theorem adapted to this class of operators. Also, for F_a-factorable operators T and T', we derive some equivalent conditions such that the equation $T' = TX$ has a unique solution in the class of F_a-factorable operators. Moreover, by using the concept of F_a-frame in $L^2(0,\infty)$, which is similar yet distinct from the traditional frame concept, we establish optimal F_a-frame bounds, and provide a complete characterization of all corresponding F_a-dual frames.

Keywords: F_a-inner product, F_a-frame, F_a-dual frame; F_a-factorable-function, Riesz Representation Theorem.
2020 MSC: Primary 46B15, 42C15

1. Introduction

Let H be a separable Hilbert space. A sequence $\{f_j\}_{j\in\mathbb{Z}} \subset H$ is called a frame for H if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{j\in\mathbb{Z}} |\langle f, f_j\rangle|^2 \leq B\|f\|^2, \quad \text{for all } f \in H.$$

The numbers A and B are called the lower and upper frame bounds, respectively [3], and the optimal frame bounds are defined as the supremum of all lower bounds and the infimum of all upper bounds, respectively.

Following the initial formulation of frames by Duffin and Schaeffer [5] in the context of nonharmonic Fourier series, the theory of frames has been extensively developed and various generalizations of frames have been introduced. In 1946, Gabor introduced a special frame, known as the Gabor frame or Weyl-Heisenberg frame, which is pivotal in signal analysis. Wavelet frames and continuous frames are other types of frames, and the reader can refer to [3] for their definitions and some fundamental properties.

In [8], Kamyabi Gol and Raisi Tousi studied φ-frames in $L^2(G)$, where G is a locally compact group and φ is a topological automorphism of G. They showed that φ-frames share many structural properties with classical frames and introduced associated φ-analysis and synthesis operators.

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In the context of shift-invariant systems, the bracket product

$$[f,g](x) = \sum_{\alpha\in\mathbb{Z}^d} f(x+2\pi\alpha)\overline{g(x+2\pi\alpha)}$$

for $f, g \in L^2(\mathbb{R}^d)$ has been employed by several authors [4, 12, 13], to study approximation and frame properties.

In 2003, Casazza and Lammers [2] introduced a function-valued inner product for functions $f, g \in L^2(\mathbb{R})$ depending on a fixed positive real number $a > 0$, given by

$$\langle f, g\rangle_a(x) = \sum_{n\in\mathbb{Z}} f(x-na)\, g(x-na), \quad \text{for all } x \in \mathbb{R}.$$

This inner product, which produces a real-valued function on $\mathbb{R}$, plays an important role in analyzing translation-invariant systems and is particularly useful in time-frequency analysis.

Another related generalization was proposed in [1], where the authors introduced the L-bracket product on $L^2(G)$, associated to a uniform lattice L in a locally compact abelian group G, and used it to define L-dual frames with respect to the induced function-valued inner product.

In [7], a new function-valued inner product on $L^2(0,\infty)$ was defined using the dilation operator. The authors established connections between orthonormal sequences, orthonormal bases, and frames in the context of this inner product.

In [11], the concepts of F_a-Bessel sequences and F_a-frames were introduced in $L^2(0,\infty)$ for a fixed real number $a > 1$. Several characterizations of these families were established, along with their relationships to classical Bessel sequences with respect to the standard inner product on $L^2(0,\infty)$.

In 2020, an explicit example of an F_a-frame was presented in [14], highlighting certain essential distinctions between F_a-frames and classical frames.

It is well-known that all classical results for frames do not hold for F_a-frames. While removing a single vector from a classical frame yields either another frame or an incomplete set [3]. But Example 2.1 in [14], shows that removing one vector from an F_a-frame can produce an F_a-complete set that is not a frame. We refer to this example in Section 3. These observations underline fundamental distinctions between classical frames and F_a-frames.

In this article, we investigate F_a-factorable operators and establish a new version of the Riesz Representation Theorem for such operators mapping into $L^1[1,a]$. Moreover, we study the optimal frame bounds for F_a-frames and provide a complete characterization of all corresponding F_a-dual frames.

2. F_a-Inner product

For two arbitrary functions $f, g \in L^2(0,\infty)$, it was shown in [7] that the series

$$\sum_{j\in\mathbb{Z}} a^j f(a^j x)\overline{g(a^j x)}$$

converges almost everywhere to a function in $L^1[1,a]$, where $a>1$ is a fixed real number and $L^2(0,\infty)$ and $L^1[1,a]$ are equipped with their usual norms. This leads to the definition of a function-valued inner product, referred to as the F_a-inner product, associated with the dilation parameter a, given by

$$\langle f,g\rangle_a(x) := \sum_{j\in\mathbb{Z}} a^j f(a^j x)\overline{g(a^j x)}, \qquad \text{a.e. } x\in[1,a].$$

This function-valued inner product induces a corresponding norm on $L^2(0,\infty)$, referred to as the F_a-norm, and defined by

$$\|f\|_a(x) := \sqrt{\langle f,f\rangle_a(x)}, \quad x\in(0,\infty).$$

As noted in [7], a measurable function φ on $(0,\infty)$ is said to be a-dilation periodic, if

$$\varphi(ax)=\varphi(x), \quad \text{for all } x\in(0,\infty).$$

The set of all a-dilation periodic functions in $L^\infty(0,\infty)$ is denoted by $B_a(0,\infty)$.

A useful approach for constructing a-dilation periodic functions is as follows: given a function φ defined on $[1,a]$, one can define an extended function $\widetilde{\varphi}$ on $(0,\infty)$ by

$$\widetilde{\varphi}(a^j x) := \varphi(x), \quad j\in\mathbb{Z}, \ x\in[1,a].$$

It is straightforward to verify that for all $f,g\in L^2(0,\infty)$,

$$\langle f,g\rangle_a(ax) = \frac{1}{a}\langle f,g\rangle_a(x), \quad x\in(0,\infty).$$

Therefore, while the F_a-inner product $\langle f,g\rangle_a$ of two functions $f,g\in L^2(0,\infty)$ is generally not a-dilation periodic, the function $\widetilde{\langle f,g\rangle_a}$ obtained by dilation-periodic extension is indeed an a-dilation periodic function.

We are now prepared to recall the following theorem and definition from [7], which are essential in the analysis presented in this paper.

Theorem 2.1. [7] *Suppose that $f,g\in L^2(0,\infty)$. Then the following assertions hold:*

(i) $\langle f,g\rangle = \int_1^a \langle f,g\rangle_a(x)dx$,
(ii) $\|f\|_2 = \|\|f\|_a\|_{L^2[1,a]}$, *where here and in the sequel* $\|.\|_2$ *refers to the scalar norm of* $L^2(0,\infty)$,
(iii) $\langle f,g\rangle_a = \overline{\langle g,f\rangle_a}$,
(iv) $|\langle f,g\rangle_a| \le \|f\|_a\|g\|_a$,
(v) $\|f+g\|_a \le \|f\|_a + \|g\|_a$,
(vi) $\langle \varphi f,g\rangle_a = \varphi\langle f,g\rangle_a = \langle f,\overline{\varphi}g\rangle_a$, *for all* φ *in* $B_a(0,\infty)$.

Definition 2.2. Suppose that $E\subset(0,\infty)$ is measurable and $1\le p\le\infty$. A linear operator $\Lambda: L^2(0,\infty)\to L^p(E)$ is called an F_a-factorable operator, if $\Lambda(\varphi f)=\varphi\Lambda(f)$, for all $\varphi\in B_a(0,\infty)$ and $f\in L^2(0,\infty)$.

Example 2.3. [14] *Let V be a closed linear subspace of $L^2(0,\infty)$ and $\varphi V \subset V$, for all $\varphi \in B_a(0,\infty)$. Then*

$$P_V(\varphi f) = \varphi P_V(f), \quad f \in L^2(0,\infty) \text{ and } \varphi \in B_a(0,\infty).$$

So P_V is an F_a-factorable operator on $L^2(0,\infty)$.

We conclude this section with the following result, which serves as an analogue of the Riesz Representation Theorem for F_a-factorable operators from $L^2(0,\infty)$ into $L^1[1,a]$.

Theorem 2.4. *(F_a-Riesz Representation Theorem) A linear operator Λ from $L^2(0,\infty)$ into $L^1[1,a]$ is bounded F_a-factorable if and only if there exists $g \in L^2(0,\infty)$ such that $\Lambda(f) = \langle f,g\rangle_a$, almost everywhere, for all $f \in L^2(0,\infty)$. Moreover the operator norm of Λ coincides with the usual norm of g, that is, $\|\Lambda\| = \|g\|_2$.*

Proof. Assume first that there exists a function $g \in L^2(0,\infty)$, such that

$$\Lambda(f) = \langle f,g\rangle_a, \quad \text{for all } f \in L^2(0,\infty).$$

We aim to show that the operator Λ is a bounded F_a-factorable operator. In view of Theorem 2.1, we observe that

$$\begin{aligned}
\|\Lambda(f)\|_1 &= \|\langle f,g\rangle_a\|_1 \\
&= \int_1^a |\langle f,g\rangle_a(x)|dx \\
&\le \int_1^a \|f\|_a(x)\,\|g\|_a(x)\,dx \\
&\le \left(\int_1^a \|f\|_a^2(x)\,dx\right)^{\frac{1}{2}} \left(\int_1^a \|g\|_a^2(x)\,dx\right)^{\frac{1}{2}} \\
&= \|f\|_2\,\|g\|_2\,.
\end{aligned}$$

Thus $\|\Lambda\| \le \|g\|_2$. On the other hand $\|\Lambda(g)\|_1 = \|g\|_2^2$ and so $\|\Lambda\| = \|g\|_2$. Observe that the F_a-factorable property of the operator Λ follows directly from part (vi) of Theorem 2.1.

Conversely, suppose that Λ is a bounded and F_a-factorable operator from $L^2(0,\infty)$ into $L^1[1,a]$. Define the linear functional ψ on $L^2(0,\infty)$ by

$$\psi(f) = \int_1^a \Lambda(f)(x)\,dx.$$

Then, by the classical Riesz Representation Theorem for Hilbert spaces, there exists a function $g \in L^2(0,\infty)$ such that

$$\psi(f) = \langle f,g\rangle, \quad \text{for all } f \in L^2(0,\infty).$$

Now, consider the F_a-factorable operator $\mathbf{\Lambda}_g$ defined by

$$\mathbf{\Lambda}_g(f) = \langle f,g\rangle_a, \quad \text{for all } f \in L^2(0,\infty).$$

It then follows that

$$\begin{aligned}\int_1^a \Lambda(f)(x)\,dx &= \psi(f)\\ &= \langle f, g\rangle\\ &= \int_1^a \langle f, g\rangle_a(x)\,dx\\ &= \int_1^a \mathbf{\Lambda}_g(f)(x)\,dx.\end{aligned}$$

Finally, by Proposition 4.2 in [7], we conclude that $\Lambda = \mathbf{\Lambda}_g$. □

The next result can be viewed as an analogue of Theorem 1.1 in [6], which was originally established in the framework of Hilbert C^*-modules.

We need the following function sequence space from [11], which will also be used in the next section. For each $a > 1$, define

$$l_1^2[1,a] = \left\{ \{\widetilde{g_j}\}_{j\in\mathbb{Z}} \;:\; g_j \in \mathrm{L}^\infty[1,a] \quad \textit{and} \quad \int_1^a \sum_{j\in\mathbb{Z}} |g_j(x)|^2 dx < \infty \right\}.$$

It is straightforward to check that $l_1^2[1,a]$ is a (function-valued) inner product space with respect to the following inner product

$$\langle\cdot,\cdot\rangle_{l_1^2[1,a]} : l_1^2[1,a] \times l_1^2[1,a] \longrightarrow L^1[1,a],$$

$$\left\langle \{\widetilde{g_j}\}_{j\in\mathbb{Z}}, \{\widetilde{h_j}\}_{j\in\mathbb{Z}} \right\rangle_{l_1^2[1,a]}(x) = \sum_{j\in\mathbb{Z}} g_j(x)\overline{h_j}(x).$$

Also, it is a Hilbert space with the following inner product

$$\begin{aligned}\left\langle \{\widetilde{g_j}\}_{j\in\mathbb{Z}}, \{\widetilde{h_j}\}_{j\in\mathbb{Z}} \right\rangle_{l_1^2[1,a]} &= \int_1^a \left\langle \{\widetilde{g_j}\}_{j\in\mathbb{Z}}, \{\widetilde{h_j}\}_{j\in\mathbb{Z}} \right\rangle_{l_1^2[1,a]}(x)dx\\ &= \int_1^a \sum_{j\in\mathbb{Z}} g_j(x)\overline{h_j}(x).\end{aligned}$$

As usual, one can define the pointwise norm of each $\{\widetilde{g_j}\}_{j\in\mathbb{Z}} \in l_1^2[1,a]$ by

$$\left\| \{\widetilde{g_j}\}_{j\in\mathbb{Z}} \right\|_{l_1^2[1,a]}(x) = \left(\sum_{j\in\mathbb{Z}} |g_j(x)|^2 \right)^{\frac{1}{2}}, \qquad x \in [1,a], \tag{1}$$

and its uniform norm by

$$\left\| \{\widetilde{g_j}\}_{j\in\mathbb{Z}} \right\|_{l_1^2[1,a]} = \left(\int_1^a \sum_{j\in\mathbb{Z}} |g_j(x)|^2 dx \right)^{\frac{1}{2}}. \tag{2}$$

Theorem 2.5. *Let T' be a bounded and F_a-factorable operator from $L^2(0,\infty)$ into $L^2(0,\infty)$ and let T be a bounded and F_a-factorable operator from $l_1^2[1,a]$ into $L^2(0,\infty)$. Then the following statements are equivalent.*

(i) *$T'T'^* \leq \lambda TT^*$ for some $\lambda > 0$,*
(ii) *There exists $\mu > 0$ such that $\|T'^*f\|_a(x) \leq \mu\|T^*f\|_{l_1^2[1,a]}(x)$ for all f in $L^2(0,\infty)$ and for almost all $x \in [1,a]$,*
(iii) *There exists a bounded and F_a-factorable operator D from $L^2(0,\infty)$ into $l_1^2[1,a]$ such that $T' = TD$,*
(iv) *$R(T') \subseteq R(T)$.*

If one of the conditions (i)–(iv) satisfies, then the above operator D is unique and

$$T' = TD, \quad R(D) \subseteq N(T),$$

and in this case

$$\|D\|^2 = inf\{\lambda > 0 : T'T'^* \leq \lambda TT^*\},$$

$N(D) = N(T')$ and $R(D) \subseteq \overline{R(T^*)}^{\|\cdot\|_{l_1^2[1,a]}}$, that closure is refer to uniform norm (2) in $l_1^2[1,a]$.

The operator D is called the reduced solution of the equation $T' = TX$.

Proof. *(i) ⇒ (ii)* If $T'T'^* \leq \lambda TT^*$ for some $\lambda > 0$, then for any $f \in L^2(0,\infty)$ and almost all $x \in [1,a]$, $\langle T'T'^*f, f\rangle_a(x) \leq \lambda\langle TT^*f, f\rangle_a(x)$, or

$$\|T'^*f\|_a^2(x) \leq \lambda\|T^*f\|_{l_1^2[1,a]}^2(x),$$

therefore

$$\|T'^*f\|_a(x) \leq \mu\|T^*f\|_{l_1^2[1,a]}(x),$$

where $\mu = \sqrt{\lambda}$.

(ii) ⇒ (iii) Let $\mu > 0$ be a positive number such that for any f in $L^2(0,\infty)$, $\|T'^*f\|_a(x) \leq \mu\|T^*f\|_{l_1^2[1,a]}(x)$, define

$$D' : R(T^*) \to R(T'^*), \quad T^*f \mapsto T'^*f,$$

for any f in $L^2(0,\infty)$. Then by hypothesis, D' is a well defined linear operator. Moreover, for any f in $L^2(0,\infty)$ and almost all $x \in [1,a]$,

$$\|D'(T^*f)\|_a(x) = \|T'^*f\|_a \leq \mu\|T^*f\|_{l_1^2[1,a]}(x),$$

then

$$\|D'(T^*f)\|_2 \leq \mu\|T^*f\|_{l_1^2[1,a]},$$

so we can extend D' linearly and continuously to $\overline{R(T^*)}^{\|\cdot\|_{l_1^2[1,a]}}$, which is denoted still by D' for convenience. Hence we may define the map

$$\hat{D'} : l_1^2[1,a] \to L^2(0,\infty)$$

by

$$\hat{D'}(\{\widetilde{g_j}\}_{j\in\mathbb{Z}}) = \begin{cases} D'\{\widetilde{g_j}\}_{j\in\mathbb{Z}} & if\ \{\widetilde{g_j}\}_{j\in\mathbb{Z}} \in \overline{R(T^*)}^{\|\cdot\|_{l_1^2[1,a]}} \\ 0 & if\ \{\widetilde{g_j}\}_{j\in\mathbb{Z}} \in N(T) \end{cases}$$

and so $\hat{D'}T^* = T'^*$.

Let $P : l_1^2[1,a] \to \overline{R(T^*)}^{\|\cdot\|_{l_1^2[1,a]}}$ be the orthogonal projection onto the uniform norm closure $\overline{R(T^*)}^{\|\cdot\|_{l_1^2[1,a]}}$ of $R(T^*)$. If $\{\widetilde{g_j}\}_{j\in\mathbb{Z}}, \{\widetilde{g_j}'\}_{j\in\mathbb{Z}}$ are in $l_1^2[1,a]$

such that $T(\{\widetilde{g_j}\}_{j\in\mathbb{Z}}) = T(\{\widetilde{g_j}'\}_{j\in\mathbb{Z}})$, then $\left(\{\widetilde{g_j}\}_{j\in\mathbb{Z}} - \{\widetilde{g_j}'\}_{j\in\mathbb{Z}}\right)$ belongs to $N(T) = R(T^*)^\perp$, and $P(\{\widetilde{g_j}\}_{j\in\mathbb{Z}}) = P(\{\widetilde{g_j}'\}_{j\in\mathbb{Z}})$. Now define

$$D : L^2(0,\infty) \to l_1^2[1,a], \quad D = PT^{-1}T',$$

where T^{-1} does not refer to the inverse map of T, since T is not necessarily invertible, but rather its inverse image. By the discussion in the last paragraph, the map D is well defined.

For any f, g in $L^2(0,\infty)$ and almost all x in $[1,a]$ we have

$$\begin{aligned}\langle \hat{D}'(T^*f), g\rangle_a(x) &= \langle T'^*f, g\rangle_a(x)\\ &= \langle f, T'g\rangle_a(x)\\ &= \langle f, TDg\rangle_a(x)\\ &= \langle T^*f, Dg\rangle_{l_1^2[1,a]}(x).\end{aligned}$$

so $\langle \hat{D}'(T^*f), g\rangle = \langle T^*f, Dg\rangle_{l_1^2[1,a]}$. Also, for any $\{\widetilde{g_j}\}_{j\in\mathbb{Z}}$ in $R(T^*)^\perp = N(T)$, we have

$$\langle \hat{D}'\{\widetilde{g_j}\}_{j\in\mathbb{Z}}, g\rangle = 0 = \langle \{\widetilde{g_j}\}_{j\in\mathbb{Z}}, Dg\rangle_{l_1^2[1,a]}.$$

Therefore, applying the orthogonal decomposition

$$l_1^2[1,a] = N(T) \oplus \overline{R(T^*)}^{\|\cdot\|_{l_1^2[1,a]}},$$

we obtain that for arbitrary $\{\widetilde{f_j}\}_{j\in\mathbb{Z}}$ in $l_1^2[1,a]$ and g in $L^2(0,\infty)$,

$$\langle \hat{D}'(\{\widetilde{f_j}\}_{j\in\mathbb{Z}}), g\rangle = \langle \{\widetilde{f_j}\}_{j\in\mathbb{Z}}, Dg\rangle_{l_1^2[1,a]},$$

so $(\hat{D}')^* = D$ and $TD = T'$. Also D is an F_a-factorable operator, since

$$\begin{aligned}\langle \hat{D}'(\phi\{\widetilde{g_j}\}_{j\in\mathbb{Z}}), f\rangle_a(x) &= \langle \{\phi\widetilde{g_j}\}_{j\in\mathbb{Z}}, Df\rangle_{l_1^2[1,a]}(x)\\ &= \phi\langle \{\widetilde{g_j}\}_{j\in\mathbb{Z}}, Df\rangle_{l_1^2[1,a]}(x)\\ &= \phi\langle \hat{D}'(\{\widetilde{g_j}\}_{j\in\mathbb{Z}}), f\rangle_a(x)\\ &= \langle \phi\hat{D}'(\{\widetilde{g_j}\}_{j\in\mathbb{Z}}), f\rangle_a(x),\end{aligned}$$

so $\hat{D}'$ is F_a-factorable and D is F_a-factorable too.

$(iii) \Rightarrow (iv)$ It is clear.

$(iii) \Rightarrow (i)$ If $TX = T'$ has a solution D, then for each $f \in L^2(0,\infty)$,

$$\begin{aligned}\langle T'T'^*f, f\rangle_a(x) &= \langle TDD^*T^*f, f\rangle_a(x)\\ &= \langle DD^*T^*f, T^*f\rangle_{l_1^2[1,a]}(x)\\ &\le \|D\|^2\langle T^*f, T^*f\rangle_{l_1^2[1,a]}(x)\\ &= \|D\|^2\|T^*f\|^2_{l_1^2[1,a]}(x),\end{aligned}$$

we may choose $\lambda = \|D\|^2$.

$(iv) \Rightarrow (ii)$ Since for any f in $L^2(0,\infty)$, $T'f \in R(T)$, there exists $\{\widetilde{f_j}\}_{j\in\mathbb{Z}} \in N(T)^\perp$ such that $T'f = T(\{\widetilde{f_j}\}_{j\in\mathbb{Z}})$. Set $D(f) = \{\widetilde{f_j}\}_{j\in\mathbb{Z}}$ and then $TD(f) = T'f$.

It is easy to know that D is well defined. Indeed if $\{\widetilde{f_j}\}_{j\in\mathbb{Z}}, \{\widetilde{f'_j}\}_{j\in\mathbb{Z}} \in N(T)^\perp$ with $T'f = T\{\widetilde{f_j}\}_{j\in\mathbb{Z}} = T\{\widetilde{f'_j}\}_{j\in\mathbb{Z}}$, then $\{\widetilde{f_j}\}_{j\in\mathbb{Z}} - \{\widetilde{f'_j}\}_{j\in\mathbb{Z}} \in N(T) \cap N(T)^\perp$, so $\{\widetilde{f_j}\}_{j\in\mathbb{Z}} = \{\widetilde{f'_j}\}_{j\in\mathbb{Z}}$.

Moreover, for every $f, g \in L^2(0,\infty)$ and $\alpha, \beta \in \mathbb{C}$, we get

$$\begin{aligned} T'(\alpha f + \beta g) &= \alpha T'(f) + \beta T'(g) \\ &= \alpha TDf + \beta TDg \\ &= T(\alpha Df + \beta Dg). \end{aligned}$$

Since $\alpha Df + \beta Dg \in N(T)^\perp$, by definition of D, we get $D(\alpha f + \beta g) = \alpha Df + \beta Dg$.

Finally we will prove the boundedness of D, as a linear operator from the Hilbert space $L^2(0,\infty)$ to the Hilbert space $l_1^2[1,a]$, by the closed graph Theorem (see for instance [8]). In fact, let

$$\|f_n - f\|_2 \to 0 \quad , \qquad \|Df_n - \{\widetilde{g_j}\}_{j\in\mathbb{Z}}\|_{l_1^2[1,a]} \to 0,$$

for almost all $x \in [1,a]$ as $n \to \infty$. Since T and T' are continuous and $TD(f_n) = T'\{f_n\}$, we have

$$\|TDf_n - T'f\|_2 \to 0, \ \ \textit{and} \ \ \|TDf_n - T\{\widetilde{g_j}\}_{j\in\mathbb{Z}}\|_2 \to 0,$$

as $n \to \infty$, for almost all $x \in [1,a]$.

By the uniqueness of limit, we get $T'f = T\{\widetilde{g_j}\}_{j\in\mathbb{Z}}$ and

$$T(\{\widetilde{g_j}\}_{j\in\mathbb{Z}} - Df) = T(\{\widetilde{g_j}\}_{j\in\mathbb{Z}}) - T'f = 0.$$

Since $\{\widetilde{g_j}\}_{j\in\mathbb{Z}}, Df \in N(T)^\perp$, we have $Df = \{\widetilde{g_j}\}_{j\in\mathbb{Z}}$. Furthermore for almost all $x \in [1,a]$,

$$\begin{aligned} \langle Df, T^*g\rangle_{l_1^2[1,a]}(x) &= \langle TDf, g\rangle_a(x) \\ &= \langle T'f, g\rangle_a(x) \\ &= \langle f, T'^*g\rangle_a(x). \end{aligned}$$

Replacing f by T'^*g, we have $\langle DT'^*g, T^*g\rangle_{l_1^2[1,a]}(x) = \langle T'^*g, T'^*g\rangle_a(x)$ and so

$$\begin{aligned} \|T'^*g\|_a^2(x) &= \langle DT'^*g, T^*g\rangle_{l_1^2[1,a]}(x) \\ &= \sum_{j\in\mathbb{Z}} \pi_j(DT'^*g)\overline{\pi_j(T^*g)}(x) \\ &\le \Big(\sum_{j\in\mathbb{Z}} |\pi_j(DT'^*g)(x)|^2\Big)^{\frac{1}{2}} \Big(\sum_{j\in\mathbb{Z}} |\pi_j(T^*g)(x)|^2\Big)^{\frac{1}{2}} \\ &= \|DT'^*g\|_{l_1^2[1,a]}(x)\|T^*g\|_{l_1^2[1,a]}(x). \end{aligned}$$

Thus $\|T'^*g\|_a(x) \le \|D\|\|T^*g\|_{l_1^2[1,a]}(x)$, for almost all $x \in [1,a]$.

Until now, we have proved the equivalence of (i)-(iv).

If there exists another F_a-factorable operator D_1 from $L^2(0,\infty)$ to $l_1^2[1,a]$ such that $T' = TD_1$ and $R(D_1) = N(T)^{\perp}$, then for any $f \in L^2(0,\infty)$, $TD(f) = TD_1(f) = T'f$ and $T(D-D_1)(f) = 0$. It follows that

$$R(D - D_1) \subseteq N(T) \cap N(T)^{\perp} = \{0\},$$

and so $D = D_1$ and the uniqueness of D is proved. Finally, if D is the reduced solution, for any $\{\widetilde{f}_j\}_{j\in\mathbb{Z}} \in N(T)$ and $g \in L^2(0,\infty)$, we have

$$\langle D^*(\{\widetilde{f}_j\}_{j\in\mathbb{Z}}), g\rangle_a(x) = \langle \{f_j\}_{j\in\mathbb{Z}}, Dg\rangle_{l_1^2[1,a]}(x) = 0, \ a.e.\ x \in [1,a],$$

so $\langle D^*(\{\widetilde{f}_j\}_{j\in\mathbb{Z}}), g\rangle = \int_1^a \langle D^*(\{\widetilde{f}_j\}_{j\in\mathbb{Z}}), g\rangle_a(x)dx = 0$, and $D^*(\{\widetilde{f}_j\}_{j\in\mathbb{Z}}) = 0$, for all $\{\widetilde{f}_j\}_{j\in\mathbb{Z}} \in N(T)$.

Now, for each $\lambda > 0$ with $T'T'^* \leq \lambda TT^*$, since $TD = T'$, we have

$$\|D^*T^*f\|_a^2(x) = \|T'^*f\|_a^2(x) \leq \lambda \|T^*f\|^2_{l_1^2[1,a]}(x),$$

so $\|D^*T^*f\|_2^2 \leq \lambda \|T^*f\|^2_{l_1^2[1,a]}$ for all $f \in L^2(0,\infty)$ and almost all $x \in [1,a]$. This proves that

$$\|D^*(\{\widetilde{f}_j\}_{j\in\mathbb{Z}})\|_2^2 \leq \lambda \|\{\widetilde{f}_j\}_{j\in\mathbb{Z}}\|^2_{l_1^2[1,a]},$$

for all $\{\widetilde{f}_j\}_{j\in\mathbb{Z}} \in N(T)^{\perp} = \overline{R(T^*)}$, or equivalently, for all $\{\widetilde{f}_j\}_{j\in\mathbb{Z}} \in l_1^2[1,a]$. Thus

$$\|D\|^2 = \|D^*\|^2 \leq inf\{\lambda > 0 : T'T'^* \leq \lambda TT^*\}.$$

On the other hand, $T'T'^* = TDD^*T^* \leq \|D\|^2 TT^*$. Thus, we have

$$\|D\|^2 = inf\{\lambda > 0 : T'T'^* \leq \lambda TT^*\}.$$

Since $l_1^2[1,a] = N(T) \oplus \overline{R(T^*)}^{\|\cdot\|_{l_1^2[1,a]}}$ and $R(D) \subseteq N(T)^{\perp}$, we have

$$R(D) \subseteq \overline{R(T^*)}^{\|\cdot\|_{l_1^2[1,a]}}.$$

For the equality $N(T') = N(D)$, if $f \in N(T')$, then by (iii),

$$\begin{aligned}\langle Df, T^*g\rangle_{l_1^2[1,a]} &= \langle TDf, g\rangle \\ &= \langle T'f, g\rangle \\ &= 0,\end{aligned}$$

for all $g \in L^2(0,\infty)$. Hence $Df \in \overline{R(T^*)}^{\perp}$ and the last paragraph proves that $Df = 0$ and so $N(T') \subseteq N(D)$. Since $N(D) \subseteq N(T')$, we have $N(T') = N(D)$. This completes the proof of theorem. □

3. F_a-analysis, F_a-synthesis and F_a-frame operators

According to [3], for a frame $\{f_j\}_{j\in\mathbb{Z}}$ in H, the analysis operator $T : H \to l^2(\mathbb{Z})$ is defined by

$$T(f) = \{\langle f, f_j\rangle\}_{j\in\mathbb{Z}}.$$

Its adjoint $T^* : l^2(\mathbb{Z}) \to H$ is called the synthesis operator and is given by

$$T^*(\{c_j\}_{j\in\mathbb{Z}}) = \sum_{j\in\mathbb{Z}} c_j f_j.$$

The composition $S = T^*T$ is known as the frame operator, which is bounded, positive, self-adjoint, and invertible.

In this section, we begin by recalling the definition of an F_a-frame in $L^2(0,\infty)$ with respect to the function-valued inner product introduced above, as presented in [7]. Subsequently, by introducing the F_a-analysis, F_a-synthesis, and F_a-frame operators, we characterize all F_a-dual frames associated with a given F_a-frame. These results may be regarded as function-valued analogues of the classical dual frame theory in the Hilbert space $L^2(0,\infty)$.

For each $m \in \mathbb{Z}$, define the function ψ_m on $[1,a]$ by

$$\psi_m(x) := \frac{1}{\sqrt{a-1}} e^{2\pi i \frac{m}{a-1}(a-x)}, \quad x \in [1,a].$$

In the following, the notation $\overline{\text{span}}\{\widetilde{\psi}_m\}_{m\in\mathbb{Z}}$ denotes the closed linear span of $\{\widetilde{\psi}_m\}_{m\in\mathbb{Z}}$ in $L^2(0,\infty)$ with respect to the usual norm.

Definition 3.1. [7] A sequence $F = \{f_j\}_{j\in\mathbb{Z}} \subset L^2(0,\infty)$ is said to be an F_a-frame sequence, if there exist constants $0 < A_F \leq B_F < \infty$ such that for all $f \in \overline{\text{span}}\{\widetilde{\psi}_m f_n\}_{m,n\in\mathbb{Z}}$, the following inequality holds:

$$A_F\|f\|_a^2(x) \leq \sum_{j\in\mathbb{Z}} |\langle f, f_j\rangle_a(x)|^2 \leq B_F\|f\|_a^2(x),$$

for almost all $x \in [1,a]$. If only the second inequality holds, we say that F is an F_a-Bessel sequence. The constants A_F and B_F are referred to as the F_a-frame bounds.

For example, if $f \neq 0$ in $L^2(0,\infty)$, then $\{a^{j/2} f(a^j\cdot)\}_{j\in\mathbb{Z}}$ is a Parseval F_a-frame in $L^2(0,\infty)$ [10].

As is well known in the frame theory of Hilbert spaces, the analysis, synthesis, and frame operators play critical roles. To define analogous concepts in the function-valued setting, we require the following sequence space. The sequence space $l_1^2[1,a]$ helps to introduce the mentioned operators associated to F_a-frames. Let $F = \{f_j\}_{j\in\mathbb{Z}}$ be a sequence in $L^2(0,\infty)$. The first operator is the F_a-analysis operator $T_{a,F}$, and it is defined by

$$T_{a,F} : L^2(0,\infty) \to l_1^2[1,a], \qquad T_{a,F}(g) = \{\widetilde{\langle g, f_j\rangle_a}\}_{j\in\mathbb{Z}}.$$

If it is well defined, F_a-synthesis operator is defined by

$$T^*_{a,F}: l^2_1[1,a] \to L^2(0,\infty), \qquad T^*_{a,F}\{\widetilde{g_j}\}_{j\in\mathbb{Z}} = \sum_{j\in\mathbb{Z}} f_j\widetilde{g_j},$$

for $\{\widetilde{g_j}\}_{j\in\mathbb{Z}} \in l^2_1[1,a]$.

Note that the above operators have also been introduced in [11], where the space $l^2_1[1,a]$ is equipped with the uniform norm defined in (2). In that context, the authors showed that F_a-analysis operator $T_{a,F}$ and the F_a-synthesis operator $T^*_{a,F}$ are well defined, bounded and the adjoint of each other. Here, we equipped $l^2_1[1,a]$ to the pointwise norm (1) and by another way, we generalize Theorem 2.1 of [11].

We are now at the stage where we can combine $T_{a,F}$ and $T^*_{a,F}$ to describe the operator $S_{a,F} := T^*_{a,F}T_{a,F}$. Before proceeding, we clarify the notion of F_a-boundedness.

Definition 3.2. [7] A function $f \in L^2(0,\infty)$ is called F_a-bounded, if there exists a positive constant B such that

$$\|f\|_a(x) \le B, \qquad a.e.\ x \in [1,a].$$

The collection of all F_a-bounded functions in $L^2(0,\infty)$ is denoted by $L^\infty_a(0,\infty)$.

As usual, a sequence $\{f_j\}_{j\in\mathbb{Z}}$ in $L^2(0,\infty)$ is said to be an F_a-bounded sequence if there exists a positive constant B such that

$$\|f_j\|_a(x) \le B,$$

for all integers j and for almost all $x \in [1,a]$.

As the frame theory, by a standard technique, one can see that every F_a-frame is F_a-bounded and so by Theorem 2.1(ii), it is norm bounded in $L^2(0,\infty)$. But the converse, is not true, in general. For example, the sequence $\{f_j\}_{j\in\mathbb{Z}}$ defined by

$$f_j(x) = \begin{cases} 2^{\frac{j}{2}}\chi_{[2^{-j},\,2^{-j+1})}(x), & if\ j \ge 0, \\ \chi_{[2/3,\,4/3)}(x) + (2-x)^{\frac{1}{3}}\,\chi_{[4/3,\,2)}(x), & if\ j = -1, \\ 2^{\frac{j+1}{2}}\chi_{[2^{-j-1},\,2^{-j})}(x), & if\ j \le -2. \end{cases}$$

is an F_a-frame, for $a = 2$ and so, it is F_a-bounded and norm bounded; but the subsequence $\{f_j\}_{0\neq j\in\mathbb{Z}}$ is not an F_a-frame (Example 2.1 of [14]).

Theorem 3.3. *Let $F = \{f_j\}_{j\in\mathbb{Z}}$ be an F_a-bounded sequence in $L^2(0,\infty)$. Then*

(i) *F is an F_a-Bessel sequence with bound B_F if and only if $T^*_{a,F}$ is a well defined, bounded operator from $l^2_1[1,a]$ into $L^2(0,\infty)$ and in this case, $\left\|T^*_{a,F}\right\| \le \sqrt{B_F}$.*

(ii) *The F_a-Bessel sequence F is an F_a-frame if and only if $T^*_{a,F}$ is onto.*

Proof. Let $F=\{f_j\}_{j\in\mathbb{Z}}$ be an F_a-Bessel sequence with bound B_F in $L^2(0,\infty)$. Suppose that $\{\widetilde{g_j}\}_{j\in\mathbb{Z}}\in l_1^2[1,a]$. Therefore for each $x\in[1,a]$ and $m,n\in\mathbb{Z}$, with $n>m$ (without loss of generality we may assume that $m>1$), by Lemma 4.7 of [7] we have

$$\begin{aligned}
\left\|\sum_{j=m+1}^{n} f_j\widetilde{g_j}\right\|_a(x) &= sup_{\|g\|_a(x)=1}\left|\left\langle\sum_{j=m+1}^{n} f_j\widetilde{g_j},g\right\rangle_a(x)\right| \\
&= sup_{\|g\|_a(x)=1}\left|\sum_{j=m+1}^{n}\widetilde{g_j}(x)\langle f_j,g\rangle_a(x)\right| \\
&\le sup_{\|g\|_a(x)=1}\sum_{j=m+1}^{n}\left|\widetilde{g_j}(x)\langle f_j,g\rangle_a(x)\right| \\
&\le\left(\sum_{j=m+1}^{n}|\widetilde{g_j}(x)|^2\right)^{\frac{1}{2}} sup_{\|g\|_a(x)=1}\left(\sum_{j=m+1}^{n}\left|\langle f_j,g\rangle_a(x)\right|^2\right)^{\frac{1}{2}} \\
&\le\sqrt{B_F}\left(\sum_{j=m+1}^{n}|g_j(x)|^2\right)^{\frac{1}{2}},
\end{aligned}$$

for all $g\in L^2(0,\infty)$. It follows that the series $\sum_{j=1}^{\infty} f_j\widetilde{g_j}$ is convergent in $(L^2(0,\infty),\|.\|_a)$. Therefore $T^*_{a,F}$ is well defined with $\left\|T^*_{a,F}\right\|\le\sqrt{B_F}$.

If $F=\{f_j\}_{j\in\mathbb{Z}}$ is an F_a-frame in $L^2(0,\infty)$, then by [11], $S_{a,F}$ is onto and so $T^*_{a,F}$ is onto, too.

Conversely, if $T^*_{a,F}$ is an onto operator, then by Lemma 2.2 of [14], we have

$$g=S_{a,F}S^{-1}_{a,F}g=\sum_{j\in\mathbb{Z}}\left\langle\widetilde{S^{-1}_{a,F}g,f_j}\right\rangle_a f_j, \tag{3}$$

for all $g\in L^2(0,\infty)$. So

$$\begin{aligned}
\|g\|_a^2(x) &= \langle g,g\rangle_a(x) \\
&= \left\langle\sum_{j\in\mathbb{Z}}\left\langle\widetilde{S^{-1}_{a,F}g,f_j}\right\rangle_a f_j,g\right\rangle_a(x) \\
&= \sum_{j\in\mathbb{Z}}\left\langle\widetilde{S^{-1}_{a,F}g,f_j}\right\rangle_a\langle f_j,g\rangle_a(x) \\
&\le\left(\sum_{j\in\mathbb{Z}}\left|\left\langle\widetilde{S^{-1}_{a,F}g,f_j}\right\rangle_a(x)\right|^2\right)^{\frac{1}{2}}\left(\sum_{j\in\mathbb{Z}}\left|\langle f_j,g\rangle_a(x)\right|^2\right)^{\frac{1}{2}}
\end{aligned}$$

$$= \left\| T_{a,F}\left(S_{a,F}^{-1} g\right)\right\|_{l_1^2[1,a]}(x)\left(\sum_{j\in\mathbb{Z}} \left|\langle f_j, g\rangle_a(x)\right|^2\right)^{\frac{1}{2}}$$

$$\leq \|T_{a,F}\| \left\|S_{a,F}^{-1}\right\| \|g\|_a(x)\left(\sum_{j\in\mathbb{Z}} \left|\langle f_j, g\rangle_a(x)\right|^2\right)^{\frac{1}{2}},$$

that is,

$$A_F\|g\|_a^2(x) \leq \sum_{j\in\mathbb{Z}} \left|\langle f_j, g\rangle_a(x)\right|^2,$$

where $A_F = \|T_{a,F}\|^{-2}\left\|S_{a,F}^{-1}\right\|^{-2}$. □

If $F = \{f_j\}_{j\in\mathbb{Z}}$ is an F_a-frame with bounds A_F, B_F, then the associated F_a-frame operator

$$S_{a,F}(g) = T_{a,F}^* T_{a,F}(g) = \sum_{j\in\mathbb{Z}} \widetilde{\langle g, f_j\rangle_a} f_j, \quad g \in L^2(0,\infty),$$

is bounded, self adjoint, invertible and F_a-factorable operator. Also, by Lemma 2.6 of [14], $S_{a,F}^{-1}$ is self adjoint and F_a-factorable.

Furthermore, by a standard argument, it is straightforward to see that

$$A_F I_{L^2(0,\infty)} \leq S_{a,F} \leq B_F I_{L^2(0,\infty)},$$

and

$${B_F}^{-1} I_{L^2(0,\infty)} \leq S_{a,F}^{-1} \leq {A_F}^{-1} I_{L^2(0,\infty)},$$

where here and in the sequel, $I_{L^2(0,\infty)}$ is the identity operator on $L^2(0,\infty)$.

For the proof of Theorem 3.4 below, we need the following concept of orthogonality, so called pointwise orthogonality. In fact, two sequences $\{\widetilde{f_j}\}_{j\in\mathbb{Z}}$ and $\{\widetilde{g_j}\}_{j\in\mathbb{Z}}$ in the function-valued inner product space $l_1^2[1,a]$ are pointwise orthogonal (or $\{\widetilde{f_j}\}_{j\in\mathbb{Z}} \perp \{\widetilde{g_j}\}_{j\in\mathbb{Z}}$), if

$$\left\langle \{f_j\}_{j\in\mathbb{Z}}, \{g_j\}_{j\in\mathbb{Z}}\right\rangle_{l_1^2[1,a]}(x) = 0, \quad a.e.$$

Also the pointwise orthogonal complement of a set A in $l_1^2[1,a]$ is defined by

$$A^\perp := \left\{\{\widetilde{g_j}\}_{j\in\mathbb{Z}} \in l_1^2[1,a] : \{\widetilde{f_j}\}_{j\in\mathbb{Z}} \perp \{\widetilde{g_j}\}_{j\in\mathbb{Z}}, \qquad \textit{for all } \{\widetilde{f_j}\}_{j\in\mathbb{Z}} \in A\right\}.$$

Theorem 3.4. *Let $F = \{f_j\}_{j\in\mathbb{Z}}$ be an F_a-frame in $L^2(0,\infty)$. If $f \in L^2(0,\infty)$ has a representation $f = \sum_{j\in\mathbb{Z}} \widetilde{g_j} f_j$ for some $\{\widetilde{g_j}\}_{j\in\mathbb{Z}} \in l_1^2[1,a]$, then*

$$\sum_{j\in\mathbb{Z}} |g_j(x)|^2 = \sum_{j\in\mathbb{Z}} \left|\left\langle f, S_{a,F}^{-1} f_j\right\rangle_a(x)\right|^2 + \sum_{j\in\mathbb{Z}} \left|g_j(x) - \left\langle f, S_{a,F}^{-1} f_j\right\rangle_a(x)\right|^2,$$

for almost all $x \in [1,a]$.

Proof. Suppose that $f = \sum_{j\in\mathbb{Z}} \widetilde{g_j} f_j$ and $\widetilde{h_j} := \widetilde{g_j} - \langle f, \widetilde{S_{a,F}^{-1} f_j}\rangle_a$, for each $j \in \mathbb{Z}$. Then we have

$$T_{a,F}^*(\{\widetilde{g_j}\}_{j\in\mathbb{Z}}) = T_{a,F}^*(\{\widetilde{h_j}\}) + T_{a,F}^*\Big(\Big\{\Big\langle f, \widetilde{S_{a,F}^{-1} f_j}\Big\rangle_a\Big\}_{j\in\mathbb{Z}}\Big),$$

or

$$\sum_{j\in\mathbb{Z}} \widetilde{g_j} f_j = T_{a,F}^*(\{\widetilde{h_j}\}) + \sum_{j\in\mathbb{Z}} \Big\langle f, \widetilde{S_{a,F}^{-1} f_j}\Big\rangle_a f_j.$$

Therefore

$$f = T_{a,F}^*(\{\widetilde{h_j}\}) + f,$$

thus $T_{a,F}^*(\{\widetilde{h_j}\}) = 0$, almost everywhere, and so,

$$\{\widetilde{h_j}\}_{j\in\mathbb{Z}} \in N_{T_{a,F}^*} = (R_{T_{a,F}})^\perp,$$

where $R_{T_{a,F}}$ and $N_{T_{a,F}^*}$ are the range and kernel of $T_{a,F}$ and $T_{a,F}^*$, respectively, and $(R_{T_{a,F}})^\perp$ refers to the pointwise orthogonal complement of $R_{T_{a,F}}$. By the remark before theorem, $S_{a,F}^{-1}$ is a self adjoint operator, so

$$\Big\{\Big\langle f, \widetilde{S_{a,F}^{-1} f_j}\Big\rangle_a\Big\}_{j\in\mathbb{Z}} = \Big\{\Big\langle \widetilde{S_{a,F}^{-1} f, f_j}\Big\rangle_a\Big\}_{j\in\mathbb{Z}} \in R_{T_{a,F}},$$

and we have

$$\begin{aligned}
\|\{\widetilde{g_j}\}_{j\in\mathbb{Z}}\|^2_{l_1^2[1,a]}(x) &= \Big\|\{\widetilde{h_j}\}_{j\in\mathbb{Z}} + \Big\{\Big\langle f, \widetilde{S_{a,F}^{-1} f_j}\Big\rangle_a\Big\}_{j\in\mathbb{Z}}\Big\|^2_{l_1^2[1,a]}(x) \\
&= \|\{\widetilde{h_j}\}_{j\in\mathbb{Z}}\|^2_{l_1^2[1,a]}(x) + \Big\|\Big\{\Big\langle f, \widetilde{S_{a,F}^{-1} f_j}\Big\rangle_a\Big\}_{j\in\mathbb{Z}}\Big\|^2_{l_1^2[1,a]}(x) \\
&= \sum_{j\in\mathbb{Z}} \Big| g_j(x) - \Big\langle f, S_{a,F}^{-1} f_j\Big\rangle_a(x)\Big|^2 + \sum_{j\in\mathbb{Z}} \Big|\Big\langle f, S_{a,F}^{-1} f_j\Big\rangle_a(x)\Big|^2,
\end{aligned}$$

for almost all $x \in [1, a]$, and thus the proof is completed. □

For most of the remainder of this article, we recall the following definition of the F_a-dual of F_a-frames from [9].

Definition 3.5. Suppose that $F = \{f_j\}_{j\in\mathbb{Z}}$ and $G = \{g_j\}_{j\in\mathbb{Z}}$ are two F_a-frames in $L^2(0,\infty)$. The sequence G is called F_a-dual of F_a-frame F if, for each $g \in L^2(0,\infty)$, we have

$$g = \sum_{j\in\mathbb{Z}} \widetilde{\langle g, g_j\rangle_a} f_j, \quad a.e.,$$

or equivalently, $T_{a,F}^* T_{a,G} = I_{L^2(0,\infty)}$.

By relation (3), $\widetilde{F_a} = \Big\{S_{a,F}^{-1}(f_j)\Big\}_{j\in\mathbb{Z}}$ is an F_a-dual of F, which is called the canonical F_a-dual of F.

The following Theorem 3.8 is similar to Theorem 5.7.4 of [3] and uses an almost identical proof. However, we prefer to repeat its proof here for completeness. The first step in characterizing the F_a-dual frames of a given F_a-frame is to establish a one-to-one correspondence between the F_a-duals and the bounded left inverses of $T_{a,F}$.

Note that the sequence $\widetilde{\Delta_j} := \{\widetilde{\delta_{j,k}}\}_{k\in\mathbb{Z}}$, $j \in \mathbb{Z}$, is an orthonormal basis for $l_1^2[1,a]$, where $\delta_{j,k}$ refers to the keronecker delta related to integers j and k.

Lemma 3.6. *Suppose that $F = \{f_j\}_{j\in\mathbb{Z}}$ is an F_a-frame in $L^2(0,\infty)$. Then the F_a-dual frames of F, consist of all sequences in the form*

$$G = \left\{V(\widetilde{\Delta_j})\right\}_{j\in\mathbb{Z}},$$

where V is a bounded left inverse operator of $T_{a,F}$.

Proof. If V is an arbitrary bounded left inverse of $T_{a,F}$, then V is onto. Also for all $g \in L^2(0,\infty)$ we have

$$\begin{aligned} T_{a,F}(g) &= \left\{\widetilde{\langle g, f_j\rangle_a}\right\}_{j\in\mathbb{Z}} \\ &= \sum_{j\in\mathbb{Z}} \widetilde{\langle g, f_j\rangle_a}\widetilde{\Delta_j}. \end{aligned}$$

So, Lemma 2.6 of [14] implies that

$$\begin{aligned} g &= V\, T_{a,F}(g) \\ &= V\left(\sum_{j\in\mathbb{Z}} \widetilde{\langle g, f_j\rangle_a}\ \widetilde{\Delta_j}\right) \\ &= \sum_{j\in\mathbb{Z}} \widetilde{\langle g, f_j\rangle_a}\ V\left(\widetilde{\Delta_j}\right) \\ &= \sum_{j\in\mathbb{Z}} \widetilde{\langle g, f_j\rangle_a}\ g_j, \end{aligned}$$

for all $g \in L^2(0,\infty)$, i.e., $G = \{g_j\}_{j\in\mathbb{Z}}$ is an F_a-dual frame of $\{f_j\}_{j\in\mathbb{Z}}$. For the converse, assume that $\{g_j\}_{j\in\mathbb{Z}}$ is an arbitrary F_a-dual frame of F. Since $T^*_{a,G}T_{a,F} = I_{L^2(0,\infty)}$, the F_a-synthesis operator $T^*_{a,G}$ is a bounded left inverse of $T_{a,F}$, and

$$\{g_j\}_{j\in\mathbb{Z}} = \left\{T^*_{a,G}(\widetilde{\Delta_j})\right\}_{j\in\mathbb{Z}}.$$

This completes the proof. □

Before stating the next theorem, we characterize all bounded left inverses of the operator $T_{a,F}$, in the following lemma.

Lemma 3.7. *Let $F = \{f_j\}_{j\in\mathbb{Z}}$ be an F_a-frame with F_a-synthesis operator $T^*_{a,F}$. Then the bounded left inverses of $T_{a,F}$ are precisely the operators having the form*

$$S^{-1}_{a,F}T^*_{a,F} + W\left(I_{l^2_1[1,a]} - T_{a,F}S^{-1}_{a,F}T^*_{a,F}\right),$$

where $W : l^2_1[1,a] \to L^2(0,\infty)$ is an arbitrary bounded operator.

Proof. At first, we show that each operator

$$L_W := S^{-1}_{a,F}T^*_{a,F} + W\left(I_{l^2_1[1,a]} - T_{a,F}S^{-1}_{a,F}T^*_{a,F}\right),$$

is bounded and is a left inverse of $T_{a,F}$. Since $F = \{f_j\}_{j\in\mathbb{Z}}$ is an F_a-frame, by Theorem 3.3, all operators $T^*_{a,F}$, $S^{-1}_{a,F}$ and $S_{a,F}$ are bounded. So L_W is bounded too, and

$$\begin{aligned} L_W T_{a,F} &= S^{-1}_{a,F}T^*_{a,F}T_{a,F} + WT_{a,F} - WT_{a,F}S^{-1}_{a,F}T^*_{a,F}T_{a,F} \\ &= I_{(0,\infty)} + WT_{a,F} - WT_{a,F} \\ &= I_{(0,\infty)}. \end{aligned}$$

Conversely, if U is a given left inverse of $T_{a,F}$, then we have

$$\begin{aligned} S^{-1}_{a,F}T^*_{a,F} + U\Big(I_{l^2_1[1,a]} - T_{a,F}S^{-1}_{a,F}T^*_{a,F}\Big) &= S^{-1}_{a,F}T^*_{a,F} + U - UT_{a,F}S^{-1}_{a,F}T^*_{a,F} \\ &= S^{-1}_{a,F}T^*_{a,F} + U - S^{-1}_{a,F}T^*_{a,F} \\ &= U. \end{aligned}$$

This completes the proof of lemma. □

Now, we are in position to characterize all F_a-dual frames associated with a given F_a-frame.

Theorem 3.8. *Let $F = \{f_j\}_{j\in\mathbb{Z}}$ be an F_a-frame in $L^2(0,\infty)$. Then the F_a-dual frames of F are precisely the families*

$$\{g_j\}_{j\in\mathbb{Z}} = \left\{ S^{-1}_{a,F}f_j + h_j - \sum_{k\in\mathbb{Z}} \left\langle \widetilde{S^{-1}_{a,F}f_j, f_k} \right\rangle_a h_k \right\}_{j\in\mathbb{Z}},$$

where $\{h_j\}_{j\in\mathbb{Z}}$ is an F_a-Bessel sequence in $L^2(0,\infty)$.

Proof. Since $\{h_j\}_{j\in\mathbb{Z}}$ and $\{f_j\}_{j\in\mathbb{Z}}$ are F_a-bounded sequence, by Proposition 3.9 of [7], $\left\langle \widetilde{S^{-1}_{a,F}f_j, f_k} \right\rangle_a h_k$ is well defined in $L^2(0,\infty)$. By Lemmas 3.6 and 3.7, the F_a-dual frame of F is

$$\{g_j\}_{j\in\mathbb{Z}} = \left\{ S^{-1}_{a,F}T^*_{a,F}(\widetilde{\Delta_j}) + W\Big(I_{l^2_1[1,a]} - T_{a,F}S^{-1}_{a,F}T^*_{a,F}\Big)(\widetilde{\Delta_j}) \right\}_{j\in\mathbb{Z}},$$

where $W : l_1^2[1,a] \to L^2(0,\infty)$ is an operator as the form $W\{\widetilde{k_j}\} = \sum_j \widetilde{k_j} h_j$. Then for each j,

$$\begin{aligned} g_j &= S_{a,F}^{-1} T_{a,F}^* (\widetilde{\Delta_j}) + W\Big(I_{l_1^2[1,a]} - T_{a,F} S_{a,F}^{-1} T_{a,F}^*\Big)(\widetilde{\Delta_j}) \\ &= S_{a,F}^{-1} f_j + h_j - W T_{a,F}(S_{a,F}^{-1} f_j) \\ &= S_{a,F}^{-1} f_j + h_j - W\{\widetilde{\Big\langle S_{a,F}^{-1} f_j, f_k \Big\rangle_a}\}_{k\in\mathbb{Z}} \\ &= S_{a,F}^{-1} f_j + h_j - \sum_{k\in\mathbb{Z}} \widetilde{\Big\langle S_{a,F}^{-1} f_j,\ f_k \Big\rangle_a} h_k, \end{aligned}$$

which completes the proof. □

Theorem 3.9. *Let $F = \{f_j\}_{j\in\mathbb{Z}}$ be an F_a-frame in $L^2(0,\infty)$, then*

(i) *the union of all sets $R(T_{a,F^d})$, is dense in $l_1^2[1,a]$, where $F^d = \{f_j^d\}_{j\in\mathbb{Z}}$ runs through all F_a-dual frames of F.*

(ii) *let G be another F_a-frame in $L^2(0,\infty)$. If every F_a-dual frame of F is an F_a-dual frame of G, then $G = F$.*

Proof. (i) Since the function sequence space $l_1^2[1,a]$ is a Hilbert space with respect to the usual inner product, the union of all sets $R(T_{a,F^d})$ is dense in $l_1^2[1,a]$ if and only if its orthogonal complement equals to zero.

Now, let $\{\widetilde{k_j}\}_{j\in\mathbb{Z}} \perp R(T_{a,F^d})$, for all F_a-dual frames $F^d = \{f_j^d\}_{j\in\mathbb{Z}}$ of F. Then

$$\{\widetilde{k_j}\}_{j\in\mathbb{Z}} \in R(T_{a,F^d})^{\perp} = ker(T_{a,F^d}^*),$$

and so

$$T_{a,F^d}^*(\{\widetilde{k_j}\}_{j\in\mathbb{Z}}) = 0,$$

for all F_a-dual frames $F^d = \{f_j^d\}_{j\in\mathbb{Z}}$ of F. On the other hand, by Theorem 3.8, the F_a-dual frames of F are the sequences

$$\left\{ S_{a,F}^{-1} f_j + h_j - \sum_{k\in\mathbb{Z}} \widetilde{\Big\langle S_{a,F}^{-1} f_j, f_k \Big\rangle_a} h_k \right\}_{j\in\mathbb{Z}},$$

where $\{h_j\}_{j\in\mathbb{Z}}$ is an arbitrary F_a-Bessel sequence in $L^2(0,\infty)$. Hence,

$$\sum_{j\in\mathbb{Z}} \widetilde{k_j} \left(S_{a,F}^{-1} f_j + h_j - \sum_{k\in\mathbb{Z}} \widetilde{\Big\langle S_{a,F}^{-1} f_j, f_k \Big\rangle_a} h_k \right) = 0,$$

and thus

$$\sum_{j\in\mathbb{Z}} \widetilde{k_j} \Big(h_j - \sum_{k\in\mathbb{Z}} \widetilde{\Big\langle S_{a,F}^{-1} f_j, f_k \Big\rangle_a} h_k \Big) = 0.$$

If $\{e_j\}_{j\in\mathbb{Z}}$ is an F_a-orthonormal sequence in $L^2(0,\infty)$ [7], then $\{h_j\}_{j\in\mathbb{Z}} = \{..., 0, e_1, 0, ...\}$ is an F_a-Bessel sequence in $L^2(0,\infty)$. Thus

$$\widetilde{k_1} e_1 - \sum_{j\in\mathbb{Z}} \widetilde{k_j} \left\langle \widetilde{S_{a,F}^{-1} f_j, f_1} \right\rangle_a e_1 = 0.$$

The identity $\sum_{j\in\mathbb{Z}} \widetilde{k_j}\left(S_{a,F}^{-1} f_j\right) = 0$, implies that $\widetilde{k_1} e_1 = 0$, and so $\widetilde{k_1} = 0$. Similarly, one can deduce that for each $j \geq 2$, $\widetilde{k_j} = 0$, and so $(\overline{\cup R(T_{a,F^d})})^{\perp} = \{0\}$, which completes the proof.

(ii) Assume that all F_a-dual frames F^d of F are F_a-dual frames of G. Then

$$T_{a,F}^* T_{a,F^d} = I_{L^2(0,\infty)} = T_{a,G}^* T_{a,F^d}.$$

By (i), $T_{a,F}^* = T_{a,G}^*$ and then $F = G$. □

Corollary 3.10. *Let $F = \{f_j\}_{j\in\mathbb{Z}}$ be an F_a-frame in $L^2(0,\infty)$, then the pointwise orthogonal complement of the union of $R(T_{a,F^d})$ equals to zero, where $F^d = \{f_j^d\}_{j\in\mathbb{Z}}$ runs through all F_a-dual frames of F.*

Proof. Let $\{\widetilde{k_j}\}_{j\in\mathbb{Z}} \perp R(T_{a,F^d})$ for all F_a-dual frame $F^d = \{f_j^d\}_{j\in\mathbb{Z}}$ of F, with respect to the pointwise inner product of $l_1^2[1,a]$. Then for all $f \in L^2(0,\infty)$,

$$\left\langle \{\widetilde{k_j}\}_{j\in\mathbb{Z}}, T_{a,F^d}(f) \right\rangle_{l_1^2[1,a]} = \int_1^a \left\langle \{\widetilde{k_j}\}_{j\in\mathbb{Z}}, T_{a,F^d}(f) \right\rangle_{l_1^2[1,a]} (x) dx = 0.$$

Hence $\{\widetilde{k_j}\}_{j\in\mathbb{Z}}$ belongs to the uniform orthogonal complement of the union of all $R(T_{a,F^d})$, and Theorem 3.9(i), implies that $\{\widetilde{k_j}\}_{j\in\mathbb{Z}} = 0$. This completes the proof. □

Theorem 3.11. *Let $F = \{f_j\}_{j\in\mathbb{Z}}$ be an F_a-frame for $L^2(0,\infty)$. Then for any F_a-dual frame $G = \{g_j\}_{j\in\mathbb{Z}}$ of F we have*

$$\|T_{a,G}\|^2 \geq \frac{1}{A_F^{opt}},$$

where A_F^{opt} is the optimal lower F_a-frame bound of F. Also $\widetilde{F_a}$ is the unique F_a-dual frame of F for which

$$\|T_{a,\widetilde{F_a}}\|^2 = \frac{1}{A_F^{opt}}.$$

Proof. By the definition of F_a-frame

$$\begin{aligned}\|f\|_a^2(x) &\leq \frac{1}{A_F} \sum_{j\in\mathbb{Z}} |\langle f, f_j \rangle_a (x)|^2 \\ &= \frac{1}{A_F} \|T_{a,F} f\|_{l_1^2[1,a]}^2 (x), \quad a.e. \quad x \in [1,a].\end{aligned}$$

It follows that

$$\frac{1}{A_F^{opt}} = inf\Big\{\gamma : \|f\|_a^2(x) \le \gamma \|T_{a,F}f\|^2_{l_1^2[1,a]}(x),\ \forall f \in L^2(0,\infty),\ a.e.\ x \in [1,a]\Big\}.$$

Since

$$\begin{aligned}\|f\|_2^2 &= \Big\|T^*_{a,G}T_{a,F}f\Big\|_2^2\\ &\le \|T^*_{a,G}\|^2\|T_{a,F}f\|^2_{l_1^2[1,a]},\end{aligned}$$

we have

$$\|f\|_2^2 \le \|T_{a,G}\|^2\|T_{a,F}f\|^2_{l_1^2[1,a]}.$$

Also $\|f\|_2^2 = \int_1^a \|f\|_a^2(x)dx$, and

$$\|T_{a,F}f\|^2_{l_1^2[1,a]} = \int_1^a \|T_{a,F}f\|^2_{l_1^2[1,a]}(x)dx,$$

then

$$\begin{aligned}\int_1^a \|f\|_a^2(x)dx &\le \|T_{a,G}\|^2 \int_1^a \|T_{a,F}f\|^2_{,l_1^2[1,a]}(x)dx\\ &= \|T_{a,G}\|^2 \int_1^a \sum_{j\in\mathbb{Z}} |\left\langle f, f_j\right\rangle_a(x)|^2 dx. \qquad (4)\end{aligned}$$

Since F is an F_a-frame and $\int_1^a \|f\|_a^2(x)dx < \infty$, almost every point in $[1,a]$ is a Lebesgue point of $\|f\|_a(.)$ and $\sum_{j\in\mathbb{Z}} |\left\langle f, f_j\right\rangle_a(.)|^2$. Fix an arbitrary such Lebesgue point x_0. Replace f in (4) by $\frac{1}{2\epsilon} f\chi_{\cup_{j\in\mathbb{Z}} a^j(x_0-\epsilon, x_0+\epsilon)}$ with positive ϵ such that $(x_0-\epsilon, x_0+\epsilon) \subset [1,a]$, then we have

$$\frac{1}{2\epsilon}\int_{x_0-\epsilon}^{x_0+\epsilon} \|f\|_a^2(x)dx \le \frac{\|T_{a,G}\|^2}{2\epsilon}\int_{x_0-\epsilon}^{x_0+\epsilon} \sum_{j\in\mathbb{Z}} |\left\langle f, f_j\right\rangle_a(x)|^2 dx,$$

and thus

$$\|f\|_a^2(x_0) \le \|T_{a,G}\|^2 \sum_{j\in\mathbb{Z}} |\left\langle f, f_j\right\rangle_a(x_0)|^2,$$

by letting $\epsilon \to 0^+$.

Since x_0 is arbitrary, we have

$$\|f\|_a^2(.) \le \|T_{a,G}\|^2 \sum_{j\in\mathbb{Z}} |\left\langle f, f_j\right\rangle_a(.)|^2 \quad a.e.\ x \in [1,a].$$

So by definition of A_F^{opt}, we can see

$$\|T_{a,G}\|^2 \ge \frac{1}{A_F^{opt}}.$$

In order to prove the second part of theorem, we first make use of Theorem 2.5 for surjective operators $T^*_{a,F}$ and $I_{L^2(0,\infty)}$ and find that there exist a unique operator $R : L^2(0,\infty) \longrightarrow l_1^2[1,a]$ of minimal norm for which $I_{L^2(0,\infty)} = T^*_{a,F}R$. Also, we have

$$\|R\|^2 = inf\Big\{\delta > 0 : \|f\|_a^2(x) \leq \delta \|T_{a,F}f\|^2_{l_1^2[1,a]}(x),\ \forall f \in L^2(0,\infty),\ a.e.\Big\}$$

On the other hand, an argument similar to the proof of Theorem 3.4 shows that if $f = \sum_{j\in\mathbb{Z}} \widetilde{g_j} f_j$, for some $\{\widetilde{g_j}\}_{j\in\mathbb{Z}}$, we have

$$\|\{\widetilde{g_j}\}_{j\in\mathbb{Z}}\|^2_{l_1^2[1,a]}(x) = \sum_{j\in\mathbb{Z}} \Big| g_j(x) - \Big\langle f, S^{-1}_{a,F} f_j \Big\rangle_a (x) \Big|^2 + \sum_{j\in\mathbb{Z}} \Big| \Big\langle f, S^{-1}_{a,F} f_j \Big\rangle_a (x) \Big|^2.$$

Also

$$\sum_{j\in\mathbb{Z}} \Big| \Big\langle f, S^{-1}_{a,F} f_j \Big\rangle_a (x) \Big|^2 = \Big\| T_{a,\widetilde{F_a}}(f) \Big\|^2_{l_1^2[1,a]}(x).$$

By replacing f with $T^*_{a,F}Rf$ one can see

$$\sum_{j\in\mathbb{Z}} \Big| \Big\langle f, S^{-1}_{a,F} f_j \Big\rangle_a (x) \Big|^2 = \Big\| T_{a,\widetilde{F_a}}(T^*_{a,F}Rf) \Big\|^2_{l_1^2[1,a]}(x) = \|Rf\|^2_{l_1^2[1,a]}(x).$$

Hence

$$\|Rf\|^2_{l_1^2[1,a]}(x) = \sum_{j\in\mathbb{Z}} \Big| \pi_j(Rf)(x) - \Big\langle f, S^{-1}_{a,F} f_j \Big\rangle_a (x) \Big|^2 + \|Rf\|^2_{l_1^2[1,a]}(x).$$

Then

$$\sum_{j\in\mathbb{Z}} \Big| \pi_j(Rf)(x) - \Big\langle f, S^{-1}_{a,F} f_j \Big\rangle_a (x) \Big|^2 = 0,$$

and so $R = T_{a,\widetilde{F_a}}$. Finally,

$$\begin{aligned}\|T_{a,\widetilde{F_a}}\|^2 &= inf\Big\{\delta > 0 : \|f\|_a^2(x) \leq \delta \|T_{a,F}f\|^2_{l_1^2[1,a]}(x), \forall f \in L^2(0,\infty),\ a.e.\Big\} \\ &= \frac{1}{A_F^{opt}}.\end{aligned}$$

This completes the proof of theorem. □

Theorem 3.12. *Let F be an F_a-frame with optimal upper F_a-frame bound B_F^{opt}. Then the following statements hold:*

(i) $B_F^{opt} = \|S_{a,F}\| = \|T_{a,F}\|^2$,

(ii) *If G is an F_a-dual frame of F and $\|S_{a,G}\| = B_F^{opt}$, then $G = \widetilde{F_a}$.*

Proof. (i) By definition, the optimal upper frame bound is

$$\begin{aligned}B_F^{opt} &= sup_{\|f\|_a(x)=1} \sum_{j\in\mathbb{Z}} |\langle f, f_j\rangle_a (x)|^2 \\ &= sup_{\|f\|_a(x)=1} \langle S_{a,F}f, f\rangle_a (x) \\ &= \|S_{a,F}\|.\end{aligned}$$

Also by definition, $\|S_{a,F}\| = \|T_{a,F}T^*_{a,F}\| = \|T_{a,F}\|^2$.

(ii) By Theorem 3.8, every F_a-dual frame of F is the form

$$G=\{g_j\}_{j\in\mathbb{Z}}=\left\{S_{a,F}^{-1}f_j+W(I_{l_1^2[1,a]}-T_{a,F}S_{a,F}^{-1}T_{a,F}^*)(\widetilde{\Delta_j})\right\}_{j\in\mathbb{Z}}.$$

Put $V=W\left(I_{l_1^2[1,a]}-T_{a,F}S_{a,F}^{-1}T_{a,F}^*\right)$, then

$$\begin{aligned}G&=\left\{S_{a,F}^{-1}f_j+V(\widetilde{\Delta_j})\right\}_{j\in\mathbb{Z}}\\&=\left\{\{\widetilde{F_a}\}_j+V(\widetilde{\Delta_j})\right\}_{j\in\mathbb{Z}}.\end{aligned}\tag{5}$$

Since V is an F_a-factorable operator, we have

$$\begin{aligned}T_{a,G}^*\{\widetilde{h_j}\}_{j\in\mathbb{Z}}&=\sum_{j\in\mathbb{Z}}\widetilde{h_j}g_j\\&=\sum_{j\in\mathbb{Z}}\widetilde{h_j}\left(\{\widetilde{F_a}\}_j+V(\widetilde{\Delta_j})\right)\\&=T_{a,\widetilde{F_a}}^*(\{\widetilde{h_j}\}_{j\in\mathbb{Z}})+V(\{\widetilde{h_j}\}_{j\in\mathbb{Z}}).\end{aligned}$$

That is,

$$T_{a,G}=T_{a,\widetilde{F_a}}+V^*,$$

and

$$\begin{aligned}T_{a,\widetilde{F_a}}^*V^*&=T_{a,\widetilde{F_a}}^*\left(I_{l_1^2[1,a]}-T_{a,F}S_{a,F}^{-1}T_{a,F}^*\right)W^*\\&=\left(T_{a,\widetilde{F_a}}^*-T_{a,\widetilde{F_a}}^*T_{a,F}S_{a,F}^{-1}T_{a,F}^*\right)W^*\\&=0.\end{aligned}$$

Therefore $VT_{a,\widetilde{F_a}}=0$, and so

$$\begin{aligned}S_{a,G}&=T_{a,G}^*T_{a,G}\\&=(T_{a,\widetilde{F_a}}^*+V)(T_{a,\widetilde{F_a}}+V^*)\\&=T_{a,\widetilde{F_a}}^*T_{a,\widetilde{F_a}}+T_{a,\widetilde{F_a}}^*V^*+VT_{a,\widetilde{F_a}}+VV^*\\&=S_{a,\widetilde{F_a}}+VV^*.\end{aligned}$$

This, together with the positivity of the operators $S_{a,G}$, $S_{a,\widetilde{F_a}}$ and VV^* implies that

$$\begin{aligned}\|S_{a,G}\|&=sup_{\|f\|_a(x)=1}\left\langle S_{a,G}f,f\right\rangle_a(x)\\&=sup_{\|f\|_a(x)=1}\left\langle S_{a,\widetilde{F_a}}f+VV^*f,f\right\rangle_a(x)\\&=sup_{\|f\|_a(x)=1}\left\langle S_{a,\widetilde{F_a}}f,f\right\rangle_a(x)+sup_{\|f\|_a(x)=1}\left\langle VV^*f,f\right\rangle_a(x)\\&=\|S_{a,\widetilde{F_a}}\|+\|VV^*\|.\end{aligned}$$

By hypothesis, $\|S_{a,G}\|=\|S_{a,\widetilde{F_a}}\|$, so $VV^*=0$, and by (5), $G=\widetilde{F_a}$. □

Conclusion

In this work, we established a new version of the Riesz Representation Theorem for F_a-factorable operators. We also investigated conditions for the existence and uniqueness of solutions to the equation $T' = TX$ within this class of operators. Additionally, we provided a complete characterization of all F_a-dual frames corresponding to a given F_a-frame and derived the relationship between the optimal bounds of an F_a-frame and the operators associated with its F_a-duals.

As a continuation of this research, we intend to further investigate the structure of F_a-duals and present explicit examples illustrating how to determine the F_a-duals of a given F_a-frame. We also aim to explore potential applications of this theory in functional analysis and signal processing.

This line of research is not only well-motivated but also essential for achieving a deeper understanding of F_a-frames and their duals. The distinctions highlighted in this study, together with the interplay between scalar-valued and function-valued inner products, pave the way for future developments in duality theory within this framework.

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Authors' contributions

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