Journal of Mahani Mathematical Research
Shahid Bahonar University of Kerman
Print ISSN: 2251-7952 Online ISSN: 2645-4505

RIESZ DUNFORD INTEGRAL AND OPERATORS ON HILBERT SPACE BASED ON UNIVALENT FUNCTIONS WITH FIXED RESIDUE

L. Nobakht, Sh. Najafzadeh, and M.R. Foroutan ✉

Article type: Research Article
(Received: 04 June 2025, Received in revised form 15 January 2026)
(Accepted: 02 February 2026, Published Online: 02 February 2026)

Abstract. In the present paper, we introduce and investigate a new class of meromorphic functions analytic in the open unit disk and applying a $q-$derivative and $q-$differential integral operator associated with quantum calculus. Furthermore, by using the familiar Riesz-Dunford integral of a linear operator on Hilbert space H, a new class of univalent functions with a fixed point is introduced. Coefficient estimate, distortion bound and extreme points are obtained.

Keywords: Hilbert space, meromorphic function, $q-$derivative, Coefficient estimate, Distortion bound and extreme points.
2020 MSC: 30C45, 30C50.

1. Introduction and Preliminaries

The study of univalent functions is one of the leading branch of geometric function theory and quantum calculus. One of the fundamental problems in this theory is coefficient estimate of such functions. After solving this problem, we may find many interesting geometric properties. The $q-$differential integral operator on Hilbert space related to meromorphic functions were studied by many researchers. Also the $q-$derivative is introduced in 1909. In this paper , we define a new subclass of meromorphic functions associated with q-derivative. Furthermore, operators on Hilbert space are considered and some geometric structures are investigated.

Let ω be a fixed point and τ_ω denote the class of functions of the form

$$f(z) = \frac{A}{z-\omega} + \sum_{n=1}^{+\infty} a_n (z-\omega)^n, \tag{1}$$

✉ mr_forootan@pnu.ac.ir, ORCID: 0000-0002-7373-617X
https://doi.org/10.22103/jmmr.2026.25383.1812
Publisher: Shahid Bahonar University of Kerman
How to cite: L. Nobakht, Sh. Najafzadeh, M.R. Foroutan, *Riesz Dunford integral and operators on Hilbert space based on univalent functions with fixed residue*, J. Mahani Math. Res. 2026; 15(2): 209-217.

where $0 \leq A \leq 1$ is the Residue of $f(z)$ in $z = \omega$. Also, $\mathcal{N}_\omega$ is the subclass of τ_ω consisting of functions with negative coefficients of the type

$$f(z) = \frac{A}{z-\omega} - \sum_{n=1}^{+\infty} a_n (z-\omega)^n, \qquad a_n \geq 0. \tag{2}$$

In the theory of Geometric functions the concept of q-derivative is an important tool. The q-derivative operator arises in the study of q-series and q-hypergeometric functions and are used in quantum mechanics, combinatorics and special functions. This operator was first defined by Jackson [5]. See also [3].
Defined the $q-$derivative for meromorphic functions $f(z) \in \mathcal{N}_\omega$ by

$$\partial_q f(z) = \frac{f(z) - f(qz)}{z(1-q)} = -\frac{1}{qz^2} + \sum_{n=1}^{+\infty} [n]_q a_n z^{n-1}, \tag{3}$$

where

$$[n]_q = \frac{1-q^n}{1-q}.$$

When $q \to 1^-$, $[n]_q = n$ and $\partial_q f(z) = f'(z)$.
See also [5]. For $f(z) \in \mathcal{N}_\omega$ Saleh and Mustafa [7] introduced the operator $\mathcal{M}_q^k$ as follows:

$$\begin{aligned} \mathcal{M}_q^0 f(z) &= f(z), \\ \mathcal{M}_q^1 f(z) &= z\partial_q f(z) + \frac{A\left((q+1)z-\omega\right)}{(z-\omega)(qz-\omega)}, \\ \mathcal{M}_q^2 f(z) &= z\partial_q \left(\mathcal{M}_q^1 f(z)\right), \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}_q^k f(z) &= z\partial_q \left(\mathcal{M}_q^{k-1} f(z)\right) + \frac{A((q+1)z-\omega)}{(z-\omega)(qz-\omega)} \\ &= \frac{A}{z-\omega} - \sum_{n=1}^{+\infty} [n]_q^k a_n z^n. \end{aligned} \tag{4}$$

For achieving to the geometric properties based on univalent functions and operators on Hilbert spaces, we need to define a new subclass. For achieving to geometric properties specially the coefficient bound we need to define a new subclass of univalent functions based of operators on Hilbert space and Riesz-Dunford integral, so we have the following definition:

Definition 1.1. The function $f \in \mathcal{N}_\omega$ is said to be a member of the class $\mathcal{N}_\omega^k(\alpha, \beta, \gamma)$ if it satisfies

$$\left| \frac{(z-\omega)^4 \left(\mathcal{M}_q^k(f(z))\right)''' + (z-\omega)^3 \left(\mathcal{M}_q^k f(z)\right)'' + (z-\omega)^2 \left(\mathcal{M}_q^k f(z)\right)' + 5A}{2(z-\omega)\left(\mathcal{M}_q^k f(z)\right) - \alpha(1+\gamma)A} \right| < \beta,$$

where $\alpha, \beta, \gamma \in [0, 1)$.

The Riesz-Dunford integral [1] is defined by

$$2\pi i f(T) = \int_c f(z)(zI - T)^{-1} dz, \tag{5}$$

where T is a linear operator on Hilbert space H, c is a positively oriented simple rectifiable cantor baying in $\Delta = \{z \in \mathbb{C} : \ |z| < 1\}$ and containing the spectrum of T in its interior domain. Also, I is the identity operator on H, see [1].

Definition 1.2. A function $f(z)$ given by (2) is in the class $\mathcal{N}_\omega^k(\alpha,\beta,\gamma,T)$ if for all operator T such that $\|T\| < 1$ and $T \neq 0$, it satisfies the following inequality:

$$\begin{aligned} &\left\| T^4 \left(\mathcal{M}_q^k f(T)\right)''' + T^3 \left(\mathcal{M}_q^k f(T)\right)'' + T^2 \left(\mathcal{M}_q^k f(T)\right)' - 5A \right\| \\ &< \beta \left\| -2T \left(\mathcal{M}_q^k f(T)\right) - \alpha(1+\gamma)A \right\|, \end{aligned} \tag{6}$$

where $\alpha, \beta, \gamma \in [0, 1)$.

For more details about the operators on Hilbert space see [2, 4, 6].

2. Coefficient estimates and distortion bounds

In this section, we investigate about the coefficient estimates and distortion bounds for functions in the class $\mathcal{N}_\omega^k(\alpha,\beta,\gamma,T)$.

Theorem 2.1. *A function $f(z)$ given by (2) is in the class $\mathcal{N}_\omega^k(\alpha,\beta,\gamma,T)$ for all $T \neq 0$, if and only if*

$$\sum_{n=1}^{\infty} \frac{n(n^2+2n-2)+2\beta}{\beta A(2-\alpha(1+\gamma))} [n]_q^k a_n \leq 1. \tag{7}$$

The result is sharp for the function $G(z)$ given by

$$G(z) = \frac{A}{z-\omega} - \frac{\beta A(2-\alpha(1+\alpha))}{n(n^2-2n+2)+2\beta} [n]_q^k (z-\omega)^n, \tag{8}$$

where $n \geq 1$.

Proof. Suppose that (7) holds. By replacing $M_q^k f(z)$ and its derivatives in (6), we have

$$\begin{aligned}
&\left\|T^4\left(M_q^k f(T)\right)''' + T^3\left(M_q^k f(T)\right)'' + T^2\left(M_q^k f(T)\right)' - 5A\right\| \\
&-\beta\left\|2T\left(M_q^k f(T)\right) - \alpha(1+\gamma)A\right\| \\
= &\left\|-\sum_{n=1}^{+\infty} n(n^2-2n+2)[n]_q^k a_n T^{n+1}\right\| \left\|T^4 f'''(T) + T^3 f''(T) + T^2 f'(T) + 5A\right\| \\
&-\beta\left\|2Tf(T) - \alpha(1+\gamma)A\right\| \\
= &\left\|-\sum_{n=1}^{+\infty} n(n^2-2n+2)[n]_q^k a_n T^{n+1}\right\| - \beta\left\|A(2-\alpha(1+\gamma)) - \sum_{n=1}^{+\infty} 2[n]_q^k a_n T^{n+1}\right\| \\
\leq &\sum_{n=1}^{+\infty}\left(n(n^2-2n+2)+2\beta\right)a_n - \beta A\left(2-\alpha(1+\gamma)\right) \leq 0.
\end{aligned}$$

Hence $f \in \mathcal{N}_\omega^k(\alpha,\beta,\gamma,T)$.
Conversely, suppose that

$$\left\|T^4 f'''(T) + T^3 f''(T) + T^2 f'(T) + 5A\right\| \leq \beta\left\|2Tf(T) - \alpha(1+\gamma)A\right\|,$$

so

$$\left\|-\sum_{n=1}^{+\infty} n(n^2-2n+2)[n]_q^k a_n T^{n+1}\right\| < \beta\left\|A\left(2-\alpha(1+\gamma)\right) - \sum_{n=1}^{+\infty} 2[n]_q^k a_n T^{n+1}\right\|.$$

Putting $T = qI$ $(0 < q < 1)$ in the above inequality, we get

$$\sum_{n=1}^{+\infty} \frac{n(n^2-2n+2)a_n q^{n+1}[n]_q^k}{A(2-\alpha(1+\gamma)) - \sum\limits_{n=1}^{+\infty} 2[n]_q^k a_n q^{n+1}} < \beta. \tag{9}$$

Upon clearing denominator (9) and letting $q \to 1$, we conclude that

$$\sum_{n=1}^{+\infty}\left(n(n^2-2n+2)+2\beta\right)[n]_q^k a_n \leq A\beta\left(2-\alpha(1+\gamma)\right),$$

which completes the proof. □

Corollary 2.2. *If $f(z) \in \mathcal{N}_\omega^k(\alpha,\beta,\gamma,T)$, then*

$$a_n \leq \frac{A\beta(2-\alpha(1+\gamma))}{[n]_q^k\left(n(n^2-2n+2)+2\beta\right)}, \qquad n \in \mathbb{N}. \tag{10}$$

Example 2.3. *Let $f(z) = z(1-z)^{-1}$ and $g(z) = z(1+iz)^{-1}$ belong to the class of normalized univalent functions. Then $(f+g)'(\frac{1}{2}(1+i)) = 0$, so $(f+g)(z)$ is not in the same class. But for meromorphic functions we have the following theorem.*

Theorem 2.4. *If $f(z) \in \mathcal{N}_\omega^k(\alpha, \beta, \gamma, T)$, $\|T\| < 1$ and $\|T\| \neq 0$, then*

$$\left\|\frac{A}{T}\right\| - \frac{A\beta\,(2-\alpha(1+\gamma))}{n(n^2-2n+2)+2\beta}\|T\|^n \leq \|f(T)\| \leq \left\|\frac{A}{T}\right\| + \frac{A\beta(2-\alpha(1+\gamma))}{n(n^2-2n+2)+2\beta}\|T\|^n.$$

The result is sharp for the function $G(z)$ given by

$$G(z) = \frac{A}{z-\omega} - \frac{\beta A(2-\alpha(1+\alpha))}{n(n^2-2n+2)+2\beta}[n]_q^k(z-\omega)^n.$$

Proof. According to Theorem 2.1, we get

$$\sum_{n=1}^{+\infty} a_n \leq \frac{A\beta(2-\alpha(1+\gamma))}{[n]_q^k\,(n(n^2-2n+2)+2\beta)}, \tag{11}$$

and by using the definition of $f(z)$ in (2), we get

$$\begin{aligned} \|f(T)\| &\geq \left\|\frac{A}{T}\right\| - \|T\|^n \sum_{n=1}^{+\infty} a_n \\ &\geq \left\|\frac{A}{T}\right\| - \frac{A\beta(2-\alpha(1+\gamma))}{[n]_q^k\,(n(n^2-2n+2)+2\beta)}\|T\|^n, \end{aligned}$$

and also

$$\begin{aligned} \|f(T)\| &\leq \left\|\frac{A}{T}\right\| + \|T\|^2 \sum_{n=1}^{+\infty} a_n \\ &\leq \left\|\frac{A}{T}\right\| + \frac{A\beta(2-\alpha(1+\gamma))}{[n]_q^k\,(n(n^2-2n+2)+2\beta)}. \end{aligned}$$

Hence, we get the required result. □

3. Extreme points and convolution structure

In this section, we discuss about extreme points of $\mathcal{N}_\omega^k(\alpha, \beta, \gamma, T)$. Also, the preserving of this class under an operator is investigated.

Theorem 3.1. *Let $f_0(z) = \dfrac{A}{z-\omega}$ and*

$$f_n(z) = \frac{A}{z-\omega} - \frac{A\beta(2-\alpha(1+\gamma))}{[n]_q^k\,(n(n^2-2n+2)+2\beta)}(z-\omega)^n, \qquad n \geq 1.$$

Then $f(z) \in \mathcal{N}_\omega^k(\alpha, \beta, \gamma, T)$ if and only if it can be expressed by

$$f(z) = \sum_{n=0}^{+\infty} p_n f_n(z),$$

where $p_n \geq 0$ and $\sum_{n=0}^{\infty} p_n = 1$.

Proof. Let

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} p_n f_n(z) \\ &= \frac{A}{z-\omega} + \sum_{n=1}^{+\infty} p_n \frac{A\beta(2-\alpha(1+\gamma))}{[n]_q^k\,(n(n^2-2n+2)+2\beta)}(z-\omega)^n. \end{aligned}$$

Since

$$\sum_{n=1}^{+\infty} \frac{[n]_q^k\left(n(n^2-2n+2)+2\beta\right)}{A\beta\,(2-\alpha(1+\gamma))} \times \frac{A\beta\,(2-\alpha(1+\gamma))}{[n]_q^k\,(n(n^2-2n+2)+2\beta)} p_n$$
$$= \sum_{n=1}^{+\infty} p_n = 1 - p_n \leq 1,$$

so, by Theorem 2.1, we get $f(z) \in \mathcal{N}_\omega^k(\alpha,\beta,\gamma,T)$.
Conversely, suppose that $f(z) \in \mathcal{N}_\omega^k(\alpha,\beta,\gamma,T)$. Then by (10), we have

$$a_n \leq \frac{A\beta\,(2-\alpha(1+\gamma))}{[n]_q^k\,(n(n^2-2n+2)+2\beta)}.$$

Setting

$$p_n = \frac{[n]_q^k\left(n(n^2-2n+2)+2\beta\right)}{A\beta\,(2-\alpha(1+\gamma))} a_n,$$

and $p_0 = 1 - \sum_{n=1}^{+\infty} p_n$, we get the required result. □

Theorem 3.2. *If $f(z) \in \mathcal{N}_\omega^k(\alpha,\beta,\gamma,T)$, then the function*

$$\mathcal{L}_t\left(f(z)\right) = t\int_0^1 \left[\theta^t f\left(z\theta + \omega(1-\theta)\right) d\theta\right], \qquad t \geq 1$$

is also in the same class.

Proof. Since $f(z) \in \mathcal{N}_\omega^k(\alpha,\beta,\gamma,T)$, so $f(z)$ is of the form (2). Hence

$$\begin{aligned} \mathcal{L}_t\left(f(t)\right) &= t\int_0^1 \left\{\theta^t \left[\frac{A}{\theta(z-\omega)} - \sum_{n=1}^{+\infty} a_n \left(\theta(z-\omega)\right)^n\right]\right\} d\theta \\ &= \frac{A}{z-\omega} - \sum_{n=1}^{+\infty} \frac{t}{t+n+1} a_n (z-\omega)^n. \end{aligned}$$

Since $\dfrac{t}{t+n+1} < 1$, so by Theorem 2.1, we conclude that

$$\mathcal{L}_t\left(f(z)\right) \in \mathcal{N}_\omega^k(\alpha,\beta,\gamma,T).$$

□

Theorem 3.3. *Let*

$$f(z) = \frac{A}{z-\omega} + \sum_{n=1}^{+\infty} a_n (z-\omega)^n$$

and

$$g(z) = \frac{A}{z-\omega} - \sum_{n=1}^{+\infty} b_n (z-\omega)^n$$

be in the class $\mathcal{N}_\omega^k(\alpha, \beta, \gamma, T)$*. Then* $(f * g)(z)$ *defined by*

$$(f * g)(z) = \frac{A}{z-\omega} - \sum_{n=1}^{+\infty} a_n b_n (z-\omega)^n,$$

belongs to $\mathcal{N}_\omega^k(\alpha, \beta, \gamma^*, T)$*, where*

$$\gamma^* \leq \frac{2}{\alpha} - \{1 + \frac{\beta A (2-\alpha(1+\gamma))^2}{\alpha(n(n^2+2n-2)+2\beta)[n]_q^k}\}.$$

Proof. By Theorem 2.1, it is sufficient to show that

$$\sum_{n=1}^{+\infty} \frac{n(n^2+2n-2)+2\beta}{\beta A\left(2-\alpha(1+\gamma^*)\right)} [n]_q^k a_n b_n \leq 1.$$

By using the Cauchy-Schwarz inequality from equation (7), we obtain

$$\sum_{n=1}^{+\infty} \frac{n(n^2+2n-2)+2\beta}{\beta A\left(2-\alpha(1+\gamma)\right)} [n]_q^k \sqrt{a_n b_n} \leq 1.$$

Hence, we find the largest γ^* such that

$$\sum_{n=1}^{+\infty} \frac{n(n^2+2n-2)+2\beta}{\beta A\left(2-\alpha(1+\gamma^*)\right)} [n]_q^k a_n b_n \leq$$
$$\sum_{n=1}^{+\infty} \frac{n(n^2+2n-2)+2\beta}{\beta A\left(2-\alpha(1+\gamma)\right)} [n]_q^k \sqrt{a_n b_n} \leq 1,$$

or equivalently

$$\sqrt{a_n b_n} \leq \frac{2-\alpha(1+\gamma^*)}{2-\alpha(1+\gamma)}, \qquad n \geq 1.$$

This inequality holds if

$$\frac{\beta A\left(2-\alpha(1+\gamma)\right)}{\left(n(n^2+2n-2)+2\beta\right)[n]_q^k} \leq \frac{2-\alpha(1+\gamma^*)}{2-\alpha(1+\gamma)},$$

or equivalently

$$\gamma^* \leq \frac{2}{\alpha} - \{1 + \frac{\beta A (2-\alpha(1+\gamma))^2}{\alpha(n(n^2+2n-2)+2\beta)[n]_q^k}\}.$$

□

Theorem 3.4. *With the same assumptions of Theorem 3.3, $(f*g)(z)$ belongs to $\mathcal{N}_{\omega}^{k}(\alpha^*,\beta,\gamma,T)$, where*

$$\alpha^* \le \frac{2}{1+\gamma} - \frac{\beta A(2-\alpha(1+\gamma))^2}{(n(n^2+2n-2)+2\beta)[n]_q^k(1+\gamma)}.$$

Proof. It is sufficient to show that

$$\sum_{n=1}^{+\infty} \frac{n(n^2+2n-2)+2\beta}{\beta A\left(2-\alpha^*(1+\gamma)\right)}[n]_q^k a_n b_n \le 1.$$

Using a similar process of the proof of Theorem 3.3, we find the largest α^* such that

$$\sqrt{a_n b_n} \le \frac{2-\alpha^*(1+\gamma)}{2-\alpha(1+\gamma)}, \qquad n \ge 1.$$

This inequality holds if

$$\frac{\beta A\left(2-\alpha(1+\gamma)\right)}{\left(n(n^2+2n-2)+2\beta\right)[n]_q^k} \le \frac{2-\alpha^*(1+\gamma)}{2-\alpha(1+\gamma)},$$

or equivalently

$$\alpha^* \le \frac{2}{1+\gamma} - \frac{\beta A(2-\alpha(1+\gamma))^2}{(n(n^2+2n-2)+2\beta)[n]_q^k(1+\gamma)}.$$

□

Conclusion

In this article, we used the concepts of $q-$calculus notations and introduced the $q-$differential integral operator for meromorphic functions. We used this newly defined operator and familiar Riesz-Dunford integral of a linear operator to establish a new class of meromorphic functions. Furthermore, we investigated some useful properties, such as coefficient estimates, distortion bound and extreme points for the functions belonging to the newly defined class of meromorphic functions. Further mathematical work may be done using the operator of this article and the subordinations approach, which enables for the definition of several further subclasses for meromorphic functions. For these classes, a number of new properties can be investigated, such as Feketo-Szego inequality, Upper bound, subordination results, etc.

References

[1] Dunford, NE, & Schwartz, TJ. (1958). Linear operator part I: General theory, Interscience publisheres, New York-London. Inter Science., 58.

[2] Frasin, BA, & Darus, M. (2004). On certain meromorphic functions with positive coefficients, Southeast Asian Bull. Math. 28(4), 615-623.

[3] Gasper, G., & Rahman, M. (2011). Basic Hypergeometric Series, Vol. 96. Cambridge University Press.

[4] Ghanim, F., & Darus, M. (2010). On a certain subclass of meromorphic univalent functions with fixed second positive coefficients, Surv. Math. Appl., 5, 49-60. http://eudml.org/doc/232334.

[5] Jakson, FH. (1909). On $q-$functions and a certain difference operator, Trans. Royal Soc. Edinburg, 46(2), 253-281. http://dx.doi.org/10.1017/S0080456800002751.

[6] Najafzadeh, Sh., & Ebadian, A. (2011). Operators on Hilbert space and its application to certain univelent functions with a fixed point, Acta Univ. Apulensis, 27, 51-56. http://auajournal.uab.ro/upload/26-809-Paper5-Acta27-2011.pdf.

[7] Saleh, ZM, & Mustafa, AO. (2022). Class of meromorphic univalent functions with fixed second positive coefficients defined by $q-$difference operator, Proc. Pak. Acad. Sci., A. Physical and Computational Sciences 59(1), 29-36. https://doi.org/10.53560/PPASA(59-1)640.

Lida Nobakht
Orcid number: 0009-0000-7889-3978
Department of Mathematics
Payame Noor University
Tehran, Iran
Email address: l.nobakt91@gmail.com

Shahram Najafzadeh
Orcid number: 0000-0002-8124-8344
Department of Mathematics
Payame Noor University
Tehran, Iran
Email address: shnajafzadeh44@pnu.ac.ir

Mohammadreza Foroutan
Orcid number: 0000-0002-7373-617X
Department of Mathematics
Payame Noor University
Tehran, Iran
Email address: mr_forootan@pnu.ac.ir