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ON ARTINIANNESS, ANNIHILATORS AND COASSOCIATED PRIMES OF FORMAL LOCAL COHOMOLOGY MODULES

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Abstract. In this paper, we obtain some results concerning vanishing, finiteness and artinianness of formal local cohomology modules. Also, we determine annihilators, cosupport and the set of coassociated primes of these modules in some special cases.

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1. **Introduction**

Throughout this paper, $(R, \mathfrak{m})$ is a commutative Noetherian local ring with identity, $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of R and M is a finite (finitely generated) R-module. Recall that the i-th local cohomology module of M with respect to $\mathfrak{a}$ is denoted by $\mathrm{H}^i_{\mathfrak{a}}(M)$ (see [4]). For each $i \geq 0$; i-th formal local cohomology of M with respect to an ideal $\mathfrak{a}$ is defined as

$$\mathfrak{F}^i_{\mathfrak{a}}(M) := \varprojlim_n \mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^n M).$$

Formal local cohomology is a useful tool in commutative algebra and algebraic geometry. It describes cohomology on formal neighborhoods and it is essential for studying formal schemes.

Formal local cohomology modules were used by Peskine and Szpiro in [9] when R is a regular ring in order to solve a conjecture of Hartshorne in prime characteristic. Another kind of investigations about formal cohomology has been done by Faltings (cf. [6]).

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Let $x = x_1, ..., x_r$ denote a system of elements such that $\mathfrak{m} = \mathrm{Rad}(x)$. In [12], Schenzel has studied the formal local cohomology module via the following isomorphism

$$\varprojlim_n \mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^n M) = \mathrm{H}^i(\varprojlim_n(\check{C}_x \otimes M/\mathfrak{a}^n M))$$

where $\check{C}_x$ denotes the Čech complex of R with respect to x.

For basic results about formal local cohomology, we refer the reader to [1], [2], [5] and [12].

It is a well known result of M. Brodmann [3] that, if M is a finite module over the commutative Noetherian ring R (with identity) and $\mathfrak{a}$ is an ideal of R, then the sequence of sets $(\mathrm{Ass}_R(\mathfrak{a}^n M/\mathfrak{a}^{n+1}M))_{n\in\mathbb{N}}$ is ultimately constant. Thus, if M is a finite R-module of finite dimension, then sequence $(\dim(\mathfrak{a}^n M/\mathfrak{a}^{n+1}M))_{n\in\mathbb{N}}$ is ultimately constant. Here, we denote its ultimate constant value by l^*, also we denote $\dim(M/\mathfrak{a}M)$ by l.

Let $k \geq 0$ be an integer and let $j := \dim \mathfrak{a}^k M/\mathfrak{a}^{k+1}M$. Our main goal is to obtain some properties of formal local cohomology modules $\mathfrak{F}^j_{\mathfrak{a}}(M)$.

In Section 2, we will prove the following results:

i) $\mathfrak{F}^j_{\mathfrak{a}}(M) \neq 0$ and if $j > 0$, then $\mathfrak{F}^j_{\mathfrak{a}}(M)$ is not finite,

ii) If $\mathfrak{F}^j_{\mathfrak{a}}(M)$ is Artinian, then $\mathrm{Att}_R\, \mathfrak{F}^j_{\mathfrak{a}}(M) = \mathrm{Att}_R(\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^{k+1}M))$,

iii) For any integer $i > j$, we have $\mathfrak{F}^i_{\mathfrak{a}}(M) \simeq \mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^k M)$,

iv) If $\mathfrak{b}\mathfrak{F}^j_{\mathfrak{a}}(M)$ is Artinian, then there exists an integer s such that

$$\mathrm{Ann}_R(\mathfrak{b}\mathfrak{F}^j_{\mathfrak{a}}(M)) = \mathrm{Ann}_R(\mathfrak{b}\,\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^s M)).$$

Then we obtain similar results for the formal local cohomology module $\mathfrak{F}^{l^*}_{\mathfrak{a}}(M)$. Also, we determine the sets $\mathrm{Ann}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M))$, $\mathrm{Att}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M))$ and $\mathrm{Cosupp}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M))$.

In Section 3, we investigate the set $\mathrm{Coass}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M))$ and we find some relations between $\mathrm{Coass}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M))$ and $\mathrm{Assh}_R(\mathfrak{b}(M/\mathfrak{a}^k M))$ for some $k \in \mathbb{N}$. Also, we will prove that if $\mathrm{Supp}_R(\mathfrak{b}M \cap \mathfrak{a}M)$ is a finite set, then $\mathrm{Coass}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M))$ is a finite set. As our last result, we will show that, whenever $l > 0$, $\mathfrak{F}^l_{\mathfrak{a}}(M)$ is Artinian if and only if $\mathrm{Coass}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)) = \mathrm{Assh}_R\, \mathfrak{F}^l_{\mathfrak{a}}(M)$.

2. Artinianness of formal local cohomology modules

A prime ideal $\mathfrak{p}$ of R is said to be an attached prime of M if $\mathfrak{p} = (N :_R M)$ for some submodule N of M. If M admits a reduced secondary representation,

$M = S_1 + S_2 + \cdots + S_n$, then the set of attached primes $\mathrm{Att}_R(M)$ of M is equal to $\{\sqrt{0 :_R S_i} : i = 1, \ldots, n\}$ (see [7]).

An R-module L is cocyclic if L is a submodule of $\mathrm{E}(R/\mathfrak{m})$ for some $\mathfrak{m} \in \operatorname{Max} R$. The cosupport of M is defined as the set of prime ideals $\mathfrak{p}$ such that there exists a cocyclic homomorphic image L of M such that $\mathrm{Ann}(L) \subseteq \mathfrak{p}$ and this set is denoted by $\mathrm{Cosupp}_R(M)$.

Let $\mathfrak{p} \in \operatorname{Spec} R$. If there is a cocyclic homomorphic image L of M such that $\mathfrak{p} = \mathrm{Ann}_R L$, then $\mathfrak{p}$ is called coassociated to M. The set of coassociated primes of M is denoted by $\mathrm{Coass}_R(M)$. In [13] we can see that $\mathrm{Coass}_R(M) \subseteq \mathrm{Cosupp}_R(M)$ and every minimal element of the set $\mathrm{Cosupp}_R(M)$ belongs to $\mathrm{Coass}_R(M)$.

Recall that, for an R-module M, the set $\mathrm{Assh}_R M$ is defined as

$$\{\mathfrak{p} \in \mathrm{Ass}_R M : \dim R/\mathfrak{p} = \dim M\}.$$

If M and N are finite R-modules such that $\mathrm{Supp}_R M = \mathrm{Supp}_R N$, then $\mathrm{Assh}_R M = \mathrm{Assh}_R N$.

Lemma 2.1. *Let $(R, \mathfrak{m})$ be a local ring, $\mathfrak{a}$ an ideal of R and M a finite R-module. Let $k \in \mathbb{N}_0$ and let $j := \dim \dfrac{\mathfrak{a}^k M}{\mathfrak{a}^{k+1} M} > 0$. Then for all integers $i \geq j$ and $t \geq k$, there exists an exact sequence*

$$\mathfrak{F}^i_{\mathfrak{a}}(\mathfrak{a}^t M) \to \mathfrak{F}^i_{\mathfrak{a}}(M) \to \mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^t M) \to 0,$$

and if $j > 0$, then there exists the following exact sequence

$$\mathrm{H}^{i-1}_{\mathfrak{m}}(M/\mathfrak{a}^t M) \to \mathfrak{F}^i_{\mathfrak{a}}(\mathfrak{a}^t M) \to \mathfrak{F}^i_{\mathfrak{a}}(M) \to \mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^t M) \to 0.$$

Proof. By [12, Theorem 3.11], the exact sequence

$$0 \to \mathfrak{a}^t M \to M \to M/\mathfrak{a}^t M \to 0$$

induces the following long exact sequence

$$\mathfrak{F}^{i-1}_{\mathfrak{a}}(M/\mathfrak{a}^t M) \to \mathfrak{F}^i_{\mathfrak{a}}(\mathfrak{a}^t M) \to \mathfrak{F}^i_{\mathfrak{a}}(M) \to \mathfrak{F}^i_{\mathfrak{a}}(M/\mathfrak{a}^t M) \to \mathfrak{F}^{i+1}_{\mathfrak{a}}(\mathfrak{a}^t M).$$

Since $t \geq k$ we have

$$\mathrm{Supp}_R(\mathfrak{a}^t M/\mathfrak{a}^{t+1} M) \leq \mathrm{Supp}_R(\mathfrak{a}^k M/\mathfrak{a}^{k+1} M)$$

and so $\dim(\mathfrak{a}^t M/\mathfrak{a}^{t+1} M) \leq \dim(\mathfrak{a}^k M/\mathfrak{a}^{k+1} M) = j < i + 1$. Thus, by [12, Theorem 4.5], $\mathfrak{F}^{i+1}_{\mathfrak{a}}(\mathfrak{a}^t M) = 0$. Now, by using [2, Lemma 2.1] and from the above sequence we get the result. □

Theorem 2.2. *Let $(R,\mathfrak{m})$ be a local ring, $\mathfrak{a}$ an ideal of R and M a finite R-module. Let $k \in \mathbb{N}_0$ and $j := \dim \frac{\mathfrak{a}^k M}{\mathfrak{a}^{k+1} M}$. Then $\mathfrak{F}^j_{\mathfrak{a}}(M) \neq 0$. Moreover, if $j > 0$ then $\mathfrak{F}^j_{\mathfrak{a}}(M)$ is not finite.*

Proof. By [4, Theorem 6.1.4] $\mathrm{H}^j_{\mathfrak{m}}(\mathfrak{a}^k M/\mathfrak{a}^{k+1}M) \neq 0$. Thus $\mathrm{Att}_R(\mathrm{H}^j_{\mathfrak{m}}(\mathfrak{a}^k M/\mathfrak{a}^{k+1}M)) \neq \emptyset$. Assume that $\mathfrak{p} \in \mathrm{Att}_R(\mathrm{H}^j_{\mathfrak{m}}(\mathfrak{a}^k M/\mathfrak{a}^{k+1}M))$. By [4, Theorem 7.3.2] $\mathfrak{p} \in \mathrm{Ass}_R(\mathfrak{a}^k M/\mathfrak{a}^{k+1}M)$ and $\dim(R/\mathfrak{p}) = j$. Thus $\mathfrak{p} \in \mathrm{Ass}_R(M/\mathfrak{a}^{k+1}M)$. By [4, 11.3.9], it follows that $\mathfrak{p} \in \mathrm{Att}_R(\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^{k+1}M))$ and so $\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^{k+1}M) \neq 0$. On the other hand, by Lemma 2.1, we have the exact sequence

$$\mathfrak{F}^j_{\mathfrak{a}}(M) \to \mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^{k+1}M) \to 0.$$

Since $\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^{k+1}M) \neq 0$, the above exact sequence shows that $\mathfrak{F}^j_{\mathfrak{a}}(M) \neq 0$. Now, assume that $j > 0$. Since by [4, Theorem 7.1.3] $\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^{k+1}M)$ is Artinian and $\dim R/\mathfrak{p} = j > 0$ by [4, Corollary 7.2.12] $\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^{k+1}M)$ is not finite. Now, the above exact sequence completes the proof. □

Lemma 2.3. *Let $(R,\mathfrak{m})$ be a local ring, $\mathfrak{a}$ an ideal of R and M a finite R-module. Let $k \in \mathbb{N}_0$ and $j := \dim \frac{\mathfrak{a}^k M}{\mathfrak{a}^{k+1} M}$. Then for all $t \geq k$, there exists the following exact sequence*

$$\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^{t+1}M) \to \mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^{t}M) \to 0.$$

Proof. The short exact sequence

$$0 \to \mathfrak{a}^t M/\mathfrak{a}^{t+1}M \to M/\mathfrak{a}^{t+1}M \to M/\mathfrak{a}^t M \to 0$$

induces a long exact sequence

$$\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^{t+1}M) \to \mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^{t}M) \to \mathrm{H}^{j+1}_{\mathfrak{m}}(\mathfrak{a}^t M/\mathfrak{a}^{t+1}M).$$

Since $\dim(\mathfrak{a}^t M/\mathfrak{a}^{t+1}M) < j+1$ by [4, Theorem 6.1.2], $\mathrm{H}^{j+1}_{\mathfrak{m}}(\mathfrak{a}^t M/\mathfrak{a}^{t+1}M) = 0$, and so we get the result. □

Theorem 2.4. *Let $(R,\mathfrak{m})$ be a local ring, $\mathfrak{a}$ an ideal of R and M a finite R-module. Let $k \in \mathbb{N}_0$ and $j := \dim \frac{\mathfrak{a}^k M}{\mathfrak{a}^{k+1} M}$. If $\mathfrak{F}^j_{\mathfrak{a}}(M)$ is Artinian, then $\mathrm{Att}_R \mathfrak{F}^j_{\mathfrak{a}}(M) = \mathrm{Att}_R(\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^{k+1}M))$.*

Proof. By Lemma 2.1, if $j = 0$, then there exists the following exact sequence:

$$0 \to \mathfrak{F}^j_{\mathfrak{a}}(\mathfrak{a}^k M) \to \mathfrak{F}^j_{\mathfrak{a}}(M) \to \mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^k M) \to 0,$$

and if $j > 0$, then there exists the following exact sequence:

$$\mathrm{H}^{j-1}_{\mathfrak{m}}(M/\mathfrak{a}^kM) \to \mathfrak{F}^j_{\mathfrak{a}}(\mathfrak{a}^kM) \to \mathfrak{F}^j_{\mathfrak{a}}(M) \to \mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^kM) \to 0.$$

By [4, Theorem 7.1.3] $\mathrm{H}^{j-1}_{\mathfrak{m}}(M/\mathfrak{a}^kM)$ is Artinian and by the assumption $\mathfrak{F}^j_{\mathfrak{a}}(M)$ is Artinian. Thus, from the above sequences it follows that $\mathfrak{F}^j_{\mathfrak{a}}(\mathfrak{a}^kM)$ is Artinian and so by [10, Theorem 2.5], $\mathrm{Att}_R(\mathfrak{F}^j_{\mathfrak{a}}(\mathfrak{a}^kM)) = \mathrm{Assh}(\mathfrak{a}^kM/\mathfrak{a}^{k+1}M)$. Now, from the above exact sequences and by using [4, 7.2.6] and Lemma 2.3, we have

$$\begin{aligned}\mathrm{Att}_R(\mathfrak{F}^j_{\mathfrak{a}}(M)) &\subseteq \mathrm{Att}_R(\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^kM)) \cup \mathrm{Att}_R(\mathfrak{F}^j_{\mathfrak{a}}(\mathfrak{a}^kM))\\ &\subseteq \mathrm{Att}_R(\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^{k+1}M)) \cup \mathrm{Assh}(\mathfrak{a}^kM/\mathfrak{a}^{k+1}M)\\ &= \mathrm{Att}_R(\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^{k+1}M)) \cup \{\mathfrak{p} \in \mathrm{Ass}(\mathfrak{a}^kM/\mathfrak{a}^{k+1}M) : \dim R/\mathfrak{p} = j\}\\ &\subseteq \mathrm{Att}_R(\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^{k+1}M)) \cup \{\mathfrak{p} \in \mathrm{Ass}(M/\mathfrak{a}^{k+1}M) : \dim R/\mathfrak{p} = j\}.\end{aligned}$$

But, by [4, 11.3.9]

$$\{\mathfrak{p} \in \mathrm{Ass}(M/\mathfrak{a}^{k+1}M) : \dim R/\mathfrak{p} = j\} \subseteq \mathrm{Att}_R(\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^{k+1}M))$$

and so $\mathrm{Att}_R(\mathfrak{F}^j_{\mathfrak{a}}(M)) \subseteq \mathrm{Att}_R(\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^{k+1}M))$. On the other hand, by using Lemma 2.1, $\mathrm{Att}_R(\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^{k+1}M)) \subseteq \mathrm{Att}_R\,\mathfrak{F}^j_{\mathfrak{a}}(M)$. Hence, it follows that $\mathrm{Att}_R\,\mathfrak{F}^j_{\mathfrak{a}}(M) = \mathrm{Att}_R(\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^{k+1}M))$. □

Lemma 2.5. *Let $(R,\mathfrak{m})$ be a local ring, $\mathfrak{a}$ an ideal of R and M a finite R-module. Let $k \in \mathbb{N}_0$ and $j := \dim \dfrac{\mathfrak{a}^kM}{\mathfrak{a}^{k+1}M}$. Then for any integer $i > j$ we have $\mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^{t+1}M) \simeq \mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^tM)$ for all $t \geq k$.*

Proof. From the short exact sequence

$$0 \to \mathfrak{a}^tM/\mathfrak{a}^{t+1}M \to M/\mathfrak{a}^{t+1}M \to M/\mathfrak{a}^tM \to 0$$

we get the following exact sequence

$$\mathrm{H}^i_{\mathfrak{m}}(\mathfrak{a}^tM/\mathfrak{a}^{t+1}M) \to \mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^{t+1}M) \to \mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^tM) \to \mathrm{H}^{i+1}_{\mathfrak{m}}(\mathfrak{a}^tM/\mathfrak{a}^{t+1}M).$$

But, $\dim(\mathfrak{a}^tM/\mathfrak{a}^{t+1}M) \leq j < i$ and so by [4, Theorem 6.1.2],

$$\mathrm{H}^i_{\mathfrak{m}}(\mathfrak{a}^tM/\mathfrak{a}^{t+1}M) = \mathrm{H}^{i+1}_{\mathfrak{m}}(\mathfrak{a}^tM/\mathfrak{a}^{t+1}M) = 0.$$

Now, the above exact sequence completes the proof. □

Theorem 2.6. *Let $(R,\mathfrak{m})$ be a local ring, $\mathfrak{a}$ an ideal of R and M a finite R-module. Let $k\in\mathbb{N}_0$ and $j:=\dim\frac{\mathfrak{a}^kM}{\mathfrak{a}^{k+1}M}$. Then for any integer $i>j$ we have $\mathfrak{F}^i_{\mathfrak{a}}(M)\simeq \mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^kM)$.*

Proof. By Lemma 2.5,

$$\mathfrak{F}^i_{\mathfrak{a}}(M)\simeq \varprojlim_{n\in\mathbb{N}}\mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^nM)\simeq \varprojlim_{n\geq k}\mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^nM)\simeq \mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^kM).$$

□

It is well known that, there exists an integer k_0 such that $\mathrm{Ass}_R(\mathfrak{a}^kM/\mathfrak{a}^{k+1}M)=\mathrm{Ass}_R(\mathfrak{a}^{k_0}M/\mathfrak{a}^{k_0+1}M)$ for all $k\geq k_0$ (cf. [3]). Thus, $\dim(\mathfrak{a}^kM/\mathfrak{a}^{k+1}M)=\dim_R(\mathfrak{a}^{k_0}M/\mathfrak{a}^{k_0+1}M)$ for all $k\geq k_0$. We use l^* to denote the stable value of $\dim(\mathfrak{a}^kM/\mathfrak{a}^{k+1}M)$.

In the next we obtain an upper bound for $\mathrm{fgrade}(\mathfrak{a},M)$. Recall that

$$\mathrm{fgrade}(\mathfrak{a},M)=\inf\{i\in\mathbb{N}_0:\mathfrak{F}^i_{\mathfrak{a}}(M)\neq 0\}.$$

Theorem 2.7. *Let $(R,\mathfrak{m})$ be a local ring, $\mathfrak{a}$ an ideal of R and M a finite R-module. Let l^* be the stable value of $\dim\frac{\mathfrak{a}^kM}{\mathfrak{a}^{k+1}M}$. Then*

i) $\mathrm{fgrade}(\mathfrak{a},M)\leq l^*$,

ii) If $l^*>0$, then $\mathfrak{F}^{l^*}_{\mathfrak{a}}(M)$ is not finite,

iii) There exists an integer k_0 such that $\mathfrak{F}^i_{\mathfrak{a}}(M)\simeq \mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^{k_0}M)$ for all $i>l^*$,

iv) $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is Artinian for all $i>l^*$.

Proof. i) and ii): By Theorem 2.2.

iii) Let k_0 be an integer such that $l^*=\dim\frac{\mathfrak{a}^{k_0}M}{\mathfrak{a}^{k_0+1}M}$. Then the assertion follows by Theorem 2.6.

iv) By (iii) and [4, Theorem 7.1.3]. □

Theorem 2.8. *Let $(R,\mathfrak{m})$ be a local ring, $\mathfrak{a}$ and $\mathfrak{b}$ ideals of R and M a finite R-module. Let $k\in\mathbb{N}_0$ and $j:=\dim\frac{\mathfrak{a}^kM}{\mathfrak{a}^{k+1}M}$. Then*

$$\mathrm{Ann}_R(\mathfrak{b}\mathfrak{F}^j_{\mathfrak{a}}(M))=\cap_{t\geq k}\mathrm{Ann}_R(\mathfrak{b}\,\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^tM)).$$

Proof. By Lemma 2.1, for any integer $t\geq k$ there exists the following exact sequence:

$$\mathfrak{F}^j_{\mathfrak{a}}(M)\to \mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^tM)\to 0.$$

Thus, $\mathrm{Ann}_R(\mathfrak{b}\mathfrak{F}^j_{\mathfrak{a}}(M)) \subseteq \cap_{t\geq k} \mathrm{Ann}_R(\mathfrak{b}\,\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^t M))$.

Conversely, assume that $x \in \cap_{t\geq k} \mathrm{Ann}_R(\mathfrak{b}\,\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^t M))$. Since, for any inverse system $\{L_t\}$,

$$\mathfrak{b}\varprojlim_t L_t \subseteq \varprojlim_t \mathfrak{b} L_t,$$

we have

$$x\mathfrak{b}\mathfrak{F}^j_{\mathfrak{a}}(M) \simeq x\mathfrak{b}\varprojlim_{n\in\mathbb{N}} \mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^n M) \simeq x\mathfrak{b}\varprojlim_{n\geq k} \mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^n M) \subseteq \varprojlim_{n\geq k} x\mathfrak{b}\,\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^n M) = 0.$$

Therefore $x \in \mathrm{Ann}_R(\mathfrak{b}\mathfrak{F}^j_{\mathfrak{a}}(M))$. □

Corollary 2.9. *Let $(R, \mathfrak{m})$ be a local ring, $\mathfrak{a}$ and $\mathfrak{b}$ ideals of R and M a finite R-module. Assume that* $\dim M/\mathfrak{a}M = l$. *Then*

$$\mathrm{Ann}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) = \cap_{t\geq 0} \mathrm{Ann}_R(\mathfrak{b}\,\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^t M)).$$

Corollary 2.10. *Let $(R, \mathfrak{m})$ be a local ring, $\mathfrak{a}$ and $\mathfrak{b}$ ideals of R and M a finite R-module. Assume that* $\dim M/\mathfrak{a}M = l$. *Then*

$$\mathrm{Cosupp}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) \subseteq \mathrm{Supp}_R\,\mathfrak{b}M.$$

Proof. For any R-module L, we have $\mathrm{Cosupp}_R(L) \subseteq \mathrm{V}(\mathrm{Ann}_R(L))$. Thus, $\mathrm{Cosupp}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) \subseteq \mathrm{V}(\mathrm{Ann}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)))$. Since for any integer k, $\mathrm{Ann}_R\,\mathfrak{b}M \subseteq \mathrm{Ann}_R(\mathrm{H}^l_{\mathfrak{m}}(\mathfrak{b}(M/\mathfrak{a}^k M))$ by using Corollary 2.9, we have

$$\mathrm{Cosupp}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) \subseteq \mathrm{V}(\cap_{k\in\mathbb{N}} \mathrm{Ann}_R(\mathrm{H}^l_{\mathfrak{m}}(\mathfrak{b}(M/\mathfrak{a}^k M)))) \subseteq \mathrm{V}(\mathrm{Ann}_R\,\mathfrak{b}M) = \mathrm{Supp}_R\,\mathfrak{b}M.$$

□

Theorem 2.11. *Let $(R, \mathfrak{m})$ be a local ring, $\mathfrak{a}$ and $\mathfrak{b}$ ideals of R and M a finite R-module. Let $k \in \mathbb{N}_0$ and $j := \dim \dfrac{\mathfrak{a}^k M}{\mathfrak{a}^{k+1}M}$. If $0 \neq \mathfrak{b}\mathfrak{F}^j_{\mathfrak{a}}(M)$ is Artinian, then there exists an integer $s \in \mathbb{N}$ such that $\mathrm{Ann}_R(\mathfrak{b}\mathfrak{F}^j_{\mathfrak{a}}(M)) = \mathrm{Ann}_R(\mathfrak{b}\,\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^s M))$.*

Proof. Using Lemma 2.3, for any integer $t \geq k$, there exists the following epimorphism

$$\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^{t+1}M) \to \mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^t M) \to 0.$$

Thus $\mathrm{Ann}_R\,\mathfrak{b}\,\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^{t+1}M) \subseteq \mathrm{Ann}_R\,\mathfrak{b}\,\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^t M)$. Take $U_t := \mathrm{Ann}_R\,\mathfrak{b}\,\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^t M)$. We have $U_{t+1} \subseteq U_t$. But,

$$\cap_{t\geq k}(U_t\mathfrak{b}\mathfrak{F}^j_{\mathfrak{a}}(M)) = \varprojlim_{t\geq k} U_t \mathfrak{b}\varprojlim_n \mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^n M)$$

$$\subseteq \varprojlim_{t\geq k}\varprojlim_{n} U_t \mathfrak{b}\,\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^n M)$$

$$= \varprojlim_{n}\varprojlim_{t\geq k} U_t \mathfrak{b}\,\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^n M) = 0,$$

as $U_t \mathfrak{b}\,\mathrm{H}^j_{\mathfrak{m}}(M/\mathfrak{a}^n M) = 0$ for all $t \geq n \geq k$. Thus, $\cap_{t\geq k}(U_t \mathfrak{b}\mathfrak{F}^j_{\mathfrak{a}}(M)) = 0$. But, by the assumption $\mathfrak{b}\mathfrak{F}^j_{\mathfrak{a}}(M)$ is Artinian and so the descending chain

$$\cdots \subseteq U_{k+2}\mathfrak{b}\mathfrak{F}^j_{\mathfrak{a}}(M) \subseteq U_{k+1}\mathfrak{b}\mathfrak{F}^j_{\mathfrak{a}}(M) \subseteq U_k\mathfrak{b}\mathfrak{F}^j_{\mathfrak{a}}(M)$$

of submodules of $\mathfrak{b}\mathfrak{F}^j_{\mathfrak{a}}(M)$ is stable. Therefore, there exists $s \in \mathbb{N}$ such that $\cap_{t\geq k} U_t\mathfrak{b}\mathfrak{F}^j_{\mathfrak{a}}(M) = U_s\mathfrak{b}\mathfrak{F}^j_{\mathfrak{a}}(M)$. Thus $U_s\mathfrak{b}\mathfrak{F}^j_{\mathfrak{a}}(M) = 0$ and so $U_s \subseteq \mathrm{Ann}_R(\mathfrak{b}\mathfrak{F}^j_{\mathfrak{a}}(M))$. On the other hand, by Theorem 2.8 $\mathrm{Ann}_R(\mathfrak{b}\mathfrak{F}^j_{\mathfrak{a}}(M)) \subseteq U_s$. Therefore, $\mathrm{Ann}_R(\mathfrak{b}\mathfrak{F}^j_{\mathfrak{a}}(M)) = U_s$, as required. □

Corollary 2.12. *Let $(R, \mathfrak{m})$ be a local ring, $\mathfrak{a}$ and $\mathfrak{b}$ ideals of R and M a finite R-module. Assume that $\dim M/\mathfrak{a}M = l$ and $0 \neq \mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)$ be Artinian. Then there exists $s \in \mathbb{N}$ such that*

$$\mathrm{Ann}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) = \mathrm{Ann}_R(\mathfrak{b}\,\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^s M)).$$

Proof. The assertion follows by putting $k = 0$ in Theorem 2.11. □

Corollary 2.13. *Let $(R, \mathfrak{m})$ be a local ring, $\mathfrak{a}$ and $\mathfrak{b}$ ideals of R and M a finite R-module. Let $l = \dim M/\mathfrak{a}M$ and $0 \neq \mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)$ be Artinian. Then there exists $k \in \mathbb{N}$ such that*

$$\mathrm{Ann}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) = \mathrm{Ann}_R(\frac{\mathfrak{b}(M/\mathfrak{a}^k M)}{\mathfrak{b}(M/\mathfrak{a}^k M) \cap T_R(M/\mathfrak{a}^k M)})$$

where $T_R(M/\mathfrak{a}^k M) = \cup\{N : N \leq M/\mathfrak{a}^k M$ and $\dim N < \dim M/\mathfrak{a}^k M\}$.

Proof. The result follows by Corollary 2.12 and [11, Theorem 2.8].

Corollary 2.14. *Let $(R, \mathfrak{m})$ be a local ring, $\mathfrak{a}$ and $\mathfrak{b}$ ideals of R and M a finite R-module. Let $l := \dim M/\mathfrak{a}M$. If $0 \neq \mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)$ is Artinian, then there exists $k \in \mathbb{N}$ such that*

$$\mathrm{Cosupp}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) = \mathrm{Supp}_R(\frac{\mathfrak{b}(M/\mathfrak{a}^k M)}{\mathfrak{b}(M/\mathfrak{a}^k M) \cap T_R(M/\mathfrak{a}^k M)}).$$

Proof. By Corollary 2.12, there exists $k \in \mathbb{N}$ such that

$$\mathrm{Ann}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) = \mathrm{Ann}_R(\mathfrak{b}\,\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M)).$$

But, $\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)$ and $\mathfrak{b}\,\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^kM)$ are Artinian and so by [13, Proposition 2.3], $\mathrm{Cosupp}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) = \mathrm{V}(\mathrm{Ann}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)))$ and $\mathrm{Cosupp}_R(\mathfrak{b}\,\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^kM)) = \mathrm{V}(\mathrm{Ann}_R((\mathfrak{b}\,\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^kM))$. Thus, $\mathrm{Cosupp}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) = \mathrm{Cosupp}_R(\mathfrak{b}\,\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^kM))$ and so the assertion follows by [11, Corollary 2.14]. □

3. Coassociated primes of the module $\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)$

Let $(R, \mathfrak{m})$ be a local ring, $\mathfrak{a}$ and $\mathfrak{b}$ ideals of R and M be a finite R-module. Let $l := \dim M/\mathfrak{a}M$. In this section, we obtain some results concerning coassociated primes of formal local cohomology module $\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)$. We find some relations between $\mathrm{Coass}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M))$ and $\mathrm{Assh}_R(\mathfrak{b}(M/\mathfrak{a}^kM))$ for some $k \in \mathbb{N}$. Also, as a main result, we will prove that, whenever $l > 0$, $\mathfrak{F}^l_{\mathfrak{a}}(M)$ is Artinian if and only if $\mathrm{Coass}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)) = \mathrm{Assh}_R\,\mathfrak{F}^l_{\mathfrak{a}}(M)$.

Theorem 3.1. *Let $(R, \mathfrak{m})$ be a local ring, $\mathfrak{a}$ and $\mathfrak{b}$ ideals of R and M a finite R-module. Let $l := \dim M/\mathfrak{a}M$ and $0 \neq \mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)$ be Artinian. Then there exists $k \in \mathbb{N}$ such that*

$$\mathrm{Att}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) = \mathrm{Assh}_R(\mathfrak{b}(M/\mathfrak{a}^kM)).$$

Proof. By [10, Theorem 2.3 (i)],

$$\mathrm{Att}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) \subseteq \cup_{k\in\mathbb{N}}\,\mathrm{Att}_R(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^kM)).$$

But, by [4, Theorem 3.7.3], for any integer k

$$\mathrm{Att}_R(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^kM)) = \mathrm{Assh}_R(M/\mathfrak{a}^kM) = \mathrm{Assh}_R(M/\mathfrak{a}M)$$

and so

$$\mathrm{Att}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) \subseteq \mathrm{Assh}_R(M/\mathfrak{a}M).$$

Since $\mathrm{Assh}_R(M/\mathfrak{a}M) = \{\mathfrak{p} \in \mathrm{Ass}\,M/\mathfrak{a}M : \dim R/\mathfrak{p} = l\}$ it follows that

$$\mathrm{Att}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) = \mathrm{Min}(\mathrm{Att}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M))).$$

But, for an Artinian R-module L, the sets $V(\mathrm{Ann}_R\,L)$ and $\mathrm{Att}_R\,L$ have the same minimal elements. Thus, by Corollary 2.12, there exists an integer $k \in \mathbb{N}$ such that

$$\mathrm{Min}(\mathrm{Att}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M))) = \mathrm{Min}\,\mathrm{V}(\mathrm{Ann}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M))) = \mathrm{Min}\,\mathrm{V}(\mathrm{Ann}_R(\mathfrak{b}\,\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^kM))).$$

Since by [4, Theorem 7.1.3] $\mathfrak{b}\,\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^kM)$ is Artinian, by using [11, Theorem 2.15], we have

$$\operatorname{Min} \mathrm{V}(\operatorname{Ann}_R(\mathfrak{b}\,\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^kM))) = \operatorname{Min}(\operatorname{Att}_R(\mathfrak{b}\,\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^kM))) = \operatorname{Min}(\operatorname{Att}_R(\mathrm{H}^l_{\mathfrak{m}}(\mathfrak{b}(M/\mathfrak{a}^kM)))).$$

By the assumption $\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M) \neq 0$ and so by Corollary 2.12, it follows that $\mathfrak{b}\,\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^kM) \neq 0$. Now [11, Corollary 2.4] implies that $\dim(\mathfrak{b}(M/\mathfrak{a}^kM)) = l$ and so by [4, Theorem 3.7.3] we have
$\operatorname{Att}_R(\mathrm{H}^l_{\mathfrak{m}}(\mathfrak{b}(M/\mathfrak{a}^kM))) = \operatorname{Assh}_R(\mathfrak{b}(M/\mathfrak{a}^kM))$. Therefore,

$$\operatorname{Min}(\operatorname{Att}_R(\mathrm{H}^l_{\mathfrak{m}}(\mathfrak{b}(M/\mathfrak{a}^kM)))) = \operatorname{Assh}_R(\mathfrak{b}(M/\mathfrak{a}^kM)),$$

and the proof is complete.

Theorem 3.2. *Let $(R,\mathfrak{m})$ be a local ring, $\mathfrak{a}$ and $\mathfrak{b}$ ideals of R and M a finite R-module. Let $l = \dim M/\mathfrak{a}M$ and $\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M) \neq 0$. Then there exists $k \in \mathbb{N}$ such that*

$$\operatorname{Coass}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) \cap \operatorname{Supp}_R(\mathfrak{b}(M/\mathfrak{a}^kM)) = \operatorname{Assh}_R(\mathfrak{b}(M/\mathfrak{a}^kM)).$$

Proof. We have

$$\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M) = \mathfrak{b}\varprojlim_k(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^kM)) \subseteq \varprojlim_k(\mathfrak{b}\,\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^kM)).$$

But, by the assumption $\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M) \neq 0$ and so the above inclusion shows that there exists $k \in \mathbb{N}$ such that $\mathfrak{b}\,\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^kM) \neq 0$. Now, [11, Corollary 2.4] implies that $\dim \mathfrak{b}(M/\mathfrak{a}^kM) = l$. Take $\mathfrak{p} \in \operatorname{Coass}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) \cap \operatorname{Supp}_R(\mathfrak{b}(M/\mathfrak{a}^kM))$. By [11, Lemma 2.2] $\operatorname{Coass}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) \subseteq \operatorname{Coass}_R(\mathfrak{F}^l_{\mathfrak{a}}(M))$. Thus $\mathfrak{p} \in \operatorname{Coass}_R(\mathfrak{F}^l_{\mathfrak{a}}(M))$. By [8, Theorem 2.8], there exists $\mathfrak{q} \in \operatorname{Assh}_R(M/\mathfrak{a}M)$ such that $\mathfrak{p} \subseteq \mathfrak{q}$. Thus $\dim R/\mathfrak{p} \geq \dim R/\mathfrak{q} = l$. But, $\mathfrak{p} \in \operatorname{Supp}_R(\mathfrak{b}(M/\mathfrak{a}^kM))$ implies that $\dim R/\mathfrak{p} \leq \dim(\mathfrak{b}(M/\mathfrak{a}^kM)) = l$ and so $\dim R/\mathfrak{p} = l$. Since $\dim(\mathfrak{b}(M/\mathfrak{a}^kM)) = l$ it follows that $\mathfrak{p} \in \operatorname{Ass}_R(\mathfrak{b}(M/\mathfrak{a}^kM))$ and so $\mathfrak{p} \in \operatorname{Assh}_R(\mathfrak{b}(M/\mathfrak{a}^kM))$.

Conversely, assume that $\mathfrak{p} \in \operatorname{Assh}_R(\mathfrak{b}(M/\mathfrak{a}^kM))$. By [11, Corollary 2.16] $\mathfrak{p} \in \operatorname{Att}_R(\mathfrak{b}\,\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^kM))$. On the other hand, by [8, Lemma 2.4] $\mathfrak{b}\,\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^kM)$ is a homomorphic image of $\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)$. Thus, by [13, Theorem 1.10], we have

$$\operatorname{Att}_R(\mathfrak{b}\,\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^kM)) = \operatorname{Coass}_R(\mathfrak{b}\,\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^kM)) \subseteq \operatorname{Coass}(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)).$$

Thus, it follows that $\mathfrak{p} \in \operatorname{Coass}(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M))$. But, $\operatorname{Assh}_R(\mathfrak{b}(M/\mathfrak{a}^kM)) \subseteq \operatorname{Supp}_R(\mathfrak{b}(M/\mathfrak{a}^kM))$ and so $\mathfrak{p} \in \operatorname{Assh}_R(\mathfrak{b}(M/\mathfrak{a}^kM))$ implies that $\mathfrak{p} \in \operatorname{Supp}_R(\mathfrak{b}(M/\mathfrak{a}^kM))$. Therefore, $\mathfrak{p} \in \operatorname{Coass}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) \cap \operatorname{Supp}_R(\mathfrak{b}(M/\mathfrak{a}^kM))$ and the proof is complete.
□

Theorem 3.3. *Let* $(R, \mathfrak{m})$ *be a local ring,* $\mathfrak{a}$ *and* $\mathfrak{b}$ *ideals of* R *and* M *a finite* R*-module. Assume that* $\dim M/\mathfrak{a}M = l$ *and* $\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M) \neq 0$*. Then there exists* $k \in \mathbb{N}$ *such that*

$$\mathrm{Coass}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) = \mathrm{Assh}_R(\mathfrak{b}(M/\mathfrak{a}^k M)) \cup (\mathrm{Coass}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) \cap \mathrm{Supp}_R(\mathfrak{b}M \cap \mathfrak{a}^k M)).$$

Proof. By Theorem 3.2, there exists $k \in \mathbb{N}$ such that

$$\mathrm{Coass}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) \cap \mathrm{Supp}_R(\mathfrak{b}(M/\mathfrak{a}^k M)) = \mathrm{Assh}_R(\mathfrak{b}(M/\mathfrak{a}^k M)).$$

Since $\mathrm{Coass}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) \subseteq \mathrm{Cosupp}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M))$, Corollary 2.10 implies that $\mathrm{Coass}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) \subseteq \mathrm{Supp}_R(\mathfrak{b}M)$ and so

$$\mathrm{Coass}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) = \mathrm{Coass}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) \cap \mathrm{Supp}_R(\mathfrak{b}M).$$

Thus

$$\mathrm{Coass}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) = \mathrm{Coass}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) \cap (\mathrm{Supp}_R(\mathfrak{b}M/\mathfrak{b}M \cap \mathfrak{a}^k M) \cup \mathrm{Supp}_R(\mathfrak{b}M \cap \mathfrak{a}^k M)).$$

Since $\mathfrak{b}M/\mathfrak{b}M \cap \mathfrak{a}^k M \simeq \mathfrak{b}(M/\mathfrak{a}^k M)$ it follows that

$$\mathrm{Coass}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) = \mathrm{Coass}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) \cap (\mathrm{Supp}_R(\mathfrak{b}(M/\mathfrak{a}^k M) \cup \mathrm{Supp}_R(\mathfrak{b}M \cap \mathfrak{a}^k M)).$$

Thus,

$$\mathrm{Coass}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) = \mathrm{Assh}_R(\mathfrak{b}(M/\mathfrak{a}^k M)) \cup (\mathrm{Coass}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) \cap \mathrm{Supp}_R(\mathfrak{b}M \cap \mathfrak{a}^k M)),$$

as required. □

Corollary 3.4. *Let* $(R, \mathfrak{m})$ *be a local ring,* $\mathfrak{a}$ *and* $\mathfrak{b}$ *ideals of* R *and* M *a finite* R*-module. Let* $l := \dim M/\mathfrak{a}M$ *and* $\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M) \neq 0$*. If* $\mathrm{Supp}_R(\mathfrak{b}M \cap \mathfrak{a}M)$ *is a finite set, then* $\mathrm{Coass}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M))$ *is a finite set. In particular, if* $\mathrm{Supp}_R(\mathfrak{a}M)$ *is a finite set, then* $\mathrm{Coass}_R \mathfrak{F}^l_{\mathfrak{a}}(M)$ *is a finite set.*

Proof. Let $k \in \mathbb{N}$ be an arbitrary integer. Since $\mathrm{Assh}_R(\mathfrak{b}(M/\mathfrak{a}^k M)) \subseteq \mathrm{Ass}_R(\mathfrak{b}(M/\mathfrak{a}^k M))$, we can see that $\mathrm{Assh}_R(\mathfrak{b}(M/\mathfrak{a}^k M))$ is a finite set. But, the assumption implies that $\mathrm{Supp}_R(\mathfrak{b}M \cap \mathfrak{a}^k M)$ is a finite set. Now, the assertion follows by Theorem 3.3. □

In the next example, we show that the converse of the above result is not true.

Example 3.5. *Let* $(R, \mathfrak{m})$ *be a complete Noetherian regular local ring of dimension* $d > 1$. *We have* $l := \dim R/\mathfrak{m} = 0$. *Since* R *is an integral domain* $\mathrm{Supp}_R(\mathfrak{m}R) = \mathrm{Spec}\, R$ *is an infinite set. But,* R *is complete and so by [1, Theorem 2.6 9 (i)],* $\mathfrak{F}^0_{\mathfrak{m}}(R)$ *is a finite* R*-module. Thus, by [13, Theorems 4.5, 1.9],* $\mathrm{Coass}_R(\mathfrak{F}^0_{\mathfrak{m}}(R)) = \{\mathfrak{m}\}$ *is a finite set.*

Theorem 3.6. *Let* $(R, \mathfrak{m})$ *be a local ring,* $\mathfrak{a}$ *and* $\mathfrak{b}$ *ideals of* R *and* M *a finite* R*-module. Let* $l := \dim M/\mathfrak{a}M > 0$. *If* $\mathfrak{a} \nsubseteq \sqrt{(0 :_R \mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M))}$, *then* $\mathrm{Coass}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) \nsubseteq \mathrm{V}(\mathfrak{a})$.

Proof. Assume that $\mathrm{Coass}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) \subseteq \mathrm{V}(\mathfrak{a})$ and so there exists an integer k such that $\mathfrak{a}^k\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)$ is finite by [14, Satz 1.2]. Now, by [10, Theorem 2.17], we have $\mathfrak{a}^k\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M) = 0$. Thus $\mathfrak{a} \subseteq \sqrt{(0 :_R \mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M))}$, as required. □

Theorem 3.7. *Let* $(R, \mathfrak{m})$ *be a local ring,* $\mathfrak{a}$ *and* $\mathfrak{b}$ *ideals of* R *and* M *a finite* R*-module. Let* $l := \dim M/\mathfrak{a}M > 0$. *If* $\mathfrak{a} \nsubseteq \sqrt{(0 :_R \mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M))}$, *then for any* $k \in \mathbb{N}$ *we have* $\mathrm{Coass}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) \neq \mathrm{Assh}_R(\mathfrak{b}(M/\mathfrak{a}^kM))$.

Proof. Assume that $\mathfrak{a} \nsubseteq \sqrt{(0 :_R \mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M))}$. By Theorem 3.6, $\mathrm{Coass}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) \nsubseteq \mathrm{V}(\mathfrak{a})$. But, for any integer k we have

$$\mathrm{Assh}_R(\mathfrak{b}(M/\mathfrak{a}^kM)) \subseteq \mathrm{Supp}_R(\mathfrak{b}(M/\mathfrak{a}^kM)) \subseteq \mathrm{Supp}_R(M/\mathfrak{a}^kM) \subseteq \mathrm{V}(\mathfrak{a}).$$

and so it follows that $\mathrm{Coass}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) \neq \mathrm{Assh}_R(\mathfrak{b}(M/\mathfrak{a}^kM))$. □

Theorem 3.8. *Let* $(R, \mathfrak{m})$ *be a local ring,* $\mathfrak{a}$ *an ideal of* R *and* M *a finite* R*-module. Let* $l := \dim M/\mathfrak{a}M > 0$. *Then* $\mathfrak{F}^l_{\mathfrak{a}}(M)$ *is Artinian if and only if* $\mathrm{Coass}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)) = \mathrm{Assh}_R(M/\mathfrak{a}M)$.

Proof. If $\mathfrak{F}^l_{\mathfrak{a}}(M)$ is Artinian, then by [10, Theorem 2.5]

$$\mathrm{Coass}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)) = \mathrm{Att}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)) = \mathrm{Assh}_R(M/\mathfrak{a}M).$$

Assume that $\mathfrak{F}^l_{\mathfrak{a}}(M)$ is not Artinian. Thus, [2, Corollary 2.10] implies that $\mathfrak{a} \nsubseteq \sqrt{(0 :_R \mathfrak{F}^l_{\mathfrak{a}}(M))}$. By putting $\mathfrak{b} = R$ in Theorem 3.7, it follows that there exists $k \in \mathbb{N}$ such that

$$\mathrm{Coass}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)) \neq \mathrm{Assh}_R(M/\mathfrak{a}^kM).$$

Since $\mathrm{Assh}_R(M/\mathfrak{a}^kM) = \mathrm{Assh}_R(M/\mathfrak{a}M)$ it follows that

$$\mathrm{Coass}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)) \neq \mathrm{Assh}_R(M/\mathfrak{a}M),$$

as required. □

Question 3.9. In Corollary 3.4, we showed that if $\mathrm{Supp}_R(\mathfrak{a}M)$ is a finite set, then the set $\mathrm{Coass}_R\, \mathfrak{F}^l_{\mathfrak{a}}(M)$ is finite. Now it is natural to ask the following questions:

For any arbitrary integer i, when is the set of coassociated primes of formal local cohomology module $\mathfrak{F}^i_{\mathfrak{a}}(M)$ finite? Is it possible to determine $\mathrm{Coass}_R\, \mathfrak{F}^i_{\mathfrak{a}}(M)$?

Conclusions. In this article, we can relate the theory of commutative algebra to the theory of formal local cohomology modules. With the results of the article, we show the importance of formal local cohomology theory as a study tool within commutative algebra theory. These results contribute to a deeper understanding of the behavior of formal local cohomology modules.
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References

[1] Asgharzadeh, M., & Divaani-Aazar, K. (2011). Finiteness properties of formal local cohomology modules and Cohen-Macaulayness, Comm. Algebra., 39, 1082-1103. https://doi.org/10.1080/00927871003610312

[2] Bijan-Zadeh, M. H., & Rezaei. Sh. (2014). Artinianness and attached primes of formal local cohomology modules, Algebra. Colloq. 21(2), 307-316. https://doi.org/10.1142/S1005386714000261

[3] Brodmann, M. (1979) Asymptotic stability of $\mathrm{Ass}(M/I^nM)$, Proc. Amer. Math. Soc. 74, 16-18. https://doi.org/10.1090/S0002-9939-1979-0521865-8

[4] Brodmann, M., & Sharp, R. Y. (1988). Local cohomology: an algebraic introduction with geometric applications, Camb. Univ. Press, United Kingdom. https://doi.org/10.1017/CBO9780511629204

[5] Eghbali, M. (2013) On Artinianness of formal local cohomology, colocalization and coassociated primes, Math. Scand., 113(1), 5-19. https://doi.org/10.7146/math.scand.a-15478

[6] Faltings, G. (1979) Algebraization of some formal vector bundles, Ann. of Math., 110, 501-514.

[7] Macdonald, I. G. (1973). Secondary representations of modules over a commutative ring, in Symposia Mat., Istituto Nazionale di alta Matematica Roma, 23-43.

[8] Rezaei, Sh., & Ghasemi-Kalemasihi, M. (2023). On the finiteness of formal local cohomology modules, J. Alg. Systems, 11(1), 171-185. https://doi.org/10.22044/jas.2022.11072.1549 l

[9] Peskine, C., & Szpiro, L. (1972). Dimension projective finie etcohomologie locale, Publ. Math. I. H. E. S., 42, 47-119.

[10] Rezaei, Sh., & Riahini-komachali, A. (2022). Finiteness properties of formal local cohomology modules, J. Alg. Systems, 9(2), 323-333. https://doi.org/10.22044/jas.2021.9875.1484

[11] Rezaei, Sh., & Sadeghi, B. (2023). On the top local cohomology and formal local cohomology modules, Bull. Korean Math. Soc., 60(1), 149-160. https://doi.org/10.4134/BKMS.b220003

[12] Schenzel, P. (2007). On formal local cohomology and connectedness, J. Algebra., 315(2), 894-923. https://doi.org/10.1016/j.jalgebra.2007.06.015

[13] Yassemi, S. (1995). Coassociated primes, Comm. Alg. 23, 1473-1498. https://doi.org/10.1080/00927879508825288

[14] Zöschinger, H. (1988). Über koassoziierte Primideale, Math. Scand., 63, 196-211. https://doi.org/10.7146/math.scand.a-12233

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