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APPLICATION OF HÖHLE'S SQUARE ROOTS ON HOOP ALGEBRAS

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Abstract. In this article, we use the square root as a tool to study hoop algebras. To do so, we define the square root and make the first attempt to explore the significant properties of this concept in this setting. Then, due to the key role of square roots in obtaining new hoop algebras, we apply them to the filters of hoop algebras and show that the formation of square roots on quotient structures of hoop algebras by their filters is well-behaved. In addition, a new class of hoop algebras having square roots, so-called good hoop algebras, is introduced, and some relationships with other classes of ordered algebras, such as Boolean algebras and Gödel algebras are explored. Several examples are provided as well. Ultimately, it is shown that the class of all (good) bounded $\vee$-hoop algebras with square roots is a variety.

Keywords: Hoop algebra, Wajsberg hoop, Basic hoop, Good hoop, Gödel algebra, (Strict) Square root, n-th root.
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1. Introduction

Hoop algebras are naturally ordered commutative residuated integral monoids, introduced by B. Bosbach [8, 9]. He showed that the resulting class of structures can be viewed as an equational class and that the class is congruence distributive and congruence permutable. In a manuscript by J. R. Büchi and T. M. Owens ([10]) devoted to a study of Bosbach's algebras, written in the mid-seventies, the commutative members of this equational class were given the name *hoops*. Among the researchers who have done research in this field so far, the following authors can be mentioned. Blok and Ferririm [4,5], regarding on varieties and quasi-varieties of hoops and their reducts and the structure of hoop algebras, Berman and Blok [3], regarding the free Łukasiewicz and hoop residuation algebras, Agliano, Ferririm and Montagna [1], regarding the basic hoops, Georgescu et al. [13], regarding the pseudo-hoops, Kondo [15], regarding some types of filters in hoops. Square roots are valuable tools for algebraic structures with non-idempotent binary operations. They can provide us useful information about the elements and some properties of these types of algebras,

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and they help us to build new operations from binary operations according to our needs. For this reason, square roots have many applications in the theory of semigroups, groups, and rings.

In [14], Höhle also attempted to explore the significance of square roots in the class of residuated lattices; integral, commutative, residuated, ℓ-monoids. He classified the class of MV-algebras with a square root and presented a decomposition for each MV-algebra as a direct product of a Boolean algebra and a strict MV-algebra. His study showed that a complete MV-algebra with square roots is either a complete Boolean algebra or a Boolean valued model of the real unit interval viewed as an MV-algebra or a product of both. Ambrosio [2] continued to study square roots on MV-algebras. She investigated the homomorphic image of MV-algebra with square roots.

The main goals of this paper are to provide a study on hoop algebras with square roots and present their main properties:

- Define square roots on hoop algebras and review some examples.
- Studying properties of square roots on variant hoop algebras, such as bounded and basic hoops, and hoops with double negation property (DNP, for short), and etc.
- Define n-th roots as extending square root and examine some of their properties.
- The application of square root on the filters of hoop algebras is investigated and under some circumstances, a structural theorem regarding the effect of the square root on quotient hoop algebras has been proved (see Theorem 4.5 below).
- Define a good hoop algebra and explore some of its characteristics and then some relationships with Boolean algebras and Gödel algebras have been investigated (see Theorems 4.19, 4.24, and Remark 4.20).
- In the end, it is shown that the class of all bounded hoop algebras with square roots is a variety.

The paper is organized as follows: Section 2 contains basic definitions, properties, and results about hoop algebras that will be used in the next sections. Section 3 while defining square roots on hoop algebras, we study properties of square roots on very hoop algebras. In Section 4, firstly applications of square roots on the filters of hoop algebras are provided, and then we define the concept of a good hoop algebra and a strict square root and classify the class of some hoop algebras with square roots.

2. **Preliminaries**

In this section, while providing definitions and some properties of hoop algebras, we collect the prerequisites used for the rest of the paper.

Definition 2.1. [1,15] A hoop algebra is an algebra $\mathbf{H} = (H; \odot, \rightarrow, 1)$ of type $(2, 2, 0)$ such that

(H1) $(H; \odot, 1)$ is a commutative monoid;

(H2) $x \to x = 1$;
(H3) $x \odot (x \to y) = y \odot (y \to x)$;
(H4) $(x \odot y) \to z = x \to (y \to z)$,
for every $x, y, z \in H$.

Let $\mathbf{H} = (H; \odot_H, \to_H, 1_H)$ and $\mathbf{G} = (G; \odot_G, \to_G, 1_G)$ be two hoop algebras. We say that $\mathbf{G}$ is a subalgebra of $\mathbf{H}$, denoted by $\mathbf{G} \leq \mathbf{H}$, if $G \subseteq H$ and every fundamental operation in G is a restriction of the corresponding operation in H, meaning that $1 \in G$, $\odot_G = \odot_H |_G$, and $\to_G = \to_H|_G$.

In a hoop algebra $\mathbf{H} = (H; \odot_H, \to_H, 1_H)$, define $x \leq y$ if and only if $x \to y = 1$. It is easily seen that $\leq$ defines a partial order relation on H (see also [1]). We define $x^0 = 1$ and $x^n = x^{n-1} \odot x$ for every $n \in \mathbb{N}$. An element $x \in H$ is called idempotent if and only if $x \odot x = x$. The set of idempotent elements in H is denoted by $\mathrm{Id}(\mathbf{H})$. $\mathbf{H}$ is called idempotent if each of its elements is idempotent. $\mathbf{H}$ is called bounded if it has a least element denoted by 0. We define the unary operation " $'$ " for every $x \in H$ as $x' := x \to 0$. We also set $(x')' = x''$. An element $x \in H$ is called dense if and only if $x' = 0$. The set of all dense elements in H is denoted by $D(\mathbf{H})$.

An element $x \in H$ is called regular if $x'' = x$. The set of regular elements in H is denoted by $\mathrm{reg}(\mathbf{H})$. If for every $x \in H$, $x'' = x$, then we say that the bounded hoop algebra $\mathbf{H}$ has the "double negation property", or simply (DNP). Let $\mathbf{H}$ be a bounded hoop algebra. For every $x \in H$ with $x \neq 1$, the order of x, denoted by $\mathrm{ord}(x)$, is the smallest positive integer n such that $x^n = 0$, if such an n exists. In this case, x is called a nilpotent element. The set of all nilpotent elements in H is denoted by $N(\mathbf{H})$.

Example 2.2. *a)* [5] *Let* $\mathbf{G} = (G; +, -, 0, \wedge, \vee)$ *be an Abelian* l*-group. Consider the positive cone* $P(\mathbf{G})$ *of* $\mathbf{G}$*, defined by* $P(\mathbf{G}) = \{x \in G : x \vee 0 = x\}$*. It is clear that* $\mathbf{P}(\mathbf{G}) := (P(G); \cdot, \to, 1)$ *is an algebra with the operations* $x \cdot y := x + y, x \to y := (y - x) \vee 0, 1_{\mathbf{P}(\mathbf{G})} := 0_{\mathbf{G}}$*. It can be easily seen that* $\mathbf{P}(\mathbf{G})$ *satisfies the conditions of Definition 2.1. Therefore, it is a hoop algebra. Notice that the partial order on* $\mathbf{P}(\mathbf{G})$ *is the converse of the partial order induced by* $\mathbf{G}$*.*
b) Consider the free monoid generated by a single element a*. Put* $C_a = \{1 = a^0, a, a^2, \cdots\}$ *and define the operations* $\odot$ *and* $\to$ *on* C_a *for any* $n, m < w$ *as:* $a^n \odot a^m = a^{n+m}, \quad a^n \to a^m = a^{\max(m-n,0)}$*. It is evident that* $\mathbf{C}_a = (C_a, \odot, \to, 1)$ *satisfies the conditions of Definition 2.1. Therefore, it is a hoop algebra. For more information, see [5].*
c) Let $\mathbf{G} = (G; +, -, 0, \wedge, \vee)$ *be an Abelian* l*-group,* $u \in G$ *be an arbitrary element with* $u \geq 0$*. Put* $[0, u] = \{x \in G : 0 \leq x \leq u\}$*. On the set* $[0, u]$*, define the operations* $\odot$ *and* $\to$ *as:* $x \odot y = (x + y - u) \vee 0$*,* $x \to y = (y - x + u) \wedge u$*, for every* $x, y \in [0, u]$*. It is easy to see that* $\mathbf{\Gamma}(\mathbf{G}, \mathbf{u}) := ([0, u]; \odot, \to, 0, u)$ *satisfies the conditions of Definition 2.1. Therefore, it is a bounded hoop algebra (see also [13]).*

Next, we present a finite hoop algebra.
d) [16] *Let* $H = \{0, a, b, c, d, 1\}$. *Define the operations* $\odot$ *and* $\to$ *on* H *as follows:*

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	b	b	d	0	a
b	0	b	b	0	0	b
c	0	d	0	c	d	c
d	0	0	0	d	0	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	a	c	c	1
b	c	1	1	c	c	1
c	b	a	b	1	a	1
d	a	1	a	1	1	1
1	0	a	b	c	d	1

It can be easily checked that the algebra is a bounded hoop algebra.

Let **H** and **G** be two hoop algebras. A map $f : H \to G$ is called a hoop homomorphism if $f(x \odot y) = f(x) \odot f(y)$, $f(x \to y) = f(x) \to f(y)$ for all $x, y \in H$. Then one can prove that it is order-preserving and $f(1) = 1$. In addition, if f is one to one then, it is an order embedding. Indeed, for $x, y \in H$ we have $x \leq y$ iff $x \to y = 1$ iff $f(x \to y) = f(1) = 1$ iff $f(x) \to f(y) = 1$ iff $f(x) \leq f(y)$. Also if the hoop algebras are bounded, then we assume also $f(0) = 0$.

Theorem 2.3. ([8,9]) *Let* $\mathbf{H} = (H; \odot, \to, 1)$ *be a hoop algebra. Then for every* $x, y, z \in H$:

(i) $(H; \leq)$ *is a* $\wedge$*-semilattice with* $x \wedge y = x \odot (x \to y)$;
(ii) $x \odot y \leq z$ *if and only if* $x \leq y \to z$;
(iii) $x \odot y \leq x, y$;
(iv) $x \odot (x \to y) \leq x, y$;
(v) $x \leq y \to x$;
(vi) $1 \to x = x$;
(vii) $x \to 1 = 1$;
(viii) $x \to y \leq (y \to z) \to (x \to z)$;
(ix) $x \leq y$ *implies* $x \odot z \leq y \odot z$;
(x) $x \leq y$ *implies* $z \to x \leq z \to y$;
(xi) $x \leq y$ *implies* $y \to z \leq x \to z$;
(xii) $((y \to x) \to x) \to x = y \to x$.

Proposition 2.4. [6, 13] *Let* **H** *be a hoop algebra. Define* $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$ *for every* $x, y \in H$. *Then the following conditions are equivalent:*

(i) *The operation* $\vee$ *is associative;*
(ii) $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$ *for all* $x, y, z \in H$;
(iii) $x \leq y$ *implies* $x \vee z \leq y \vee z$ *for all* $x, y, z \in H$;
(iv) *The operation* $\vee$ *is a join operation on* H.

A hoop algebra H is called $\vee$-hoop [6, Definition 2.7] if $\vee$ is a join operation on H.

Proposition 2.5. (See [13, Proposition 2.8 and Lemma 2.9]) *Let* **H** *be a hoop algebra, and let $x, y, z \in H$. If arbitrary joins exist in H then,*
(i) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$;
(ii) $x \odot (\bigvee_{i\in I} y_i) = \bigvee_{i\in I}(x \odot y_i)$, *for every* $\{y_i \mid i \in I\} \subseteq H$;
(iii) $x \wedge (\bigvee_{i\in I} y_i) = \bigvee_{i\in I}(x \wedge y_i)$, *for every* $\{y_i \mid i \in I\} \subseteq H$.

Proposition 2.6. [13, Proposition 2.15] *Let* **H** *be a $\vee$-hoop algebra and $x, y, z, t \in H$. Then:*
(i) *If* $x \vee y = 1$, *then* $x \odot y = x \wedge y$;
(ii) *If* $x \vee y = 1$ *and* $x \le z$, $y \le t$, *then* $z \vee t = 1$;
(iii) *If* $x \vee y = 1$, *then* $x^n \vee y^n = 1$ *for every* $n \in \mathbb{N}$.

Proposition 2.7. [8, 9] *Let* **H** *be a bounded hoop algebra. Then for every $x, y \in H$, we have:*
(i) $x \le x''$, $x \odot x' = 0$, $x''' = x'$, $x' \le x \to y$;
(ii) *If* $x \le y$, *then* $y' \le x'$;
(iii) $x \to y' = y \to x' = (x \odot y)'$;
(iv) $x \odot y = 0$ *if and only if* $x \le y'$.

Lemma 2.8. *Let H be a bounded hoop algebra. Then $x'' \odot y'' \le (x \odot y)''$, for every $x, y \in H$.*

Proof. Let $x, y \in H$. By Proposition 2.7(i), $x \odot y \le (x \odot y)''$. Now according to Theorem 2.3(ii), $x \le y \to (x \odot y)''$. Therefore, by Proposition 2.7(i, iii), $x \le y \to (x \odot y)'' = (x \odot y)' \to y' = (x \odot y)' \to y''' = y'' \to (x \odot y)''$. Now, in view of Theorem 2.3(ii), $x \odot y'' \le (x \odot y)''$. In a similar way, we obtain $x'' \odot y'' \le (x \odot y)''$. □

Proposition 2.9. [13] *Let* **H** *be a bounded hoop algebra. Then* **H** *has (DNP) if and only if $(x \to y) \to y = (y \to x) \to x$, for every $x, y \in H$.*

Remark 2.10. Let **H** be a bounded $\vee$-hoop algebra with (DNP). Then, according to Proposition 2.9, we have $x \vee y = (x \to y) \to y = (y \to x) \to x$.

Definition 2.11. [5, Definition 1.12] A *Wajsberg hoop* is a hoop algebra that satisfies the condition of antipode (T): $(x \to y) \to y = (y \to x) \to x$. Any hoop which satisfies (T) is in fact a lattice, and the operation $\vee$ is equal to $x \vee y = (x \to y) \to y$ for every $x, y \in H$ (As mentioned already $x \wedge y = x \odot (x \to y)$ for all $x, y \in H$).

It is a direct consequence of Proposition 2.9 and Definition 2.11 that every bounded hoop algebra with (DNP), is a bounded Wajsberg hoop and conversely.

Definition 2.12. [5] Let **H** be a hoop algebra. A subset F of H is called a *filter* of **H** if it satisfies the following conditions:
(F1) $1 \in F$,
(F2) $x \odot y \in F$ for all $x, y \in F$,
(F3) if $x \le y$ and $x \in F$, then $y \in F$ for all $x, y \in H$.

Let **H** be a hoop algebra and $\emptyset \neq X \subseteq H$. The intersection of all filters of **H** containing X is denoted by $\langle X\rangle$ or $F_g(X)$ and is equal to

$$\langle X\rangle = \{a \in H : \exists n \in \mathbb{N}, x_1, x_2, \cdots, x_n \in X,\ x_1 \odot x_2 \odot \cdots \odot x_n \leq a\}.$$

In particular, for each member $x \in H$ we have $\langle x\rangle = \{a \in H : \exists n \in \mathbb{N}, x^n \leq a\}$.

A filter F of **H** is called proper if $F \neq H$. It is clear that H and $\{1\}$ are filters of **H**, we call them trivial filters of **H**. It can be seen that if **H** is a bounded hoop algebra, then a filter is proper if and only if it does not contain 0. We denote the set of all filters of **H** by $\mathcal{F}(\mathbf{H})$.

Recall [15] that given a filter F of a hoop algebra **H**, one can define a congruence $\equiv_F$ on **H** by $x \equiv_F y$ if and only if $x \to y$ and $y \to x \in F$. We will denote the class of an element $x \in H$ by x/F or just by $[x]$. Since the class of all hoops forms a variety, any quotient structure $H/F = \{x/F : x \in H\}$ of **H** by $\equiv_F$ is also a hoop algebra by the following definitions:

$$x/F \odot y/F := (x \odot y)/F, \quad x/F \to y/F := (x \to y)/F, \quad \text{and} \quad 1 := 1/F$$

for all $x/F, y/F \in H/F$. In the hoop algebra $\mathbf{H}/F = (H/F; \odot, \to, 1)$, we also have $x/F \leq y/F$ if and only if $x \to y \in F$.

Definition 2.13. [6] Let **H** be a $\vee$-hoop algebra. A proper filter P of **H** is prime if for every $x, y \in H$, $x \vee y \in P$ implies $x \in P$ or $y \in P$. A proper filter M of the hoop algebra **H** is called a *maximal* filter if it is not properly contained in any other filter.

Let L be a bounded lattice with the smallest element 0 and the greatest element 1. For $a \in L$, we say $b \in L$ is a *complement* of a if $a \vee b = 1$ and $a \wedge b = 0$. If a has a unique complement, we denote this complement by a^*. The set of all complemented elements in L is denoted by $B(L)$ and is called the Boolean center of L.

Lemma 2.14. [17, Lemma 1.13] *Let* **H** *be a bounded hoop algebra, $x \in B(H)$, and x^* be the complement of x. Then, $x' = x^*$ and $x'' = x$.*

Lemma 2.15. [17, Proposition 1.17] *Let* **H** *be a bounded hoop algebra and $x \in B(H)$. In this case, the following statements hold:*
(i) *$x^2 = x$ and $x'' = x$;*
(ii) *$x^2 = x$ and $x' \to x = x$;*
(iii) *For all $a \in H$, $(x \to a) \to x = x$;*
(iv) *$x' \wedge x = 0$.*

Definition 2.16. [13] We call a hoop algebra **H** *basic* if the following condition holds for every $x, y, z \in H$:

$$\text{(BH)} \quad (x \to y) \to z \leq ((y \to x) \to z) \to z.$$

Lemma 2.17. [13] *Assume* **H** *is a basic hoop algebra. Then, for every $x, y, z \in H$:*
(i) *$(x \to y) \vee (y \to x) = 1$;*
(ii) *$x \odot (y \wedge z) = (x \odot y) \wedge (x \odot z)$.*

Proposition 2.18. [13, Proposition 4.9] *Any Wajsberg hoop is a basic hoop.*

Corollary 2.19. *Every bounded hoop algebra with (DNP) is a basic hoop.*

Proof. By using Definition 2.11 and Proposition 2.18, the result follows. □

Remark 2.20. It is substantial to mention that, in view of Proposition 2.18 and Corollary 2.19, the propositions which hold for basic hoop algebras are also valid for Wajsberg hoops and bounded hoops with (DNP).

3. Hoop algebras and square roots

In 1995, Höhle [14] defined the square root as a unary operation on an l-groupoid, which is commutative, residuated, and integral, and investigated some properties in this type of algebras. He then studied the influence of the square root on MV-algebras and ultimately proposed a new classification for these algebras based on the square root. In 1999, Ambrosio [2] presented other properties for strict MV-algebras. In 2015, Chen and Dudek [11] studied the representation and properties of square roots in pseudo-MV-algebras. Recently, in 2023, A. Dvurečenskij and Zahiri [12] presented representations and other properties of the square root on pseudo-MV-algebras. This section introduces the definitions of the square root and the n-th root in hoop algebras, and provides properties of these operations.

Definition 3.1. Let $\mathbf{H}$ be a hoop algebra. A mapping $s : H \to H$ is called a *square root* on $\mathbf{H}$ if it satisfies the following conditions:
(S1) for all $x \in H$, $s(x) \odot s(x) = x$,
(S2) for all $x, y \in H$, if $y \odot y \leq x$, then $y \leq s(x)$.

Example 3.2. *a) Let $I = [0,1]$ be the unit closed interval of real numbers, and let $a \in I$. Let $G_a = \{a^t : t \in \mathbb{Q}^+ \cup \{0\}\}$ where $a^0 = 1$. We define the multiplication of two elements as $a^t \odot a^s = a^{t+s}$ and the operation $\to$ is given by $a^t \to a^s = a^{\max(s-t,0)}$. It can be easily verified that $\mathbf{G_a} = (G_a; \odot, \to, 1)$ is a hoop algebra. We define the unary operation $s : G_a \to G_a$ by $s(a^{\frac{m}{n}}) = a^{\frac{m}{2n}}$, for every $a^{\frac{m}{n}} \in G_a$. We check the conditions of Definition 3.1. We have $s(a^{\frac{m}{n}}) \odot s(a^{\frac{m}{n}}) = a^{\frac{m}{2n}} \odot a^{\frac{m}{2n}} = a^{\frac{m}{n}}$. Thus, condition (S1) holds. Additionally, $a^t \odot a^t \leq a^{\frac{m}{n}} \Rightarrow a^{2t} \leq a^{\frac{m}{n}} \Rightarrow a^t \leq a^{\frac{m}{2n}} \Rightarrow a^t \leq s(a^{\frac{m}{n}})$. Therefore, condition (S2) is also satisfied. Hence, the operation s is a square root on $\mathbf{G_a}$.*
b) Let $I = [0,1]$ be the unit closed interval of real numbers. We define the operations $\odot$ and $\to$ on I as follows. For any $x, y \in I$,

$$x \odot y = \min(x, y) \quad \text{and} \quad x \to y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } x > y \end{cases}.$$

*The structure **Goa** $:= (I; \odot, \to, 1)$ with these operations is a hoop algebra. This algebraic structure is known as the Gödel algebra (see [17]).*

*In **Goa**, we define the unary operation $s : I \to I$ as $s(x) = x$ for every $x \in I$. It is evident that the conditions of Definition 3.1 hold. Therefore, s is*

*a square root on **Goa**.*
c) Let $I = [0,1]$ be the unit closed interval of real numbers. We define the operations $\odot$ and $\to$ on I as follows. For any $x, y \in I$,

$$x \odot y = x \cdot y \quad \text{and} \quad x \to y = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x} & \text{if } x > y \end{cases}.$$

*The structure $\boldsymbol{Pra} := (I; \odot, \to, 1)$ defined with these operations is a hoop algebra (see [17]). This algebraic structure is known as the product algebra. In this algebra, we define the unary operation $s : I \to I$ as $s(x) = \sqrt{x}$, for all $x \in I$. In this case, we have $s(x) \odot s(x) = \sqrt{x} \cdot \sqrt{x} = x$. Thus, the condition (S1) of Definition 3.1 is satisfied. Also, $y \odot y \leq x \Rightarrow y^2 \leq x \Rightarrow y \leq \sqrt{x} \Rightarrow y \leq s(x)$. Therefore, (S2) is also met. Hence, s is a square root on **Pra**.*
d) Let $(B; \vee, \wedge, ', 0, 1)$ be a Boolean algebra. Define two operations on B as $x \odot y = x \wedge y$, and $x \to y = x' \vee y$ for all $x, y \in B$. Then, $\mathbf{B} = (B; \odot, \to, 0, 1)$ is a bounded hoop algebra. In $\mathbf{B}$, we define the unary operation $s : B \to B$ as $s(x) = x$ for all $x \in B$. In this case, we have $s(x) \odot s(x) = x \wedge x = x$. Also, $y \odot y \leq x \Rightarrow y \wedge y \leq x \Rightarrow y \leq x \Rightarrow y \leq s(x)$. Therefore, s is a square root on $\mathbf{B}$.

Remark 3.3. Let $(H = \{0, a, b, 1\}, \leq)$ be a chain. Define the operations $\odot$ and $\to$ on H as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then, $\mathbf{H} = (H; \odot, \to, 0, 1)$ is a bounded hoop algebra. It is easily seen that $\mathbf{H}$ does not have a square root. In fact, the value for $s(a)$ does not exist.

Proposition 3.4. *Let $\mathbf{H} = (H; \odot_H, \to_H, 1_H)$ and $\mathbf{G} = (G; \odot_G, \to_G, 1_G)$ be two hoop algebras and $s_1 : H \to H$ and $s_2 : G \to G$ be square roots on $\mathbf{H}$ and $\mathbf{G}$, respectively. Then*

(i) *$\mathbf{H} \times \mathbf{G} = (H \times G; \odot_{H\times G}, \to_{H\times G}, 1_{H\times G})$ is a hoop algebra with the operations*

$$(x_1, x_2) \odot_{H\times G} (y_1, y_2) := (x_1 \odot_H y_1, x_2 \odot_G y_2),$$
$$(x_1, x_2) \to_{H\times G} (y_1, y_2) := (x_1 \to_H y_1, x_2 \to_G y_2),$$

and $1_{H\times G} := (1_H, 1_G)$ for every $x_1, y_1 \in H$ and $x_2, y_2 \in G$.

(ii) *The function $s : H \times G \to H \times G$ defined by $s(x, y) = (s_1(x), s_2(y))$ is a square root on $\mathbf{H} \times \mathbf{G}$.*

Proof. (i) It is evident.
(ii) We examine the validity of the conditions of Definition 3.1. For every $x_1 \in H$ and $x_2 \in G$ we have $s(x_1, x_2) \odot s(x_1, x_2) = (s_1(x_1), s_2(x_2)) \odot (s_1(x_1), s_2(x_2)) =$

$(s_1(x_1) \odot s_1(x_1), s_2(x_2) \odot s_2(x_2)) = (x_1, x_2)$. Thus, condition (S1) holds. To prove condition (S2), suppose $y_1 \in H$ and $y_2 \in G$ such that $(y_1, y_2) \odot (y_1, y_2) \leq (x_1, x_2)$. Therefore, we have $y_1 \odot y_1 \leq x_1$ and $y_2 \odot y_2 \leq x_2$. Since s_1 and s_2 are the square roots on **H** and **G** respectively, we have $y_1 \leq s_1(x_1)$ and $y_2 \leq s_2(x_2)$. Then, $(y_1, y_2) \leq (s_1(x_1), s_2(x_2))$. Hence, the condition (S2) is also satisfied. □

Theorem 3.5. *Let* **H** *be a hoop algebra and* $s : H \to H$ *be a square root on* **H**. *Then for every* $x \in H$

(i) $x \leq s(x)$. *In particular, if* **H** *is bounded and* $s(x) = 0$, *then* $x = 0$;
(ii) $s(1) = 1$.

Proof. (i) By Theorem 2.3(iii), for every $x \in H$, we have $x = s(x) \odot s(x) \leq s(x)$.
(ii) By Theorem 2.3(vii) and item (i), it is clear. □

Theorem 3.6. *Let* **H** *be a hoop algebra and* $s : H \to H$ *be a square root on* **H**. *Then the square root is unique.*

Proof. Assume s and r are two square roots on **H**. According to Definition 3.1, for every $x \in H$, we have $s(x) \odot s(x) \leq x \Rightarrow s(x) \leq r(x)$. Similarly, one has $r(x) \leq s(x)$. Therefore, $r(x) = s(x)$. □

Proposition 3.7. *Let* **H** *be a hoop algebra and* $s : H \to H$ *be a square root on* **H**. *Then,*
(i) *the square root is one-to-one;*
(ii) *If* $s(x) = x$, *then* $x^2 = x$, *for every* $x \in H$. *Conversely, if* $x^2 = x$ *holds for every* $x \in H$, *then* $s(x) = x$, *for every* $x \in H$;
(iii) $x = 1$ *if and only if* $s(x) = 1$.

Proof. (i) Let $x, y \in H$ be such that $s(x) = s(y)$. Then, we have $x = s(x) \odot s(x) = s(y) \odot s(y) = y$.
(ii) Assume $x^2 = x$ for every $x \in H$. In this case, $x = s(x) \odot s(x) = s(x)$. Conversely, assume $s(x) = x$ for some $x \in H$. Therefore, $x = s(x) \odot s(x) = x \odot x = x^2$.
(iii) If $x = 1$, then by Theorem 3.5, $s(x) = 1$. Conversely, if $s(x) = 1$, then by Theorem 3.5(ii), $s(x) = s(1)$. Therefore, by item (i), $x = 1$. □

Remark 3.8. Regarding Proposition 3.7, we observe the following:
(i) The equivalence

$$s(x) = x \quad \text{if and only if} \quad x \in B(\mathbf{H})$$

does not hold in general. It may happen that $s(x) = x$ while $x \notin B(\mathbf{H})$; see the Gödel algebra in Example 3.2(b). Conversely, it is possible that $x \in B(\mathbf{H})$ but $s(x) \neq x$; see the Łukasiewicz algebra in Example 4.15(a).

(ii) The equivalence

$$s(x) = x \text{ for all } x \in H \quad \text{if and only if} \quad H = B(\mathbf{H})$$

is also not valid in general. If $H = B(\mathbf{H})$, then by Example 3.2(d) we have $s(x) = x$ for all $x \in H$. However, the converse does not hold in general. In the Gödel algebra presented in Example 3.2(b), we have $s(x) = x$ for all $x \in H$, while $H \neq B(\mathbf{H})$.

(iii) If $s(x) \in B(\mathbf{H})$, then $s(x) = x$, since $s(x) = s(x)^2 = x$. However, the converse does not hold in general; see again the Gödel algebra in Example 3.2(b).

Lemma 3.9. *Let* $\mathbf{H}$ *be a hoop algebra,* $a \in \mathrm{Id}(H)$ *and* $H[a] = \{x \in H : a \leq x\}$. *Then,* $\mathbf{H[a]}$ *is a subalgebra and a filter of* $\mathbf{H}$.

Proof. The operations $\odot$ and $\to$ on $H[a]$ are those operations of H restricted to $H[a]$. It is clear that $1 \in H[a]$. Let $x, y \in H[a]$. Then, due to $a \leq x$ and $a \leq y$, and by Theorem 2.3(ix), we have $a = a \odot a \leq x \odot y$. Thus, $x \odot y \in H[a]$. Therefore, $H[a]$ is closed under multiplication. Also, according to Theorem 2.3(v,x), we have, $a \leq x \to a \leq x \to y$, i.e., $x \to y \in H[a]$. Hence, $\mathbf{H[a]}$ is a subalgebra of $\mathbf{H}$. Considering the conditions (F1) and (F2) of Definition 2.12 from the above explanations, and with the fulfillment of condition (F3), $H[a]$ is a filter of H. □

Theorem 3.10. *Let* $\mathbf{H}$ *be a hoop algebra and* $s : H \to H$ *be a square root on* $\mathbf{H}$. *Then, for every* $x, y \in H$*:*

(i) $x \leq y$ *if and only if* $s(x) \leq s(y)$*;*

(ii) $x \leq s(x^2) \leq s(x)$ *and* $s(x^2) \odot s(x^2) = s(x) \odot s(x) \odot s(x) \odot s(x) = x^2$*;*

(iii) $x \wedge y \leq s(x) \odot s(y)$*;*

(iv) $s(x \wedge y) \leq s(x) \wedge s(y)$*;*

(v) $s(x) \odot s(y) \leq s(x \odot y)$*;*

(vi) $s(x) \to s(y) = s(x \to y)$*;*

(vii) $x^2 \leq s(x) \odot s(x^2) \leq x$*;*

(viii) *If* $y \leq s(x)$*, then* $y^n \leq x$*, for every* $n \in \mathbb{N}$ *with* $n \geq 2$*;*

(ix) *If* $a \in \mathrm{Id}(H)$*, then* $s_a : H[a] \to H[a]$ *induced by* s *is a square root on* $H[a]$*;*

(x) *If* $x \leq y$*, then* $s([x, y]) \subseteq [s(x), s(y)]$*, where* $[x, y] = \{z \in H : x \leq z \leq y\}$.

Proof. (i) If $x \leq y$, then we have $s(x) \odot s(x) = x \leq y$, and by the condition (S2) of Definition 3.1, $s(x) \leq s(y)$. The converse is clear.
(ii) We have $x \odot x \leq x \odot x$, therefore by (S2), $x \leq s(x \odot x) = s(x^2)$ and by (i), $s(x^2) \leq s(x)$. Moreover, by (S1), $s(x \odot x) \odot s(x \odot x) = x \odot x = s(x) \odot s(x) \odot s(x) \odot s(x)$.
(iii) By (i) and the compatibility of the partial order with multiplication, we have $x \wedge y = s(x \wedge y) \odot s(x \wedge y) \leq s(x) \odot s(y)$.
(iv) The proof is straightforward by (i).

(v) We have $(s(x) \odot s(y)) \odot (s(x) \odot s(y)) = (s(x) \odot s(x)) \odot (s(y) \odot s(y)) = x \odot y$. Then, by (S2), $s(x) \odot s(y) \leq s(x \odot y)$.
(vi) We have $x \odot (s(x) \to s(y)) \odot (s(x) \to s(y)) = s(x) \odot s(x) \odot (s(x) \to s(y)) \odot (s(x) \to s(y)) = s(x) \odot (s(x) \wedge s(y)) \odot (s(x) \to s(y)) = (s(x) \wedge s(y)) \odot s(x) \odot (s(x) \to s(y)) \leq (s(x) \wedge s(y)) \odot s(y) \leq s(y) \odot s(y) = y$. Therefore, by Theorem 2.3(ii), $(s(x) \to s(y)) \odot (s(x) \to s(y)) \leq x \to y$. From Definition 3.1, we get $s(x) \to s(y) \leq s(x \to y)$. On the other hand, by the part (v), one has $s(x) \odot s(x \to y) \leq s(x \odot (x \to y)) = s(x \wedge y) \leq s(y)$. Therefore, according to Theorem 2.3(ii), $s(x \to y) \leq s(x) \to s(y)$. Hence, $s(x \to y) = s(x) \to s(y)$.
(vii) According to Theorem 3.5(i), $x \leq s(x)$. Also, by part (ii), $x \leq s(x^2)$. Therefore, $x^2 \leq s(x) \odot s(x^2)$. Furthermore, as s is monotonic, we have $s(x^2) \leq s(x)$. Therefore, $s(x) \odot s(x^2) \leq s(x) \odot s(x) = x$.
(viii) Suppose $y \leq s(x)$. For $n \geq 2$, we have $y^n \leq y \odot y \leq s(x) \odot s(x) = x$.
(ix) According to Lemma 3.9, the proof is easily done.
(x) The proof is straightforward according to part (i). □

Corollary 3.11. *Let* **H** *be a hoop algebra,* $s : H \to H$ *be a square root on* **H** *and* $x \in H$ *and* $n \in \mathbb{N}$. *Then*

(i) $(s(x)^n)^2 = (s(x)^2)^n = s(x)^{2n} = x^n$;
(ii) $s(x)^n \leq s(x^n)$.

Proof. (i) The proof is straightforward.
(ii) According to Theorem 3.10(v), it is clear. □

Proposition 3.12. *Let* **H** *be a hoop algebra and* $s : H \to H$ *be a square root on* **H**. *Then:*

(i) $[(\forall x)(s(x) = x)] \Longrightarrow s(x \odot y) = s(x) \odot s(y)$;
(ii) $[(\forall x, y)(x \odot y = x \wedge y)] \Longrightarrow s(x \odot y) = s(x) \odot s(y)$;
(iii) $[(\forall x, y)(s(x \odot y) = s(x) \odot s(y))] \Longrightarrow s(x \wedge y) = s(x) \wedge s(y)$;
(iv) $[(\forall x, y)(s(x \odot y) = s(x) \odot s(y))] \Longrightarrow s(x^2) = x$. *In particular, if* **H** *is bounded, then* $s(0) = 0$.

Proof. (i) It is fairly trivial, if $s(x) = x$ for any $x \in H$, then $s(x \odot y) = x \odot y = s(x) \odot s(y)$.
(ii) By Theorem 3.10(v), we know that $s(x) \odot s(y) \leq s(x \odot y)$. On the other hand, by part (i) of the same Theorem, we have $s(x \odot y) \leq s(x)$ and $s(x \odot y) \leq s(y)$. Therefore, $s(x \odot y) \leq s(x) \wedge s(y)$. According to the assumption, we have $s(x) \wedge s(y) = s(x) \odot s(y)$. Then, $s(x \odot y) \leq s(x) \odot s(y)$. Thus, we get the result.
(iii) According to Theorem 2.3(i), we know $x \wedge y = x \odot (x \to y)$. Now, according to the hypothesis and Theorem 3.10(vi), we have $s(x \wedge y) = s(x \odot (x \to y)) = s(x) \odot s(x \to y) = s(x) \odot (s(x) \to s(y)) = s(x) \wedge s(y)$.
(iv) Suppose $s(x) \odot s(y) = s(x \odot y)$ and $x = y$. In this case, we have $x = s(x) \odot s(x) = s(x \odot x) = s(x^2)$. Also, we have $0 = s(0) \odot s(0) = s(0 \odot 0) = s(0)$, since, according to Theorem 2.3(iii), $0 \odot 0 = 0$. □

In the following examples, we show that for $\mathbf{G_a}$ and **Goa**, the operation s, preserves the operation $\odot$.

Example 3.13. *a) In Example 3.2(a),* $s(a^{\frac{m}{n}}) = a^{\frac{m}{2n}}$. *Now, if* $x = a^{\frac{m}{n}}$ *and* $y = a^{\frac{u}{v}}$, *then* $s(x) \odot s(y) = a^{\frac{m}{2n}+\frac{u}{2v}} = a^{\frac{mv+nu}{2nv}} = s(a^{\frac{mv+nu}{nv}}) = s(a^{\frac{m}{n}+\frac{u}{v}}) = s(a^{\frac{m}{n}} \odot a^{\frac{u}{v}}) = s(x \odot y)$. *Therefore,* $s(x) \odot s(y) = s(x \odot y)$ *holds for every* $x, y \in G_a$.
*b) In the Gödel hoop **Goa**, in Example 3.2(b), let* $s : I \to I$ *be the identity function. Then clearly the relation* $s(x) \odot s(y) = s(x \odot y)$ *holds for every* $x, y \in I$.

Similarly, this holds for Examples 3.2(c) and 3.2(d) as well.

Theorem 3.14. *Let* $\mathbf{H}$ *be a basic hoop algebra and* $s : H \to H$ *be a square root on* $\mathbf{H}$. *Then, for every* $x, y \in H$:

(i) $s(x \wedge y) = s(x) \wedge s(y)$;
(ii) *If* $y \leq s(x) \odot s(y)$, *then* $y \leq x$.

Proof. (i) By Theorem 3.10(iv), we have $s(x \wedge y) \leq s(x) \wedge s(y)$. On the other hand, based on Lemma 2.17(ii), $(s(x) \wedge s(y)) \odot (s(x) \wedge s(y)) = (s(x) \odot s(x)) \wedge (s(x) \odot s(y)) \wedge (s(y) \odot s(x)) \wedge (s(y) \odot s(y)) \leq x \wedge y$. Therefore, we have $s(x) \wedge s(y) \leq s(x \wedge y)$. Thus, the equality $s(x \wedge y) = s(x) \wedge s(y)$ holds.
(ii) Let $x, y \in H$ such that $y \leq s(x) \odot s(y)$. Then, we have $y = y \wedge (s(x) \odot s(y)) = (s(y) \odot s(y)) \wedge (s(x) \odot s(y)) = s(y) \odot (s(x) \wedge s(y)) = s(y) \odot (s(y) \odot (s(y) \to s(x))) = y \odot (s(y) \to s(x)) \leq s(x) \odot s(y) \odot (s(y) \to s(x)) \leq s(x) \odot s(x) = x$ and we have done. □

Theorem 3.15. *Let* $\mathbf{H}$ *be a bounded hoop algebra and* $s : H \to H$ *be a square root on* $\mathbf{H}$. *Then, for every* $x, y \in H$:

(i) $s(x) \odot s(0) \leq x$;
(ii) *If* $x \in \mathrm{Id}(\mathbf{H})$ *such that* $x \leq s(0)$, *then* $x = 0$;
(iii) $x \wedge x' \leq s(0)$;
(iv) $(s(x) \odot s(0)')' \to s(y) \leq s(x' \to y)$;
(v) $s(x) \leq x' \to s(0)$;
(vi) $s(x') \otimes s(0) = s(x')$ *where* $x \otimes y = (x \to s(0)) \to y$;
(vii) $s(x') \odot s(x) \leq s(0)$;
(viii) $s(x')' \leq s(x'')$;
(ix) $s(0) = \max\{x \wedge x' : x \in H\}$;
(x) $s(x) \to s(x') = s((x^2)')$. *Moreover, if* $x = x^2$, *then* $s(x) \to s(x') = s(x')$;
(xi) $s(x)' \leq s(x')$;
(xii) *If* $x \in \mathrm{reg}(\mathbf{H})$, *then* $s(x) \in \mathrm{reg}(\mathbf{H})$;
(xiii) $s(x \odot y) \leq ((s(x) \odot s(y)) \to s(0)) \to s(0)$;
(xiv) $s(0) \leq s(0)'$;
(xv) *Let* $s(x \odot y) = s(x) \odot s(y)$, *for all* $x, y \in H$. *If* $x \leq y$, *then* $s([x, y]) = [s(x), s(y)]$. *In particular,* $s(\mathbf{H}) = [s(0), s(1)]$.

Proof. (i) For every $x \in H$, we have $0 \leq x$. Therefore, by Theorem 3.10(i) and Theorem 2.3(ix), we have $s(0) \leq s(x) \Rightarrow s(x) \odot s(0) \leq s(x) \odot s(x) = x$.
(ii) Suppose $x \in \mathrm{Id}(H)$ such that $x \leq s(0)$. Then $x = x \odot x \leq s(0) \odot s(0) = 0$.
(iii) By Theorem 2.3(i, iii), we have $(x \wedge x') \odot (x \wedge x') \leq x \odot x' = 0$. Therefore, $x \wedge x' \leq s(0)$.
(iv) Using Theorem 2.3(viii), we get $s(x) \to s(0) \leq (s(0) \to 0) \to (s(x) \to 0) = s(0)' \to (s(x) \to 0)$. Now, by Theorem 2.3(xi), we have $(s(0)' \to (s(x) \to 0)) \to s(y) \leq (s(x) \to s(0)) \to s(y)$. On the other hand, by Theorem 3.10(vi), we have $s(x' \to y) = s(x') \to s(y) = (s(x) \to s(0)) \to s(y)$. Thus, we conclude $(s(0)' \to (s(x) \to 0)) \to s(y) \leq s(x' \to y)$. Now, using Proposition 2.7(iii), the result is obtained.
(v) Using Theorems 3.5(i), 3.10(vi), and 2.3(ii), we have $x' \leq s(x') = s(x \to 0) = s(x) \to s(0) \Rightarrow s(x) \leq x' \to s(0)$.
(vi) By Theorem 2.3(xii) and Theorem 3.10(vi), we have $s(x') \otimes s(0) = (s(x') \to s(0)) \to s(0) = ((s(x) \to s(0)) \to s(0)) \to s(0) = s(x) \to s(0) = s(x')$.
(vii) Using Theorem 3.10(vi), we have $s(x') \odot s(x) = s(x) \odot (s(x) \to s(0)) \leq s(0)$.
(viii) According to Theorem 3.5(i), for every $x \in H$, $x' \leq s(x')$. Now, by Proposition 2.7(ii), we get $s(x')' \leq x'' \leq s(x'')$.
(ix) By part (iii), for every $x \in H$, $x \wedge x' \leq s(0)$. So, $s(0)$ is an upper bound for the set $\{x \wedge x' : x \in H\}$. Also, since $s(0) \odot s(0) = 0$, by part (vi), we have $s(0) \leq s(0)'$. Hence, $s(0) = s(0) \wedge s(0)' \in \{x \wedge x' : x \in H\}$. Therefore, $s(0) = \max\{x \wedge x' : x \in H\}$.
(x) One has $s(x) \to s(x') = s(x \to x') = s(x \to (x \to 0)) = s(x^2 \to 0) = s((x^2)')$. The second assertion is clear.
(xi) We know that $0 \leq s(0)$. Therefore, by Theorem 2.3(x) and Theorem 3.10(vi), for every $x \in H$, we have $s(x) \to 0 \leq s(x) \to s(0)$ and so $s(x)' \leq s(x')$.
(xii) Let $x \in \mathrm{reg}(H)$. Then $x'' = x$. According to Proposition 2.7(i), we know $s(x) \leq s(x)''$. Now by Lemma 2.8, we have $s(x)'' \odot s(x)'' \leq (s(x) \odot s(x))'' = x'' = x$. Therefore, according to (S2), $s(x)'' \leq s(x)$ and we are done.
(xiii) We have $s(x \odot y) \leq s((x \odot y)'') = s(((x \odot y) \to 0) \to 0) = s((x \odot y) \to 0) \to s(0) = s(x \to (y \to 0)) \to s(0) = (s(x) \to s(y \to 0)) \to s(0) = (s(x) \to (s(y) \to s(0))) \to s(0) = ((s(x) \odot s(y)) \to s(0)) \to s(0)$.
(xiv) From Proposition 2.7(iv) it follows that $s(0) \odot s(0) \leq 0$, implies $s(0) \leq s(0)'$.
(xv) Let $x \leq y$. By Theorem 3.10(x), $s([x, y]) \subseteq [s(x), s(y)]$. Let $z \in H$ such that $s(x) \leq z \leq s(y)$. By Proposition 3.12(iv), $z = s(z \odot z)$ since $s(0) \leq s(x) \leq z$. Now, we have $x = s(x) \odot s(x) \leq z \odot z \leq s(y) \odot s(y) = y$. Therefore, $z = s(z \odot z) \in s([x, y])$. □

Corollary 3.16. *Let* $\mathbf{H}$ *be a bounded hoop algebra and* $s : H \to H$ *be a square root in* $\mathbf{H}$. *Then* $x \in N(\mathbf{H})$ *if and only if* $s(x) \in N(\mathbf{H})$ *for every* $x \in H$.

Proof. Assume $x \in N(\mathbf{H})$, so there exists $n \in \mathbb{N}$ such that $x^n = 0$. On the other hand, since $s(x)^2 = x$, we have $s(x)^{2n} = (s(x)^2)^n = 0$. Therefore,

$s(x) \in N(\mathbf{H})$. Conversely, suppose $s(x) \in N(\mathbf{H})$. Thus, there exists an $n \in \mathbb{N}$ such that $s(x)^n = 0$. Now, by Theorem 3.5(i), we have $x^n = 0$. Hence, $x \in N(\mathbf{H})$. □

Proposition 3.17. *Let* $\mathbf{H}$ *be a bounded hoop algebra and* $s : H \to H$ *be a square root in* $\mathbf{H}$. *Then for every* $x \in H$, $x \in D(\mathbf{H})$ *if and only if* $s(x) \in D(\mathbf{H})$.

Proof. Assume $x \in D(\mathbf{H})$, so $x' = 0$. By Theorem 3.5(i) and Proposition 2.7(ii), $s(x)' \leq x'$. Then, $s(x) \in D(\mathbf{H})$. Conversely, if $s(x) \in D(\mathbf{H})$ then, $s(x)' = 0$. By the condition (H4) of Definition 2.1, we have $x' = x \to 0 = (s(x) \odot s(x)) \to 0 = s(x) \to (s(x) \to 0) = s(x) \to s(x)' = s(x) \to 0 = s(x)' = 0$. Therefore, $x \in D(\mathbf{H})$. □

Theorem 3.18. *Let* $\mathbf{H}$ *be a bounded* $\vee$*-hoop algebra and* $s : H \to H$ *be a square root in* $\mathbf{H}$. *Then for every* $x, y \in H$:

(i) $x \vee s(0) \leq s(x^2)$. *If* $x = x^2$, *for all* $x \in H$, *then* $x \vee s(0) = s(x^2) = x$;
(ii) $(s(x) \vee s(0))^2 = x$;
(iii) $s(x) \vee s(y) \leq s(x \vee y)$. *If* $\mathbf{H}$ *is basic or* $s(x \odot y) = s(x) \odot s(y)$, *for all* $x, y \in H$, *then* $s(x) \vee s(y) = s(x \vee y)$;
(iv) *If* $x \vee y = 1$, *then* $s(x^n) \vee s(y^n) = s(x)^n \vee s(y)^n = 1$, *for every* $n \in \mathbb{N}$;
(v) *If* $x \in B(\mathbf{H})$ *and* s *preserves the multiplication operation, then* $s(x^2) = (x \to s(0)) \to s(0)$.

Proof. (i) By Theorems 3.5(i) and 3.10(ii), the proof is straightforward. For the next part, from Proposition 3.7(ii), it follows that $s(x^2) = x$, which implies equality.
(ii) We know that $s(x) \vee s(0) = s(x)$. Therefore, we have$(s(x) \vee s(0)) \odot (s(x) \vee s(0)) = s(x) \odot s(x) = x$.
(iii) Since $x \leq x \vee y$, by Theorem 3.10(i), we have $s(x) \leq s(x \vee y)$. Similarly, $s(y) \leq s(x \vee y)$. Thus, it follows that $s(x) \vee s(y) \leq s(x \vee y)$. Moreover, if $\mathbf{H}$ is basic, then using Theorems 3.10(iv,vi) and 3.14(i), we have $s(x \vee y) = s(((x \to y) \to y) \wedge ((y \to x) \to x)) = s((x \to y) \to y) \wedge s((y \to x) \to x) = (s(x \to y) \to s(y)) \wedge (s(y \to x) \to s(x)) = ((s(x) \to s(y)) \to s(y)) \wedge ((s(y) \to s(x)) \to s(x)) = s(x) \vee s(y)$. Finally, if $s(x \odot y) = s(x) \odot s(y)$ holds, for all $x, y \in H$, then by Proposition 3.12(iii), s preserves $\wedge$ and by the former argument, s preserves $\vee$.
(iv) Let $x \vee y = 1$. Since $x^n \leq s(x^n)$ and $x^n \leq s(x)^n$ for every $x \in H$, by Proposition 2.6(iii), $1 = x^n \vee y^n \leq s(x^n) \vee s(y^n)$ and $1 = x^n \vee y^n \leq s(x)^n \vee s(y)^n$. Thus, the qualities hold.
(v) Let $x \in B(H)$. Since s preserves the multiplication, and $0 \in B(H)$, according to Theorem 3.15(xii), $s(0) \in B(\mathbf{H})$. By Proposition 2.9, it can be easily shown that $(x \to s(0)) \to s(0) = (s(0) \to x) \to x$. Thus $x \vee s(0) = (x \to$

$s(0)) \to s(0)$. Since $x \in B(\mathbf{H})$, we have:

$$\begin{aligned} s(x^2) &= s(x) \\ &= s((x)'') \\ &= s((x \odot x)'') \\ &= s(((x \odot x) \to 0) \to 0) \\ &= s((x \to (x \to 0)) \to 0) \\ &= (s(x) \to (s(x) \to s(0))) \to s(0) \\ &= ((s(x) \odot s(x)) \to s(0)) \to s(0) \\ &= (x \to s(0)) \to s(0). \end{aligned}$$ □

Proposition 3.19. *Let* $\mathbf{H}$ *be a basic hoop algebra and* $s : H \to H$ *be a square root on* $\mathbf{H}$*. For every* $x \in H$*:*

(i) *If* $s(x)$ *is* $\wedge$*-irreducible, then* x *is* $\wedge$*-irreducible;*
(ii) *If* s *is onto, then the converse of (i) is also true;*
(iii) *If* $\mathbf{H}$ *is a* $\vee$*-hoop algebra and* $s(x)$ *is* $\vee$*-irreducible, then so is* x*.*

Proof. (i) Assume that $s(x)$ is $\wedge$-irreducible. Let $a, b \in H$ such that $x = a \wedge b$. Then, by Theorem 3.14(i), we have $s(x) = s(a \wedge b) = s(a) \wedge s(b)$. Since $s(x)$ is irreducible, we have $s(x) = s(a)$ or $s(x) = s(b)$. Now, by Theorem 3.7, we conclude that $x = a$ or $x = b$.
(ii) Let x be $\wedge$-irreducible and $a, b \in H$ such that $s(x) = a \wedge b$. Since s is onto, there are $u, v \in H$ such that $s(u) = a$ and $s(v) = b$. As $\mathbf{H}$ is basic, we have $s(x) = s(u) \wedge s(v) = s(u \wedge v)$. Now according to Theorem 3.7, $x = u \wedge v$. Since x is $\wedge$-irreducible, $x = u$ or $x = v$. Therefore, $s(x) = s(u) = a$ or $s(x) = s(v) = b$ and the result follows.
(iii) The proof is similar to (i). □

Remark 3.20. It is worth mentioning that if in the above proposition we are not assuming that $\mathbf{H}$ is basic, but s has the property $s(x \odot y) = s(x) \odot s(y)$, for all $x, y \in H$, then the proposition is true, as well. This is achieved by Proposition 3.12(iii).

Theorem 3.21. *Let* $\mathbf{H}$ *be a bounded* $\vee$*-hoop algebra with (DNP) and* $s : H \to H$ *be a square root on* $\mathbf{H}$*. Then for every* $x, y \in H$*:*

(i) $s(x)'' = s(x) = s(x'')$*;*
(ii) $s(x \odot y) = (s(x) \odot s(y)) \vee s(0)$*. In particular,* $x \vee s(0) = s(x^2)$*;*
(iii) $s((s(0)')^2) = s(0)'$*;*
(iv) $s(x')' \leq s(x)$*;*
(v) $s(x) \vee s(y) = s(x \vee y)$*;*
(vi) *If* $x \leq y$*, then* $s([x, y]) = [s(x), s(y)]$*. In particular,* $s(H) = [s(0), s(1)]$*.*

Proof. (i) The proof is straightforward.
(ii) By Theorem 3.10(vi), for all $x, y \in H$ we have $s(x \odot y) = s((x \odot y)'') =$

$s(((x \odot y) \to 0) \to 0) = s((x \to (y \to 0)) \to 0) = (s(x) \to (s(y) \to s(0))) \to s(0) = ((s(x) \odot s(y)) \to s(0)) \to s(0)$. By (DNP), we have $(s(x) \odot s(y)) \vee s(0) = ((s(x) \odot s(y)) \to s(0)) \to s(0)$. Thus, $s(x \odot y) = (s(x) \odot s(y)) \vee s(0)$. Now we have $s(x \odot x) = (s(x) \odot s(x)) \vee s(0) = x \vee s(0)$.
(iii) Using (ii) and Theorem 3.15(xiv) we have:

$$\begin{aligned} s((s(0) \to 0) \odot (s(0) \to 0)) &= (s(s(0) \to 0) \odot s(s(0) \to 0)) \vee s(0) \\ &= (s(0) \to 0) \vee s(0) = s(0) \to 0. \end{aligned}$$

(iv) By Theorem 3.15(viii), the proof is straightforward.
(v) By Corollary 2.19 and Theorem 3.18(iii), the proof is straightforward.
(vi) Let $x \leq y$. By Theorem 3.10(x), $s([x, y]) \subseteq [s(x), s(y)]$. Let $z \in H$ such that $s(x) \leq z \leq s(y)$. By (ii), $z = s(z \odot z)$ since $s(0) \leq s(x) \leq z$. Now, we have $x = s(x) \odot s(x) \leq z \odot z \leq s(y) \odot s(y) = y$. Therefore, $z = s(z \odot z) \in s([x, y])$. □

Definition 3.22. Let **H** be a hoop algebra and $n \in \mathbb{N}$. We say that **H** has an n-th root if there exists a unary operation $r_n : H \to H$ satisfying the following properties:

(NS1) for all $x \in H$, $r_n(x)^n = x$;

(NS2) for all $x, y \in H$, if $y^n \leq x$, then $y \leq r_n(x)$.

For the first and second roots, one has $r_1(x) = x, r_2(x) = s(x)$. Therefore, $r_2(x) \odot r_2(x) = s(x) \odot s(x) = x$.

Theorem 3.23. *Let* **H** *be a hoop algebra, and $r_n : H \to H$ and $r_m : H \to H$ be the n-th and m-th roots in* **H** *for $n, m \in \mathbb{N}$. Then for every $x, y \in H$, the following properties hold:*

(i) $x \leq r_n(x)$;
(ii) $x \leq y$ *if and only if* $r_n(x) \leq r_n(y)$;
(iii) $x \leq r_n(x^n)$;
(iv) $x \wedge y \leq r_n(x \wedge y) \leq r_n(x) \wedge r_n(y)$;
(v) $r_n(x) \odot r_n(y) \leq r_n(x \odot y)$;
(vi) $r_n(x) \to r_n(y) = r_n(x \to y)$;
(vii) $r_n(x) \leq r_n(x^{n-1}) \to r_n(x^n)$;
(viii) $r_n(x)^m \leq r_n(x^m)$;
(ix) *If* $m \leq n$, *then* $r_m(x)^m \leq r_n(x)^m$;
(x) *If* $a \in \mathrm{Id}(H)$, *then the operation* $r_{n_a} : H[a] \to H[a]$ *induced by* r_n, *is an n-th root on* $H[a]$.

Proof. (i) By the condition $(NS1)$ of Definition 3.22, we have $r_n(x)^n = x$. Therefore, $x \leq r_n(x)$.
(ii) According to the condition $(NS2)$ of Definition 3.22, since $r_n(x)^n = x \leq y$, we have $r_n(x) \leq r_n(y)$. The converse is clear.
(iii) Using $x^n = \underbrace{x \odot x \odot \cdots \odot x}_{n-\text{times}}$ and the condition $(NS2)$ of Definition 3.22, we have $x \leq r_n(x^n)$.
(iv) The proof follows easily from (i) and (ii).

(v) We have $(r_n(x) \odot r_n(y))^n = (r_n(x))^n \odot (r_n(y))^n = x \odot y$. By (NS2) of Definition 3.22, $r_n(x) \odot r_n(y) \leq r_n(x \odot y)$.
(vi) We have:

$$\begin{aligned} x \odot (r_n(x) \to r_n(y))^n &= (r_n(x))^n \odot (r_n(x) \to r_n(y))^n \\ &= (r_n(x) \odot (r_n(x) \to r_n(y))) \odot \cdots \odot (r_n(x) \odot (r_n(x) \to r_n(y))) \\ &= (r_n(x) \wedge r_n(y)) \odot \cdots \odot (r_n(x) \wedge r_n(y)) \leq \underbrace{r_n(y) \odot \cdots \odot r_n(y)}_{n-\text{times}} = y. \end{aligned}$$

Therefore, by Theorem 2.3(ii), $(r_n(x) \to r_n(y))^n \leq x \to y$. In other words, by (NS2) of Definition 3.22, $r_n(x) \to r_n(y) \leq r_n(x \to y)$. Moreover, by (v), we have $r_n(x) \odot r_n(x \to y) \leq r_n(x \odot (x \to y)) = r_n(x \wedge y) \leq r_n(y)$. Therefore, by Theorem 2.3(ii), $r_n(x \to y) \leq r_n(x) \to r_n(y)$. Hence, $r_n(x \to y) = r_n(x) \to r_n(y)$.
(vii) The proof is straightforward.
(viii) Let $n, m \in \mathbb{N}$. By (v), we have $r_n(x)^m = \underbrace{r_n(x) \odot r_n(x) \odot \cdots \odot r_n(x)}_{m-\text{times}} \leq r_n(x^m)$.
(ix) According to (NS1) of Definition 3.22 we have $r_m(x)^m = x = r_n(x)^n = r_n(x)^{n-m} \odot r_n(x)^m \leq r_n(x)^m$.
(x) The proof is straightforward. □

Example 3.24. *a) Considering Example 3.2(a), in* $\mathbf{G_a}$ *we define the unary operation* $r_n : G_a \to G_a$ *as* $r_n(a^{\frac{u}{v}}) = a^{\frac{u}{nv}}$, *for all* $a^{\frac{u}{v}} \in G_a$. *It is easily checked that the conditions of Definition 3.22 hold. For instance, for the condition (NS1), we have* $r_n(a^{\frac{u}{v}})^n = a^{\frac{u}{nv}} \odot a^{\frac{u}{nv}} \cdots \odot a^{\frac{u}{nv}} = a^{\frac{u}{nv} + \frac{u}{nv} + \cdots + \frac{u}{nv}} = a^{\frac{u}{v}}$.
b) In Example 3.2(b), the unary operation $r_n : I \to I, r_n(x) = x$ *for every* $x \in I$ *is an n-th root on* **Goa**.
c) According to Example 3.2(c), in **Pra**, *we define the unary operation* $r_n : I \to I$ *as* $r_n(x) = \sqrt[n]{x}$ *for every* $x \in I$. *It can be easily checked that the conditions of Definition 3.22 are satisfied. For instance, for condition (NS1), we have* $r_n(x)^n = (\sqrt[n]{x})^n = x$. *It is evident that the condition (NS2) also holds.*

Proposition 3.25. *Let* $\mathbf{H}$ *be a hoop algebra and* $r_n : H \to H$ *be an n-th root on* $\mathbf{H}$ *for* $n \in \mathbb{N}$. *The following statements hold:*

(i) *The n-th root is one to one;*
(ii) $r_n(1) = 1$.

Proof. (i) For any $x, y \in H$, if $r_n(x) = r_n(y)$, then $x = r_n(x)^n = r_n(y)^n = y$.
(ii) By Theorem 3.23(i), the proof is straightforward. □

Theorem 3.26. *Let* $\mathbf{H}$ *and* $\mathbf{G}$ *be two hoop algebras, and let* $f : H \to G$ *be an isomorphism. Then, for* $n \in \mathbb{N}$, *if* $r_n : H \to H$ *is an n-th root on* $\mathbf{H}$, *then* $\mathbf{G}$ *has an n-th root also.*

Proof. Suppose $a \in G$. Since f is an isomorphism, there exists a unique $x \in H$ such that $f(x) = a$. We claim that $s_n : G \to G$ defined by $s_n(a) = f(r_n(x))$ is an n-th root on G. As f is an isomorphism, it is clear that s_n is well-defined. Now for any $a \in G$, we have $(s_n(a))^n = (f(r_n(x)))^n = f((r_n(x)^n) = f(x) = a$. On the other hand, for any $a, b \in G$ such that $b \odot b \odot \ldots \odot b \leq a$, there exist unique elements $x, y \in H$ such that $f(x) = a$ and $f(y) = b$. Therefore, $f(y) \odot f(y) \odot \ldots \odot f(y) \leq f(x)$. Since f is an isomorphism, we have $f(y) \odot f(y) \odot \ldots \odot f(y) \leq f(x) \Rightarrow f(y \odot y \odot \ldots \odot y) \leq f(x) \Rightarrow y \odot y \odot \ldots \odot y \leq x \Rightarrow y \leq r_n(x) \Rightarrow f(y) \leq f(r_n(x)) \Rightarrow b \leq s_n(a)$. □

Proposition 3.27. *Let $f : H \to G$ be a monomorphism of hoop algebras, and for $n \in \mathbb{N}$, let r_n be an n-th root on $\mathbf{H}$. Then, for each $x \in H$, the map $s_n : im(f) \to im(f)$ defined by $s_n(f(x)) = f(r_n(x))$ is an n-th root on $im(f)$.*

Proof. It is clear that $\mathbf{im(f)}$ is a hoop subalgebra of $\mathbf{G}$. Now, as $\mathbf{H}$ is isomorphic to $\mathbf{im(f)}$, according to Theorem 3.26, $s_n : \mathrm{im}(f) \to \mathrm{im}(f)$ is an n-th root on $\mathbf{im(f)}$. □

4. Applications of square roots in hoop algebras

In this section, a brief discussion on the application of the square root in hoop algebras will be presented. Additionally, we will show that the class of all hoop algebras with square roots is a variety.

Proposition 4.1. *Let $\mathbf{H}$ be a hoop algebra, F a filter on $\mathbf{H}$ and $s : H \to H$ be a square root on $\mathbf{H}$. Then*

(i) $s(F) \subseteq F$,
(ii) *If $s(x) \odot s(y) = s(x \odot y)$, for all $x, y \in H$, then $s(F) = F$.*

Proof. (i) Using condition (F3) of Definition 2.12 and Theorem 3.5(i), the result is obtained.
(ii) Let $x \in F$. Then $s(x) \in s(F)$. By the assumption, we have $x = s(x) \odot s(x) = s(x \odot x) \in s(F)$. Hence $F \subseteq s(F)$. According to (i), we get the result. □

Proposition 4.2. *Let $s : H \to H$ be a square root on a hoop algebra $\mathbf{H}$ such that $s(x) \odot s(y) = s(x \odot y)$, for all $x, y \in H$. Then $s(\mathbf{H})$ is a subalgebra of $\mathbf{H}$.*

Proof. It suffices to examine the conditions of Definition 2.12. The operations $\odot$ and $\to$ on $s(H)$ are those operations of H restricted to $s(H)$. It is evident that $1 \in s(H)$. Let $x, y \in H$. It is clear that $x \odot y \in H$. Also, $s(x), s(y) \in s(H)$. Now, by the assumption, we have $s(x) \odot s(y) = s(x \odot y) \in s(H)$. Therefore, $s(x) \odot s(y) \in s(H)$. Furthermore, by Theorem 3.10(vi), $s(x) \to s(y) = s(x \to y) \in s(H)$. Thus, $\mathbf{s(H)}$ is a subalgebra of $\mathbf{H}$. □

Proposition 4.3. *Let $\mathbf{H}$ be a hoop algebra, F a filter of $\mathbf{H}$, and $s : H \to H$ is a square root on $\mathbf{H}$. If $s(x) \odot s(y) = s(x \odot y)$, for every $x, y \in H$, then $s(F)$ is a filter of $\mathbf{H}$ and $\mathbf{s(H)}$.*

Proof. We examine the conditions of Definition 2.12. Since $1 \in F$, we have $s(1) = 1 \in s(F)$. Therefore, the condition (F1) holds. Let $s(x), s(y) \in s(F)$. This implies $x, y \in F$. Since F is a filter, $x \odot y \in F$, hence $s(x \odot y) \in s(F)$. Now, according to the assumption, $s(x) \odot s(y) \in s(F)$. Therefore, the condition (F2) holds. Next, let $s(x) \in s(F)$ and $y \in H$ such that $s(x) \leq y$. By Theorem 3.5(i), $x \leq y$. Since F is a filter and $y \in F$, so $s(y) \in s(F)$. Therefore, the condition (F3) holds. Hence, $s(F)$ is a filter of $\mathbf{H}$. It is also easy to check that $s(F)$ is a filter of $s(\mathbf{H})$. □

Proposition 4.4. *Let* $\mathbf{H}$ *be a bounded hoop algebra,* F *a filter of* $\mathbf{H}$*, and* $s : H \to H$ *be a square root on* $\mathbf{H}$*. If* $s(x) \odot s(y) = s(x \odot y)$*, for every* $x, y \in H$ *then, the map* $s_F : \mathbf{H}/F \to \mathbf{H}/F$*, defined by* $s_F(x/F) = s(x)/F$ *for every* $x \in H$*, is a square root on* $\mathbf{H}/F$*.*

Proof. It is enough to check the conditions of Definition 3.1. For condition (S1), according to Definition 2.12, we have $s(x)/F \odot s(x)/F = (s(x) \odot s(x))/F = x/F$, for every $x \in H$. For condition (S2), let $x, y \in H$ such that $y/F \odot y/F \leq x/F$. By the definition of partial order on $\mathbf{H}/F$ one has $y^2 \to x \in F \Rightarrow s(y^2 \to x) \in F \Rightarrow s(y^2) \to s(x) \in F \Rightarrow s(y^2)/F \leq s(x)/F \Rightarrow y/F \leq s(x)/F$, and we are done. □

Theorem 4.5. *Let* $\mathbf{H}$ *be a bounded hoop algebra,* F *a filter of* $\mathbf{H}$*, and* $s : H \to H$ *be a square root on* $\mathbf{H}$*. If* $s(x) \odot s(y) = s(x \odot y)$*, for every* $x, y \in H$ *then,* $s_F(\mathbf{H}/F) \cong s(\mathbf{H})/s(F)$*.*

Proof. By Proposition 4.2, $\mathbf{s}(\mathbf{H})$ is a subalgebra of $\mathbf{H}$. Also according to Proposition 4.3, $s(F)$ is a filter of $\mathbf{s}(\mathbf{H})$ and then according to the paragraph before Definition 2.13, $s_F(\mathbf{H}/F)$ is a hoop algebra. In addition, by Proposition 4.4 we know that $s(\mathbf{H}/F)$ is a subalgebra of the hoop algebra $\mathbf{H}/F$. Now, define the map $f : s_F(\mathbf{H}/F) \to s(\mathbf{H})/s(F)$ by the rule $f(s_F[x]) = [s(x)]$, for every $s_F[x] \in s_F(\mathbf{H}/F)$. We show that f is an isomorphism. Let $x, y \in H$ and $s_F[x] = s_F[y]$. According to Theorem 3.7, we have $[x] = [y]$. Therefore, $x \to y \in F$ and $y \to x \in F$. By Definition 2.12 and Proposition 4.1, we have $s(x \to y) \in F$ and $s(y \to x) \in F$. Now, by Theorem 3.10(vi), we conclude that $s(x) \to s(y) \in F$ and $s(y) \to s(x) \in F$. Thus, $[s(x)] = [s(y)]$. Thus f is well-defined. It is clear that f is onto. To see it is one to one, let $f(s_F[x]) = f(s_F[y])$. Then $s(x)/s(F) = s(y)/s(F)$ which is equivalent to $s(x) \to s(y), s(y) \to s(x) \in s(F)$. Now we get $x \to y, y \to x \in F$. So $[x] = [y]$ and then $s_F[x] = s_F[y]$. Furthermore, we have $f(s_F[x] \to s_F[y]) = f(s_F[x \to y]) = [s(x \to y)] = [s(x) \to s(y)] = [s(x)] \to [s(y)] = f(s_F[x]) \to f(s_F[y])$ and $f(s_F[x] \odot s_F[y]) = f(s_F[x \odot y]) = [s(x \odot y)] = [s(x) \odot s(y)] = [s(x)] \odot [s(y)] = f(s_F[x]) \odot f(s_F[y])$. Finally, $f(s_F[1]) = [s(1)] = [1]$. Therefore, f is an isomorphism. □

Theorem 4.6. *Let* $s : H \to H$ *be a square root on a hoop algebra* $\mathbf{H}$ *and* $x \in H$*. Then* $\langle x \rangle = \langle s(x) \rangle = \langle s(x)^2 \rangle = \cdots$ *.*

Proof. By Theorem 3.5, we have $x \leq s(x)$. Therefore, by Definition 2.12, $s(x) \in \langle x \rangle$. Hence, $\langle s(x) \rangle \subseteq \langle x \rangle$. On the other hand, we know that $x = s(x) \odot s(x) \in \langle s(x) \rangle$. Then, $\langle x \rangle \subseteq \langle s(x) \rangle$. Therefore, $\langle x \rangle = \langle s(x) \rangle$. Consequently, $\langle x \rangle = \langle s(x) \rangle = \langle s(x)^2 \rangle = \cdots$. □

Theorem 4.7. *Let* $\mathbf{H}$ *be a bounded* $\vee$*-hoop algebra, and* $s : H \to H$ *be a square root on* $\mathbf{H}$*. In addition, let* $\mathbf{H}$ *be basic or* $s(x \odot y) = s(x) \odot s(y)$*, for all* $x, y \in H$*, then, a proper filter* F *of* $\mathbf{H}$ *is prime if and only if for every* $x, y \in H$*,* $s(x) \vee s(y) \in F$*, implies that* $s(x) \in F$ *or* $s(y) \in F$*.*

Proof. Let us first assume that $\mathbf{H}$ is a basic hoop. If F is a prime filter, then clearly the implication holds. Now, assume that F is a proper filter and $x \vee y \in F$. Then $s(x \vee y) \in F$. Since $\mathbf{H}$ is basic, we have $s(x) \vee s(y) \in F$. By the assumption, $s(x) \in F$ or $s(y) \in F$. Therefore, by the condition (F2) in Definition 2.12 and (S1) of Definition 3.1, $x \in F$ or $y \in F$. Thus, F is prime.

Let us now assume that $s(x \odot y) = s(x) \odot s(y)$, for all $x, y \in H$. According to Proposition 3.18(iii), the assertion follows. □

Proposition 4.8. *Let* $\mathbf{H}$ *be a bounded hoop algebra,* $s : H \to H$ *a square root on* $\mathbf{H}$*, and* F *be a proper filter of* $\mathbf{H}$*. Then, for every* $n \in \mathbb{N}$ *and* $x \in H$ *such that* $x \leq s(0)$ *we have*

(i) $s(x)^n \notin F$*;*
(ii) $s^n(x) \notin F$ *where* $s^n(x) = s(\cdots(s(x)))$ $(n - \text{times})$.

Proof. (i) Consider the assumption $x \leq s(0)$. For $n = 1$, if $s(x) \in F$, then $x = s(x) \odot s(x) \in F$, which implies that $x \odot x \in F$. Since $x \odot x \leq 0$, then $0 \in F$, is a contradiction with F being a proper filter. For $n \geq 2$, if $s(x)^n \in F$, then $s(x)^n = s(x)^2 \odot s(x)^{n-2} = 0 \in F$, is a contradiction.
(ii) This is clear. □

Let $\mathbf{H}$ be a bounded hoop algebra and $s : H \to H$ be a square root on $\mathbf{H}$. For every $x \in H$, the set S_x associated with x is defined by $S_x = \{s^n(x) : n \in \mathbb{N}\} = \{x, s(x), s^2(x), \cdots\}$.

Theorem 4.9. *Let* $\mathbf{H}$ *be a bounded hoop algebra and* $s : H \to H$ *be a square root on* $\mathbf{H}$*. Put* $S = \bigcup_{x \leq s(0)} S_x$ *and* $P = H - S$*. Then,* P *either is a maximal filter of* $\mathbf{H}$ *or* $\langle P \rangle = H$*.*

Proof. We distinguish two cases.
1) P is a filter of $\mathbf{H}$. If P is not maximal, then there exists a proper filter F of H such that $P \subsetneq F$ and hence choose $x \in F - P$. As $x \in S$, there exist $y \in H$ and $k \in \mathbb{N}$ such that $y \leq s(0)$ and $s^k(y) = x$. Thus, $x \leq s^{k+1}(0)$, because s is increasing by Proposition 3.10(i). By a simple calculation, we conclude that there is a positive integer m for which $x^m = 0$, so $0 \in \langle x \rangle$ and then $\langle x \rangle = H$. As $x \in F$, $\langle x \rangle \subseteq F$. Therefore, $F = H$, a contradiction. Therefore, P is a maximal filter and we are done.
2) The subset P of H is not a filter of $\mathbf{H}$. In this case, $P \neq \langle P \rangle$ and we

replace the filter F, as in (1), by $\langle P\rangle$ and repeat the proof of last part to obtain $\langle P\rangle = H$, as required. □

Let $\mathbf{H}$ and $\mathbf{G}$ be two hoop algebras, and $f : H \to G$ be a hoop homomorphism. Suppose $s : H \to H$ and $t : G \to G$ are square roots on $\mathbf{H}$ and $\mathbf{G}$, respectively. One has $f(s(x)) \odot f(s(x)) = f(x)$, for every $x \in H$. According to condition (S2), it is clear that $f(s(x)) \leq t(f(x))$. We say that f *preserves square roots* if $f(s(x)) = t(f(x))$, for every $x \in H$.

Theorem 4.10. *Let $f : H \to G$ be a hoop homomorphism and $s : H \to H$ and $t : G \to G$ are square roots on $\mathbf{H}$ and $\mathbf{G}$, respectively. Let $x \in H$. Then the following statements are true.*

(i) *f preserves square roots if and only if the image of f, $Im(f)$, is closed under t;*
(ii) *If f is an isomorphism, then $t = f \circ s \circ f^{-1}$;*
(iii) *If $t(f(x))$ is $\wedge$-irreducible and f is a one to one which preserves square roots, then $s(x)$ is $\wedge$-irreducible;*
(iv) *If $\mathbf{H}$ and $\mathbf{G}$ are two $\vee$-hoop algebras and $t(f(x))$ is $\vee$-irreducible and f is one to one which preserves square roots, then $s(x)$ is $\vee$-irreducible.*

Let f preserves square roots. Then one has

(v) *If $s(x) \in N(\mathbf{H})$, then $t(f(x)) \in N(\mathbf{G})$;*
(vi) *If $s(x) \in D(\mathbf{H})$, then $t(f(x)) \in D(\mathbf{G})$;*
(vii) *If $s(x) \in \mathrm{reg}(\mathbf{H})$, then $t(f(x)) \in \mathrm{reg}(\mathbf{G})$.*

If f is a one to one, then the converse of items (v), (vi) and (vii) are true, as well.

Proof. The proof of (i) and (ii) are similar to the proof of [12, Theorem 3.14].
(iii) Suppose $s(x) \neq 0$ and $a, b \in H$ such that $s(x) = a \wedge b$. Now we have $t(f(x)) = f(s(x)) = f(a \wedge b) = f(a) \wedge f(b)$. Since $t(f(x))$ is $\wedge$-irreducible, it follows that $f(s(x)) = f(a)$ or $f(s(x)) = f(b)$. Due to f is one to one, we have either $s(x) = a$ or $s(x) = b$. Therefore, $s(x)$ is also $\wedge$-irreducible.
(iv) The proof is similar to (iii).
(v) We have the implications: $s(x) \in N(\mathbf{H}) \Longrightarrow s(x)^n = 0 \Longrightarrow 0 = f(s(x)^n) = t(f(x))^n \Longleftrightarrow t(f(x)) \in N(\mathbf{H})$.
(vi) and (vii) are straightforward. Moreover, if f is one to one, then it can be easily checked that the converse of items (v), (vi) and (vii) hold. □

Corollary 4.11. *Let $\mathbf{H}$ and $\mathbf{G}$ be two $\vee$-hoop algebras, and let $f : H \to G$ be a hoop homomorphism. Suppose $s : H \to H$ and $t : G \to G$ are square roots on $\mathbf{H}$ and $\mathbf{G}$, respectively. If $\mathbf{G}$ is a basic hoop algebra and f preserves square roots, then for every $x, y \in H$, $f(s(x \wedge y)) = f(s(x)) \wedge f(s(y))$ and $f(s(x \vee y)) = f(s(x)) \vee f(s(y))$.*

Proof. Since $\mathbf{G}$ is a basic hoop algebra, in view of Theorems 3.14(i) and 3.18(iii), we have $f(s(x\wedge y)) = t(f(x\wedge y)) = t(f(x)\wedge f(y)) = t(f(x))\wedge t(f(y)) = f(s(x)) \wedge f(s(y))$. Similarly, the other equality is obtained. □

Definition 4.12. Let $s : H \to H$ be a square root on a bounded hoop algebra $\mathbf{H}$. If $s(0) = 0$, then $\mathbf{H}$ is called *good*.

As shown in Proposition 3.12(iv), if for all $x, y \in H$, we have $s(x \odot y) = s(x) \odot s(y)$, or $s(x^2) = x$, then $\mathbf{H}$ is good.

Corollary 4.13. *Let $f : H \to G$ be a hoop homomorphism between two hoop algebras $\mathbf{H}$ and $\mathbf{G}$ and $s : H \to H$ and $t : G \to G$ are square roots. If $\mathbf{H}$ is good and f preserves square roots, then $\mathbf{G}$ is also good.*

Example 4.14. *All the hoop algebras defined in Examples 3.2, are good.*

Example 4.15. *a) Let $I = [0, 1]$ be the unit closed interval of real numbers and $p \in \mathbb{N}$. Define the operations $\odot$ and $\to$ as $a \odot b = (\max(0, a^p + b^p - 1))^{\frac{1}{p}}, a \to b = \min(1, (1 - a^p + b^p)^{\frac{1}{p}})$, for all $a, b \in I$. Then, $\mathbf{L}_p = (I; \max, \min, \odot, \to, 0, 1)$ is a residuated lattice, which is called a generalized Łukasiewicz structure (see also [17, Example 1.1.2]). Assuming $p = 1$, the resulting structure is called a Łukasiewicz algebra, which is a hoop algebra. In this Łukasiewicz algebra, we have $a \odot b = \max(0, a + b - 1), a \to b = \min(1, 1 - a + b)$. Define the unitary operation $s : I \to I$ by $s(x) = \frac{x+1}{2}$, for every $x \in I$. Now, $s(x) \odot s(x) = \max(0, s(x) + s(x) - 1) = \max(0, x) = x$. Thus, the condition (S1) in Definition 3.1 holds. It can easily be shown that the condition (S2) is also satisfied. For, suppose $x, y \in I$ such that $y \odot y \leq x$. In this case, we have $y \odot y = \max(0, y + y - 1) \leq x$. If $y \odot y = 0 \leq x$, then according to Theorem 3.10(v), $s(y) \odot s(y) \leq s(y \odot y) \leq s(x)$, implying $y \leq s(x)$. If $y \odot y = y + y - 1 \leq x$, then $2y - 1 \leq x$, hence $y \leq \frac{x+1}{2} = s(x)$. However, we have $s(0) = 0.5$. Therefore, the Łukasiewicz algebra is not good. Finally, $s(x) \odot s(y) = s(x \odot y)$ does not hold.*

b) Let $\mathbf{G}$ be a 2-divisible Abelian l-group. As shown in Example 2.2(c), $\mathbf{\Gamma}(\mathbf{G}, \mathbf{u}) = ([0, u]; \odot, \to, u)$ is a bounded hoop algebra with the operations $x \odot y = (x + y - u) \vee 0, x \to y = (y - x + u) \wedge u$. In this algebra, we define the square root $s : [0, u] \to [0, u]$ by $s(x) = \frac{x+u}{2}$ for every $x \in [0, u]$. It is clear that $s(x) \in [0, u]$. Now, we have $s(x) \odot s(x) = (\frac{x+u}{2} + \frac{x+u}{2} - u) \vee 0 = (x + u - u) \vee 0 = x$. Thus, the condition (S1) of Definition 3.1 holds. It can easily be verified that the condition (S2) is also satisfied. Therefore, s is a square root on $\mathbf{\Gamma}(\mathbf{G}, \mathbf{u})$. It is worth noting that an Abelian group may be 2-divisible but not n-divisible for $n \geq 3$ in general; hence, an n-th root need not exist on $\mathbf{\Gamma}(\mathbf{G}, \mathbf{u})$. Here also, $s(0) = \frac{u}{2}$. Therefore, $\mathbf{\Gamma}(\mathbf{G}, \mathbf{u})$ is not good. Further, $s(x) \odot s(y) = s(x \odot y)$ does not hold.

Example 4.16. *a) In the hoop algebra in the former example* $\mathbf{L}_1$*, define the unitary operation* $r_n : I \to I$ *by* $r_n(x) = \frac{x+n-1}{n}$*, for every* $x \in I$*. Now,*

$$\underbrace{r_n(x) \odot r_n(x) \cdots \odot r_n(x)}_{n-times} = \underbrace{\frac{x+(n-1)}{n} \odot \frac{x+(n-1)}{n} \odot \cdots \odot \frac{x+(n-1)}{n}}_{n-times}$$
$$= (2(\frac{x+(n-1)}{n}) - 1) \odot \underbrace{\frac{x+(n-1)}{n} \odot \cdots \odot \frac{x+(n-1)}{n}}_{(n-2)-times} =$$
$$n(\frac{x+(n-1)}{n}) - (n-1) = x$$

Thus, the condition (S1) in Definition 3.22 holds. It can easily be shown that the condition (S2) is also satisfied. For, suppose $x, y \in I$ *such that* $y^n \leq x$*. In this case, we have* $y^n = \max(0, ny - (n-1)) \leq x$*. If* $y^n = 0 \leq x$*, it is evident. Because, according to Theorem 3.23(v),* $y = r_n(y) \odot r_n(y) \odot \cdots \odot r_n(y) \leq r_n(y^n) \leq r_n(x)$*. If* $y^n = ny - (n-1) \leq x$*, then* $ny \leq x + (n-1)$*, hence* $y \leq \frac{x+(n-1)}{n} = r_n(x)$*. Therefore,* r_n *is a n-th root root on* $\mathbf{L}_1$*.*

b) Let $n \geq 3$ *and* $\mathbf{G}$ *be a n-divisible Abelian l-group. In the algebra* $\mathbf{\Gamma(G, u)}$*, defined in Example 2.2(c), we define the n-th root* $r_n : [0, u] \to [0, u]$ *by* $r_n(x) = \frac{x+(n-1)u}{n}$ *for every* $x \in [0, u]$*. It is clear that* $r_n(x) \in [0, u]$*. Now, we have:*

$$\underbrace{r_n(x) \odot r_n(x) \cdots \odot r_n(x)}_{n-times} = \underbrace{\frac{x+(n-1)u}{n} \odot \frac{x+(n-1)u}{n} \odot \cdots \odot \frac{x+(n-1)u}{n}}_{n-times}$$
$$= (2(\frac{x+(n-1)u}{n}) - 2u) \odot \underbrace{\frac{x+(n-1)u}{n} \odot \cdots \odot \frac{x+(n-1)u}{n}}_{(n-2)-times} =$$
$$n(\frac{x+(n-1)u}{n}) - (n-1)u = x$$

Thus, the condition (NS1) of Definition 3.22 holds. It can easily be verified that the condition (NS2) is also satisfied. Therefore, r_n *is a n-th root root on* $\mathbf{\Gamma(G, u)}$*. Note that* $\mathbf{\Gamma(G, u)}$ *has no square root.*

We call a bounded hoop algebra $\mathbf{H}$ regular if for every $x, y \in H$ such that $x \neq 0$ and $y \neq 0$, then $x \wedge y \neq 0$.

Theorem 4.17. *Let* $\mathbf{H}$ *be a good bounded hoop algebra. Then, for every* $x \in H$*:*

(i) $x \wedge x' = 0$*;*
(ii) $s(x) \odot s(x') = 0$*;*
(iii) *If, in addition, H is a* $\vee$*-hoop algebra with (DNP), then* $s(x^2) = x$*;*
(iv) $x \in \mathrm{reg}(\mathbf{H})$ *if and only if* $s(x) \in \mathrm{reg}(\mathbf{H})$*;*
(v) *If* $\mathbf{H}$ *is basic, then* $\mathbf{H}$ *is regular if and only if for every* $x, y \in H$*,* $s(x) \neq 0$ *and* $s(y) \neq 0$*, imply* $s(x \wedge y) \neq 0$*;*

(vi) $s(x \odot y) \leq (s(x) \odot s(y))''$. *If* $\mathbf{H}$ *has (DNP), then* $s(x \odot y) = s(x) \odot s(y)$.

Proof. By Theorem 3.15(iii,vii), the proofs of (i) and (ii) are straightforward.
(iii) According to Theorem 3.21(ii), the proof is straightforward.
(iv) Let us assume $x \in \mathrm{reg}(H)$. By Theorem 3.10(vi), we have $s(x)'' = (s(x) \to 0) \to 0 = (s(x) \to s(0)) \to s(0) = s((x \to 0) \to 0) = s(x'') = s(x)$. Conversely, let's assume $s(x) \in \mathrm{reg}(H)$. By Theorem 3.10(vi), we have $s(x) = s(x)'' = (s(x) \to 0) \to 0 = (s(x) \to s(0)) \to s(0) = s(x'')$. As s is one to one, $x = x''$, hence $x \in \mathrm{reg}(H)$.
(v) Let H be a basic hoop. The *only if* part is evident. To prove the *if* part, let $x, y \in H$ be two non-zero elements. Since $s(0) = 0$ we have $s(x) \neq 0$ and $s(y) \neq 0$. As H is basic, $0 \neq s(x) \wedge s(y) = s(x \wedge y)$. Therefore, $x \wedge y \neq 0$.
(vi) Due to Theorems 3.15(xiii) and 3.21(ii), the proof is clear. □

Theorem 4.18. *Let* $s : H \to H$ *be a square root on a bounded hoop algebra* $\mathbf{H}$. *Then:*

(i) *If* $H = \mathrm{Id}(\mathbf{H})$*, then* $\mathbf{H}$ *is good;*
(ii) *If* $\mathbf{H}$ *satisfies (DNP) and is good, then* $H = \mathrm{Id}(\mathbf{H})$.

Proof. (i) By Proposition 3.7, the proof is straightforward.
(ii) Let's assume $\mathbf{H}$ is good and $x \in H$. By Theorem 3.15(v) and (DNP), we obtain $s(x) \leq x' \to 0 = (x \to 0) \to 0 = x$. Therefore, for every $x \in H$, $s(x) = x$. Thus, by Proposition 3.7, we have $H = \mathrm{Id}(H)$. □

Theorem 4.19. *Let* $\mathbf{H}$ *be a bounded* $\vee$*-hoop algebra with (DNP) and* $s : H \to H$ *be a square root on* $\mathbf{H}$. *Then* $\mathbf{H}$ *is good if and only if* $\mathbf{H}$ *is a Boolean algebra.*

Proof. By [7, Theorem 3.13], the $\vee$-bounded hoop algebra $\mathbf{H} = (H; \odot, \to, 0, 1)$ with (DNP) is equivalent to the MV-algebra $(H; \oplus, 0)$, where $x \oplus y = x' \to y$. Now, by [14, Proposition 2.19], $\mathbf{H}$ is a Boolean algebra if and only if it is good. □

Remark 4.20. Let $\mathbf{H}$ be a bounded $\vee$-hoop algebra with (DNP) and $s : H \to H$ a square root on $\mathbf{H}$.
(i) If $s(x) \odot s(y) = s(x \odot y)$ for every $x, y \in H$, then by Proposition 3.12(iv) and Theorem 4.19, $\mathbf{H}$ is a Boolean algebra.
(ii) In light of Proposition 3.12, Theorem 4.17(iii) and Theorem 4.19, for all $x \in H$, $x = s(x^2)$ if and only if $\mathbf{H}$ is a Boolean algebra.

A hoop algebra $\mathbf{H}$ is said to satisfies the *ascending chain condition* (ACC) if for every sequence $x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots$ of elements in H, there is an integer k such that $x_i = x_k$ for all $i \geq k$. $\mathbf{H}$ is said to satisfies the *descending chain condition* (DCC) if for every sequence $x_1 \geq x_2 \geq \cdots \geq x_n \geq x_{n+1} \geq \cdots$ of elements in H, there is an integer k such that $x_i = x_k$ for all $i \geq k$.

Theorem 4.21. *Let* $\mathbf{H}$ *be a finite bounded hoop algebra and* $s : H \to H$ *be a square root on* $\mathbf{H}$. *Then the following statements hold:*

(i) *The map* $s : H \to H$ *is onto;*

(ii) *For every* $x \in H$, $s(x) = x$, *and thus* **H** *is good.*

Proof. (i) It is clear that $s(H) \subseteq H$. Moreover, since s is one to one and $|s(H)| = |H|$, we have $s(H) = H$. Therefore, $s : H \to H$ is onto.
(ii) Suppose $x \in H$ such that $x < s(x)$. If $s(x) = s(s(x))$, then as s is one to one, $x = s(x)$, which is a contradiction. Thus, $s(x) < s(s(x)) = s^2(x)$. Similarly, $s^2(x) < s^3(x)$. Hence, we obtain an ascending sequence of elements in H as $x < s(x) < s^2(x) < \dots$ By the assumption of the finiteness of H, there exists $n \in \mathbb{N}$ such that $s^n(x) = s^{n+1}(x) = \dots$. Therefore, by Proposition 3.7(ii), we have $x = s(x)$, which is a contradiction. Thus, for every element in H, $s(x) = x$. Therefore, **H** is good. □

Proposition 4.22. *Let* **H** *be a bounded hoop algebra and* $s : H \to H$ *be a square root on* **H**. *If* **H** *satisfies* (ACC), *then for every* $x \in H$, $x^2 = x$.

Proof. Similar to the proof of Theorem 4.21(ii), for every $x \in H$, we have $s(x) = x$. Therefore, by Proposition 3.7(ii), for every $x \in H$, $x^2 = x$. □

Remark 4.23. It is known that idempotent hoops are the $\{\wedge,\to,1\}$-subreducts of Heyting algebras (see [5, Example 1.11]). So if a hoop algebra **H** is finite or it satisfies (ACC), then the operation $\odot$ coincides with the operation binary meet. In fact, for all $x, y \in H$, we have $x \odot y \geq x \odot (x \wedge y) = x \odot (x \odot (x \to y)) = x \odot (x \to y) = x \wedge y$. The inequality $x \odot y \leq x \wedge y$ always is true.

Theorem 4.24. *Let* **H** *be a bounded hoop algebra and* $s : H \to H$ *a square root on* **H**. *Let* **H** *be finite or it satisfies* (ACC). *Then one has*

(i) *If* **H** *is* $\vee$*-hoop with (DNP), then* **H** *is a Boolean algebra;*
(ii) *If* **H** *is totally ordered, then* **H** *has the structure of Gödel algebra.*

Proof. (i) If **H** is $\vee$-hoop with (DNP), then by using Theorem 4.21 and Proposition 4.22, **H** is good, hence is a Boolean algebra according to Theorem 4.19.
(ii) Let H be totally ordered. Based on the assumption and Remark 4.23, we have $x \odot y = x \wedge y$, for all $x, y \in H$. If $x \leq y$, it is clear that $x \to y = 1$. Let us now assume $y < x$. If $x \leq x \to y$ then we have $y = x \wedge y = x \odot (x \to y) = x \wedge (x \to y) = x$ which is a contradiction. If $x \to y \leq x$ we have $y = x \wedge y = x \odot (x \to y) = x \wedge (x \to y) = x \to y$. Thus the implication is the Gödel implication and **H** has the structure of Gödel algebra. □

Definition 4.25. Let **H** be a bounded hoop algebra and $s : H \to H$ be a square root on **H**. If $s(0) = s(0)'$, then **H** is called *strict*.

Example 4.26. *(a) In Example 4.15(a), we observed that in the Łukasiewicz algebra,* $s(0) = 0.5$. *Therefore, the Łukasiewicz algebra is not good. Now, we have* $s(0)' = s(0) \to 0 = 0.5 \to 0 = \min(1, 1 - 0.5 + 0) = 0.5 = s(0)$. *Therefore, Łukasiewicz algebra is strict.*
(b) In Example 4.15(b), we observed that in the hoop algebra $\mathbf{\Gamma(G, u)}$, $s(0) = \frac{u}{2}$. *Therefore,* $\mathbf{\Gamma(G, u)}$ *is not good. Now, we have* $s(0)' = s(0) \to 0 = \frac{u}{2} \to 0 = (0 - \frac{u}{2} + u) \wedge u = \frac{u}{2} \wedge u = \frac{u}{2} = s(0)$. *Therefore, the hoop algebra* $\mathbf{\Gamma(G, u)}$ *is*

strict.
(c) Gödel algebra, ***Goa****, and product algebra,* ***Pra****, are not strict whenever* $0 \neq 1$.

Proposition 4.27. *Let* $\mathbf{H}$ *be a bounded hoop algebra and* $s : H \to H$ *be a square root on* $\mathbf{H}$. *Assume* $\mathbf{H}$ *is good. Then* $\mathbf{H}$ *is strict if and only if* $H = \{1\}$.

Proof. Assume $\mathbf{H}$ is strict. Since $\mathbf{H}$ is good, $0 = s(0) = s(0)' = 0' = 1$. The inverse of the relationship is clear. □

Proposition 4.28. *Let* $\mathbf{H}$ *be a bounded hoop algebra and* $s : H \to H$ *be a square root on* $\mathbf{H}$. *Assume that* $\mathbf{H}$ *is strict. Then for all* $x \in H$ *we have*

(i) $s(x)' \leq s(0)$;
(ii) *For every* $x \in B(H)$, *if* $s(0) \leq x$, *then* $x = 1$.

Proof. (i) We have $0 \leq x$, for every $x \in H$. Therefore, $s(0) \leq s(x)$. Thus, by Proposition 2.7(ii), $s(x)' \leq s(0)' = s(0)$.
(ii) Let $x \in B(H)$ and $s(0) \leq x$. By Proposition 2.7(ii), $x' \leq s(0)' = s(0)$. Then, by Lemma 2.15(i) and Theorem 2.3(ix), $x' = (x')^2 \leq s(0) \odot s(0) = 0$. Hence, $x = 1$. □

Proposition 4.29. *Let* $\mathbf{H}$ *and* $\mathbf{G}$ *be two bounded hoop algebras, and* $f : H \to G$ *be a hoop homomorphism. Suppose* $s : H \to H$ *and* $t : G \to G$ *are square roots on* $\mathbf{H}$ *and* $\mathbf{G}$, *respectively. If* $\mathbf{H}$ *is strict and* f *is a preserves square roots, then* $\mathbf{G}$ *is also strict.*

Proof. We have $t(0) = t(f(0)) = f(s(0)) = f(s(0)') = f(s(0))' = t(f(0))' = t(0)'$. □

We demonstrate that the class of all bounded hoop algebras with square roots is also a variety.

Theorem 4.30. *The class of all bounded* $\vee$*-hoop algebras with square roots is a variety.*

Proof. Let $\mathcal{V}$ be the class of all bounded $\vee$-hoop algebras $\mathbf{H} = (H; \odot, \to, 0, 1)$ with square roots. Also, let $\mathcal{W}$ be the variety of all bounded $\vee$-hoop algebras $\mathbf{G} = (G; \odot, \to, s, 0, 1)$ along with the following conditions:

(sq1) $s(x) \odot s(x) = x$;
(sq2) $s((y \odot y) \vee x) \wedge y = y$;
(sq3) $s(x') = s(x) \to s(0)$.

Assume $\mathbf{H} \in \mathcal{V}$. Let $s : H \to H$ be a square root on $\mathbf{H}$. We show that $\mathbf{H} \in \mathcal{W}$. Since $\mathbf{H}$ is a hoop algebra, conditions H1 to H4, and then by Theorem 3.10(vi), (sq1) and (sq3) hold for $\mathbf{H}$. It suffices to verify condition (sq2). By Theorems 3.18(iii) and 3.10(v), one has $s((y \odot y) \vee x) \geq s(y \odot y) \vee s(x) \geq (s(y) \odot s(y)) \vee s(x) = y \vee s(x) \geq y$. Therefore, $s((y \odot y) \vee x) \wedge y = y$. Hence, $\mathbf{H} \in \mathcal{W}$.

Conversely, assume $\mathbf{G} \in \mathcal{W}$. Since $\mathbf{G}$ satisfies conditions H1 to H4, it is a hoop algebra. We only need to check condition (S2) from Definition 3.1. Suppose $y \odot y \leq x$. From condition (sq2), we obtain $y = s((y \odot y) \vee x) \wedge y = s(x) \wedge y$. Thus, $y \leq s(x)$. Therefore, s satisfies condition (S2), and hence $\mathbf{G} \in \mathcal{V}$. Thus, $\mathcal{V} = \mathcal{W}$. □

Similar to the proof of Theorem 4.30, it can be proven that the class of bounded hoop algebras possessing an n-th root is also a variety.

Theorem 4.31. *The class of all good bounded $\vee$-hoop algebras with square roots is a variety.*

Proof. Let $\mathcal{V}$ be the class of all good bounded $\vee$-hoop algebras $\mathbf{H} = (H; \odot, \rightarrow, 0, 1)$ with square roots. Also, let $\mathcal{W}$ be the variety of all bounded $\vee$-hoop algebras $\mathbf{G} = (G; \odot, \rightarrow, s, 0, 1)$ along with the following conditions:

(sq1) $s(x) \odot s(x) = x$;

(sq2) $s((y \odot y) \vee x) \wedge y = y$;

(sq4) $s(x') = s(x)'$.

Assume $\mathbf{H} \in \mathcal{V}$. Let $s : H \rightarrow H$ be a square root on $\mathbf{H}$. We show that $\mathbf{H} \in \mathcal{W}$. Since $\mathbf{H}$ is a hoop algebra, conditions H1 to H4, and then by Theorem 4.30, the conditions (sq1), (sq2) and (sq4) hold for $\mathbf{H}$. Hence, $\mathbf{H} \in \mathcal{W}$.

Conversely, assume $\mathbf{G} \in \mathcal{W}$. Since $\mathbf{G}$ satisfies conditions H1 to H4, it is a hoop algebra. We only need to check condition (S2) from Definition 3.1 and $s(0) = 0$. By Theorem 4.30, s is a square root on $\mathbf{G}$. By (sq1), $s(1) \odot s(1) = 1$, thus $s(1) \geq 1$, i.e., $s(1) = 1$. Now, we have $s(0) = s(1') = s(1)' = 1' = 0$ and hence $\mathbf{G} \in \mathcal{V}$. Thus, $\mathcal{V} = \mathcal{W}$. □

5. Conclusion and Future Research

In this paper, the concept of the square root in hoop algebras is defined, and its properties are examined across various types of hoop algebras (unbounded, bounded, $\vee$-hoop, with (DNP), basic hoop, etc.). Furthermore, the notion of the n-th roots in these algebras is introduced, along with some of its key characteristics. Later, in Section 4, several applications of the square root in hoop algebras are explored. Among these, the impact of the square root on filters in hoop algebras was examined, leading to the isomorphism theorem presented in Theorem 4.5. Additionally, hoop algebras possessing a square root such that $s(0) = 0$, were introduced and named "good hoop algebras." Their properties were examined in detail. In this section, the ascending and descending chain conditions in hoop algebras were defined. In Theorem 4.24, it was demonstrated that if a hoop algebra satisfies the ascending chain condition, then under certain circumstances, it is either a Boolean algebra or a Gödel algebra. Finally, it was established that hoop algebras possessing a square root form a variety.

Some of our future plans are to investigate the types of derivations on hoop algebras such as multiplicative derivation, f-derivation and (f, g)-derivation and the effect of the square root on the aforementioned derivation. Also, checking the properties of the created lattices will be one of our other plans.

6. Data Availability Statement

Not applicable.

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8. Conflict of interest

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