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PRIME POWER SYLOW NUMBERS

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Abstract. In this paper, we study finite groups with a prime-power number of Sylow p-subgroups. Motivated by the work of Yang et al. (2022), who characterized non-solvable groups with a prime-power number of Sylow 2-subgroups, we investigate the corresponding problem for odd primes. We prove that if a finite group G has a non-abelian composition factor whose order is divisible by an odd prime p, and the number of Sylow p-subgroups is a prime power, then p must be a Mersenne prime and $n_p(G) = 2^k$ for some integer $k \geq 2$.

Keywords: finite group, non-solvable group, Sylow number.
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1. Introduction

Let G be a finite group, and let $n_p(G)$ denote the number of Sylow p-subgroups of G, where p is a prime. By Sylow's theorems, we know that $n_p(G) \equiv 1 \pmod{p}$ and $n_p(G) \mid |G|$. While these classical results impose congruence and divisibility constraints, the precise value of $n_p(G)$ often reflects deep structural properties of the group.

For clarity, we recall the definitions of Fermat and Mersenne primes that appear in this context.

Definition 1.1. A *Fermat prime* is a prime of the form $F_n = 2^{2^n} + 1$ for some nonnegative integer n. The known Fermat primes are $3, 5, 17, 257$, and 65537.

A *Mersenne prime* is a prime of the form $M_n = 2^n - 1$, where n is itself a prime. Examples include $3, 7, 31, 127$, and 8191.

Yang et al. [6] proved that if G is a non-solvable finite group and the number of Sylow 2-subgroups of G is a power of a prime, then that prime must be a Fermat prime. Of course, the main topic of this research traces back to the question posed in the paper [1, p.1228], which asks: Is it true that the number of Sylow 2-subgroups of a simple group G is a prime p if and only if $G = \mathrm{PSL}(2, p-1)$, where p is a Fermat prime?

In this work, we study the case of odd primes in the context of Yang et al.'s theorem. Specifically, we show that if G is a finite group and for some odd

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prime p, such that G has a non-abelian composition factor S whose order is divisible by p, and the number of Sylow p-subgroups of G is a prime power, then necessarily p is a Mersenne prime, and $n_p(G) = 2^k$ for some integer $k \geq 2$. Yang et al.'s theorem can be stated as follows.

Theorem 1.2 ([6]). *Let G be a finite non-solvable group. If $n_2(G) = p^k$ is a power of a prime, then p must be a Fermat prime.*

As an immediate consequence of Theorem 1.2, we obtain the following: if G is a finite group such that the number of Sylow 2-subgroups of G is $n_2(G) = p^a$ for some prime p and integer $a \geq 1$, and if p is not a Fermat prime, then G is solvable.

This result shows that only very specific numerical configurations for $n_2(G)$ can occur in non-solvable groups.

2. Preliminaries

In this section, we collect several known results that are used in the proof of the main theorem.

Lemma 2.1 (Guralnick [3]). *Let G be a finite non-abelian simple group, and suppose $H < G$ is a proper subgroup of index $|G : H| = p^n$, a prime power. Then G is one of the following:*

(1) *$G = A_m$, $H \cong A_{m-1}$, with $m = r^a$ a prime power;*
(2) *$G = \mathrm{PSL}(m, q)$, and H is a parabolic subgroup of index $\dfrac{q^m - 1}{q - 1} = p^n$ (note m must be prime);*
(3) *$G = \mathrm{PSL}(2, 11)$, $H \cong A_5$;*
(4) *$G = M_{23}$, $H = M_{22}$, or $G = M_{11}$, $H = M_{10}$;*
(5) *$G = \mathrm{PSU}(4, 2) \cong \mathrm{PSp}(4, 3)$, and H is a parabolic subgroup of index 27.*

Throughout the paper, we examine which of the groups listed above may admit Sylow p-numbers that are prime powers, and under which conditions these numbers can be linked to special primes such as Fermat or Mersenne primes.

Lemma 2.2. *[4, Theorem 2.1] Let G be a finite group with a normal subgroup K, and let P be a Sylow p-subgroup of G. Denote by $n_p(G)$ the number of Sylow p-subgroups of G. Then $n_p(G) = a \cdot b \cdot c$, where*

- *$a = n_p(G/K)$ is the number of Sylow p-subgroups of the quotient group G/K,*
- *$b = n_p(K)$ is the number of Sylow p-subgroups of K,*
- *$c = n_p(N_G(PK)/PK)$ is the number of Sylow p-subgroups of the quotient $N_G(PK)/PK$.*

Lemma 2.3. *[2, Theorem 3.4] G is a nonsolvable group acting faithfully and 2-transitively on $\mathrm{Syl}_r(G)$ if and only if G and r are one of the following:*

(1) $G \cong A_5 = \mathrm{PSL}(2,4)$; $r = 2$.
(2) $\mathrm{PSL}(2,5) \cong A_5 \lesssim G \lesssim S_5 \cong \mathrm{PGL}(2,5)$; $r = 5$.
(3) $\mathrm{PSL}(2,9) \cong A_6 \lesssim G \lesssim \mathrm{Aut}(A_6) \cong \mathrm{P\Gamma L}(2,9)$; $r = 3$.
(4) $M \lesssim G \lesssim \mathrm{Aut}(M)$ *where* $M \cong \mathrm{PSL}(2,r^m)$ $(r^m > 3)$, $U_3(r^m)$ $(r^m > 2)$, ${}^2B_2(2^{2m+1})$ $(m > 0)$, *or* ${}^2G_2(3^{2m+1})$ $(m > 0)$; *here* r *is the characteristic of the underlying field and* $r \nmid [G : M]$.
(5) $\mathrm{PSL}(2,7) \cong \mathrm{PSL}(3,2) \lesssim G \lesssim \mathrm{Aut}(\mathrm{PSL}(3,2)) \cong \mathrm{PGL}(2,7)$; $r = 7$.
(6) $G_2(2)' \cong \mathrm{PSU}(3,3) \lesssim G < \mathrm{Aut}(\mathrm{PSU}(3,3)) \cong G_2(2)$; $r = 3$.
(7) $G \cong {}^2G_2(3) \cong \mathrm{P\Gamma L}(2,8)$; $r = 3$ *or* $r = 2$.
(8) $G \cong MG_0$ *where* M *is the group of translations of a vector space of dimension* 2 *over* $\mathrm{GF}(q)$, M *is a normal subgroup of* G, *and* $G_0 \leq \mathrm{GL}(2,q)$ *with* $G_0 = G_0^{(\infty)}C$ *where* $G_0^{(\infty)} \cong \mathrm{SL}(2,5)$, $C \leq Z(\mathrm{GL}(2,q))$, $r \mid |C|$ *and* $q = 29$ *or* 59, $r = 7$ *or* 29 *respectively.*

Lemma 2.4. *[5, Theorem 1.1] The almost simple groups which are transitive permutation groups of prime power degree* $n = p^k$ *are as follows:*

(A) (a) A_n *and* S_n *for* $n = p^k \geq 5$;
  (b) $S_{n/2}$ *for* $n = 2^k \geq 16$;
(B) (a) $\mathrm{PSL}(2,q)$ *and* $\mathrm{PGL}(2,q)$ *for any Mersenne prime* $q = 2^k - 1 > 3$, *with* $p = 2$;
  (b) $\mathrm{PSL}(2,q) \rtimes C_{2^i} \leq \mathrm{P\Gamma L}(2,q)$ *for* $q = 2^e > 2$ *and* $i = 0, \dots, f$, *with* $e = 2^f$ *and* $p^k = p = 2^e + 1$ *a Fermat prime;*
  (c) $\mathrm{PSL}(2,8)$ *and* $\mathrm{P\Gamma L}(2,8)$, *where* $p^k = 9$;
  (d) *For* $d > 2$, *any group* G *such that* $\mathrm{PSL}(d,q) \leq G \leq P\Gamma L_d(q)$ *where* $n = \frac{q^d-1}{q-1}$ *is a prime* p;
  (e) *For* $d > 2$, *any group* G *such that* $\mathrm{PSL}(d,q) \leq G \leq \mathrm{P\Gamma L}(d,q)$ *where* $n = \frac{q^d-1}{q-1}$ *is a proper prime power* p^k, *with* $k > 1$;
(C) $\mathrm{PSL}(2,11)$ *and* M_{11} *for* $n = 11$, *and* M_{23} *for* $n = 23$;
(D) $W(E_6)$ *and* $W(E_6)' \cong \mathrm{PSp}_4(3)$ *for* $n = 27$.

3. Main Result

Theorem 3.1. *Let* p *be an odd prime. Let* G *be a finite group with a non-abelian composition factor* S *of order divisible by* p. *If* $n_p(G)$ *is a power of a prime* q, *then* $q = 2$ *and* $S \cong \mathrm{PSL}(2,p)$, *where* $p \geq 7$ *is a Mersenne prime.*

Proof. By Lemma 2.2, we have $n_p(S) \mid n_p(G) = q^a$. Since S is non-abelian simple and $p \mid |S|$, we have $n_p(S) \neq 1$. As $n_p(S) \mid q^a$, it follows that $n_p(S)$ is a power of q. Hence, we may assume without loss of generality that $G = S$. Let $P \in \mathrm{Syl}_p(G)$ and set $H = N_G(P)$. Then $P \trianglelefteq H$ and, by the Sylow theorems, $|G : H| = n_p(G) = q^a$ is a prime power; since G is simple, it follows that H is a maximal subgroup of G. Applying Lemma 2.1, one of the following occurs:

(a) If $G = A_n$ and $H = A_{n-1}$ with $n = q^a \geq 5$, then for $n > 5$ the subgroup H is simple, which is impossible since $P \neq 1$ and $P \trianglelefteq H$. Thus $n = 5$ and

$G \cong A_5$. However, for A_5 the number of Sylow p-subgroups is a prime power only when $p = 2$, contradicting the assumption that p is odd.

(b) Suppose that $G = \mathrm{PSL}(n, r)$ for some prime power r, and that $|G : H| = q^a = \frac{r^n-1}{r-1}$. Observe that $\frac{r^n-1}{r-1}$ enumerates the number of one-dimensional subspaces of the n-dimensional vector space over the finite field $\mathbb{F}_r$, which corresponds precisely to the number of points in the projective space $\mathbb{P}^{n-1}(\mathbb{F}_r)$.

By Lemma 2.1, maximal subgroups of $\mathrm{PSL}(n, r)$ having prime power index must be parabolic subgroups. More specifically, these are exactly the stabilizers of either a one-dimensional subspace (a line) or its dual, an $(n-1)$-dimensional subspace (a hyperplane). Each such parabolic subgroup has index $\frac{r^n-1}{r-1}$ in $\mathrm{PSL}(n, r)$.

Consequently, the subgroup H must be a parabolic subgroup, that is, the stabilizer of a line or a hyperplane in the natural action of G on the projective space $\mathbb{P}^{n-1}(\mathbb{F}_r)$.

Assume that $G = \mathrm{PSL}(n, r)$ with $n \geq 3$. If G acted faithfully and 2-transitively on $\mathrm{Syl}_p(G)$, then by Lemma 2.3 the group G would have to appear in the classification given there. However, no group of the form $\mathrm{PSL}(n, r)$ with $n \geq 3$ occurs in Lemma 2.3. Hence this case cannot occur, and we conclude that $n = 2$.

Since $G \cong \mathrm{PSL}(2, r)$ and $|G : N_G(P)| = n_p(G) = q^a$, the group G is an almost simple group acting 2-transitively on a set of prime power size q^a. Hence Lemma 2.4 applies. We examine the possibilities listed there.

Cases (A) and (D) of Lemma 2.4 cannot occur, since these correspond to alternating, symmetric, or Weyl groups, while G is of type $\mathrm{PSL}(2, r)$. The sporadic cases in (C) are also excluded, since they correspond to the exceptional simple groups already ruled out by Lemma 2.1, where the relevant subgroups of prime power index are not normalizers of Sylow p-subgroups.

It remains to consider case (B). The subcases (B)(d) and (B)(e) require $\mathrm{PSL}(d, q)$ with $d > 2$, which contradicts the fact that $G \cong \mathrm{PSL}(2, r)$. The subcases (B)(b) and (B)(c) occur only for $p = 2$ or for degree 9, and hence do not apply when p is an odd prime. Therefore, the only remaining possibility is case (B)(a), which yields that $G \cong \mathrm{PSL}(2, p)$ with p a Mersenne prime and $n_p(G)$ a power of 2.

(c) In parts 3, 4, and 5 of Lemma 2.1, the subgroups H are specific maximal subgroups of the simple groups: $\mathrm{PSL}(2, 11)$ with $H \cong A_5$, M_{23} with $H = M_{22}$, M_{11} with $H = M_{10}$, and $\mathrm{PSU}(4, 2) \cong \mathrm{PSp}(4, 3)$ with H a parabolic subgroup of index 27. These H are distinct from the normalizer $H = N_G(P)$ defined at the beginning of Theorem 3.1.

Using computational verification with GAP, it can be checked that in these exceptional cases, the subgroups H from Lemma 2.1 do not contain a normal Sylow p-subgroup. Thus, the subgroup $H = N_G(P)$ cannot be any of these.

This excludes these isolated cases from occurring in the context of Theorem 3.1, completing the proof. □

Remark 3.2. We cannot remove assumption G has a non-abelian composition factor S of order divisible by p. For example, consider $G = A_5 \times (C_{23} : C_{11})$. Then G is a non-solvable group with $n_{11}(G) = 23$. Therefore, Theorem 3.1 does not extend to all non-solvable groups when p is odd, even under the assumption that $n_p(G)$ is a prime.

An interesting direction for future research is to understand the structure of finite *solvable* groups where the number of Sylow p-subgroups is a power of a prime. The natural question is:

Question: Is it possible to characterize solvable groups for which the number of Sylow p-subgroups is a power of a prime? In particular, what structural restrictions arise from this condition, and can these groups be classified or described in terms of identifiable patterns?

This problem remains open and could provide valuable insight into the relationship between Sylow subgroup counts and the internal structure of finite groups.

4. Conclusion

In this paper, we investigated finite groups whose number of Sylow p-subgroups is a prime power, with particular emphasis on the case where p is an odd prime and the group admits a non-abelian composition factor divisible by p.

Our main result shows that this seemingly mild arithmetic condition imposes strong structural restrictions: specifically, the prime p must be a Mersenne prime, and the corresponding non-abelian composition factor is isomorphic to $\mathrm{PSL}(2, p)$. Moreover, the number of Sylow p-subgroups is necessarily a power of 2. This highlights a striking contrast with the case $p = 2$, where Fermat primes arise instead.

The results demonstrate how the interplay between Sylow theory and the classification of finite simple groups can significantly restrict possible group structures. In particular, the condition that $n_p(G)$ is a prime power forces the group into a very narrow and well-understood class.

Finally, several natural questions remain open, especially in the solvable case. A deeper understanding of solvable groups with prime-power Sylow numbers could reveal further structural patterns and possibly lead to a broader classification framework. We hope that this work provides a foundation for such future investigations.

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6. **Conflict of interest**

The authors declare that they have no conflicts of interest regarding the publication of this article.

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