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STATISTICAL APPROACHES TO POINT ESTIMATION OF YONGTING'S CAPABILITY INDEX

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Abstract. Yongting capability index is a suitable criterion for measuring and evaluating the efficiency of industrial processes to produce items conforming the fuzzy quality. This paper develops and applies several statistical estimation approaches for evaluating Yongting's capability index based on the fuzzy quality and provides a comprehensive comparative study of their performance. This enhances the methodological toolkit available for researchers and practitioners engaged in evaluating and improving industrial production processes. The proposed and discussed approaches in this paper are: (1) Kernel density estimation, (2) Monte Carlo estimation, (3) method of moments estimation and (4) maximum likelihood estimation. The proposed estimation approaches are compared in a simulation case study to show the performance discussed approaches.

Keywords: Capability estimation, Kernel density estimation, Monte Carlo simulation, Maximum likelihood, Method of moments.
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1. Introduction

In traditional quality control, a product is classified simply as "conforming" or "non-conforming" based on whether its measurable characteristic x falls within rigid specification limits $[LSL, USL]$. This binary view is represented by the indicator function

$$I_{[LSL,USL]}(x) = \begin{cases} 1 & if \quad LSL \le x \le USL, \\ 0 & elsewhere. \end{cases} \tag{1}$$

Consider a factory producing precision bolts where the target diameter is 10.0 millimeters (mm), with lower and upper specification limits $[LSL, USL]$ of 9.8 mm and 10.2 mm, respectively. Under the traditional binary classification, two bolts with diameters of 10.19 mm and 10.21 mm are treated fundamentally differently: the first is labeled "conforming", while the second is "non-conforming" and potentially scrapped. In practice, however, both bolts are functionally identical and may be used in the same assembly without issue. This simple

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example highlights the inherent inadequacy of crisp specification limits for describing real-world quality, where performance often degrades gradually rather than abruptly [18].

However, in many real-world situations, the notion of "good quality" is not crisp. A value close to the target may be considered excellent, a value near the specification limit may be barely acceptable, and a value just outside the limit may still possess some useful quality. To model this gradual transition, Yongting [21] introduced the concept of "fuzzy quality". A fuzzy quality is defined by a membership function

$$\tilde{Q}\colon \mathbb{R} \to [0,1]$$

where $\tilde{Q}$ quantifies the degree to which a product with characteristic x satisfies the quality requirement. Thus, $\tilde{Q}=1$ means perfect conformance, $\tilde{Q}=0$ means total non-conformance, and intermediate values represent partial conformance. For instance, consider the diameter of a manufactured shaft. The nominal (target) diameter is 10.0 mm, and the traditional specification limits are $LSL = 9.8$ mm and $USL = 10.2$ mm. From a fuzzy perspective, the quality membership function can be defined as:

$$\tilde{Q}(x) = \begin{cases} 0 & & x < 9.7, \\ \frac{x-9.7}{0.1} & if & 9.7 < x < 9.8, \\ 1 & if & 9.8 \leq x \leq 10.2, \\ \frac{10.3-x}{0.1} & if & 10.2 < x < 10.3, \\ 0 & & x \geq 10.3. \end{cases} \tag{2}$$

Here, diameters between 9.8 and 10.2 mm are fully acceptable ($\tilde{Q}=1$). In the transition zones (9.7, 9.8) and (10.2, 10.3), the quality degree decreases linearly from 1 to 0, reflecting the reality that parts slightly outside the strict limits may still be usable (e.g., in less critical assemblies). This fuzzy definition is more informative and operationally flexible than the rigid binary classification. Although this definition seems simple at the first, the "fuzzy quality" is one of the useful and meaningful definitions on the basis of the fuzzy sets theory which can have various practical aspects in the industry; see [17] for a comprehensive discussion on the applications and details of fuzzy quality.

After the quality extension —from precise to fuzzy— the methods of estimating and measuring the potential of a industrial process in the production of high-quality items needed to be modified and updated. For this purpose, several process capability indices (PCIs) were defined on the basis of fuzzy quality. As one of them, Yongting [21] defined the capability index by

$$C_{\tilde{Q}} \;=\; \int_{-\infty}^{+\infty} \tilde{Q}(x)\; f(x)dx, \tag{3}$$

in which f is the probability density function of a one-dimensional continuous quality characteristic X, and $\tilde{Q}$ is the membership function of fuzzy quality to construct the degree of conformity with the standard fuzzy quality [21]. Note that $\tilde{Q}(x)$ indicates the degree of conformity with standard quality (or, more succinctly, the degree of quality) when the quality characteristic of a product is measured as x.

It is worth noting that Yongting's capability index admits a probabilistic interpretation as an expectation, namely $C_{\tilde{Q}} = E\left(\tilde{Q}(X)\right)$. Consequently, when the quality characteristic X is discrete, the integral in Eq. (3) is naturally replaced by a summation of the form $\sum_{x\in\mathcal{X}} \tilde{Q}(x)\ P(X = x)$. Hence, the definition of Yongting's capability index is directly extendable to discrete quality characteristics. In this study, however, we focus on the continuous case, since fuzzy quality is predominantly defined and applied to continuous quality characteristics in industrial practice, and the proposed estimation methods are developed within this framework.

In terms of specific contributions to the field, Parchami et al. [14] introduced a novel approach utilizing Monte Carlo simulation to evaluate the quality of a manufacturing process using Yongting's index. Parchami et al. [16] presented the structure of testing the fuzzy process capability indices to assess the capability of a normal process when the specification limits are imprecise. Iranmanesh et al. [5] proposed a statistical fuzzy quality test to analyze the manufacturing process considering fuzzy specification limits. The Monte Carlo simulation method was introduced in [6] to evaluate the quality of a production process using the widely recognized capability index C_{pk}. For additional insights and related works, Iranmanesh et al. [8] developed a new generation of process capability indices based on fuzzy measurements; they [7] explored a few robust estimators for the capability index S_{pk}; Iranmanesh et al. [9] evaluated manufacturing process performance to quality test based on fuzzy specification limits; Amirzadeh et al. [1] designed a fuzzy p-chart based on the mean degree of nonconformity; the construction of $\bar{X}$-R control charts was considered in [13] based on two different percentile-based methods (i.e., the quantiles of quality degrees and the kernel density estimation method); Parchami et al. [15] applied the simulation testing of fuzzy quality with a case study in pipe manufacturing industries.

The organization of this paper is as follows. The kernel estimation of Yongting's capability index is introduced in Section 2. Monte Carlo (MC) simulation method is presented to estimate Yongting's capability index in Section 3. In Sections 4 and 5, the method of moment (MM) and the maximum likelihood (ML) estimations are used to estimate the capability index, respectively. A comparative analysis based on the mean square error (MSE) criterion and relative efficiency (RE) is provided to asses the performance of the proposed estimators in Section 6. Section 7 presents numerical results and a comparison study to analyze and compare the proposed estimation approaches. Finally, the conclusions are presented, along with a discussion of potential directions for future research.

2. Capability estimation based on kernel method

Kernel density estimation (KDE) is a well-established nonparametric statistical method used to estimate the probability density function of a random variable based on a finite data sample through a smoothing kernel function. While KDE itself is not a novel technique, its application to Yongting's fuzzy capability index is introduced here for the first time. This approach is particularly valuable in industrial quality control where the underlying distribution of quality characteristics may not conform to standard parametric forms, such as normality, allowing for more robust and flexible estimation without strong distributional assumptions. By replacing the unknown probability density function in Eq. (3) with the kernel density estimator, the KD-based estimator for Yongting's capability index can be computed as [4]:

$$\widehat{C_{\tilde{Q}}}^{KD} = \int_{-\infty}^{+\infty} \tilde{Q}(x)\ \widehat{f}_{x_1,\ldots,x_n}(x)dx, \tag{4}$$

in which the kernel density estimation is equal to

$$\widehat{f}_{x_1,\cdots,x_n}(x) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x-x_i}{h}\right), \tag{5}$$

where $h > 0$ often referred to as the bandwidth, and K is a non-negative kernel function that satisfies $\int_{-\infty}^{+\infty} K(u)\ du = 1$ and is symmetric around zero. Some of common kernel functions are:

- The triangular kernel: $K(x) = \max\{1 - |x|, 0\}$,
- The Gaussian kernel: $K(x) = \dfrac{1}{\sqrt{2\pi}} exp(\frac{-x^2}{2})$,
- The rectangular (or uniform) kernel: $K(x) = \frac{I(|x|\leq 1)}{2}$, where I is the indicator function.

The choice of kernel function influences the smoothness of the density estimate, with the Gaussian kernel often preferred for its smooth properties and infinite support, while the triangular and rectangular kernels provide compact support and may be computationally simpler in certain scenarios. This estimator provides a flexible, nonparametric alternative for approximating Yongting's index and allows assessing the effect of the underlying distribution on the index without relying on parametric assumptions. Its performance is evaluated and compared with other estimation approaches in the simulation study, where we demonstrate its advantages in handling skewed or multimodal distributions commonly encountered in manufacturing processes.

Remark 2.1. In kernel density estimation, the bandwidth h is a key smoothing parameter that balances bias and variance to minimize the asymptotic mean integrated squared error (AMISE). Selecting an optimal h is crucial, as a small h can lead to overfitting (high variance, undersmoothing), while a large h can cause underfitting (high bias, oversmoothing). We use Silverman's rule-of-thumb (implemented as "nrd0" in R's stats package), given by [19]

$$\hat{h} = \left(\frac{4}{3}\right)^{1/5} \tilde{\sigma} n^{-1/5},$$

where $\tilde{\sigma} = \min\left(\hat{\sigma}, \frac{\text{IQR}}{1.34}\right)$, in which $\hat{\sigma}$ is the sample standard deviation, IQR the interquartile range, and n the sample size. The constant 1.34 is approximately the interquartile range, $\Phi^{-1}(0.75) - \Phi^{-1}(0.25)$, of the standard Normal distribution. Curiously, the interquartile range for the standard Normal distribution is 1.35 to two decimals accuracy, but the use of 1.34 in the estimator $\tilde{\sigma}$ has prevailed. Silverman [19] made the ad hoc suggestion to reduce the factor $\left(\frac{4}{3}\right)^{1/5} \approx 1.06$ to 0.9. This results in the bandwidth estimate

$$\hat{h} \approx 0.9\ \tilde{\sigma}\ n^{-1/5}.$$

This adjustment enhances performance across a wider range of distributions, particularly those with heavier tails or multimodality. The exponent $-1/5$ derives from asymptotic theory for optimal AMISE minimization, assuming a Gaussian reference distribution. Alternative methods for bandwidth selection, such as cross-validation or plug-in estimators, could be explored in future work, but Silverman's rule provides a computationally efficient and reliable default for our purposes in estimating fuzzy capability indices.

For more comprehensive details on KDE, including advanced bandwidth selection techniques and theoretical properties, interested readers are referred to [3] and [19]. This method's integration with fuzzy quality concepts opens new avenues for nonparametric process capability analysis in imprecise environments.

3. Capability estimation based on Monte Carlo simulation

According to Zadeh's probabilistic interpretation [22], the Yongting fuzzy capability index $C_{\tilde{Q}}$ defined in Eq. (3) can be expressed as the probability of the fuzzy quality event [17], i.e.,

$$C_{\tilde{Q}} = E\left(\tilde{Q}(X)\right) = P\left(X \in \tilde{Q}\right) \tag{6}$$

where E is the mathematical expectation. Based on a random sample $X_1, ..., X_n$ from the quality characteristic, a Monte Carlo estimator of Yongting's index can be obtained as the sample mean of the membership values:

$$\widehat{C_{\tilde{Q}}}^{MC} = \overline{\tilde{Q}(X)} = \frac{1}{n}\sum_{i=1}^{n} \tilde{Q}(X_i). \tag{7}$$

While this MC estimator is not a novel methodology, it provides a simple and interpretable benchmark for approximating $C_{\tilde{Q}}$. Its inclusion allows comparison with other parametric and nonparametric estimation approaches and facilitates evaluation of their relative performance.

Lemma 3.1. *Let $\tilde{Q}\colon \mathbb{R} \to [0,1]$ be a measurable membership function and X a random variable with probability density function $f(x)$. Assume that:*

1. $P(\tilde{Q}(X) > 0) > 0$ *(i.e., $\tilde{Q}$ is not almost everywhere zero on the support of f),*
2. $P(\tilde{Q}(X) < 1) > 0$ *(i.e., $\tilde{Q}$ is not almost everywhere one on the support of f).*

Then, we have:

$$0 < E\left(\tilde{Q}(X)\right) < 1 \text{ and } 0 < E\left(\tilde{Q}(X)^2\right) < 1. \tag{8}$$

Proof. Since $0 < \tilde{Q}(X) < 1$, we have $0 < E\left(\tilde{Q}(X)\right) < 1$. The first assumption guarantees that the expectation is strictly positive, while the second assumption ensures it is strictly less than 1. The same reasoning applies to $E\left(\tilde{Q}(X)^2\right)$ because $0 < \tilde{Q}(X)^2 \leq \tilde{Q}(X) < 1$. □

Lemma 3.2. *The estimator $\widehat{C_{\tilde{Q}}}^{MC}$ is strongly consistent for Yongting's index, i.e.,*

$$\widehat{C_{\tilde{Q}}}^{MC} \xrightarrow{a.s.} C_{\tilde{Q}}.$$

Proof. Recall that

$$\widehat{C_{\tilde{Q}}}^{MC} = \frac{1}{n}\sum_{i=1}^{n} \tilde{Q}(X_i),$$

where $\{X_i\}_{i=1}^{n}$ are i.i.d. observations. Since $0 < \tilde{Q}(X) < 1$, we have $E\left(\tilde{Q}(X)\right) < \infty$. Therefore, by the Strong Law of Large Numbers (SLLN),

$$\widehat{C_{\tilde{Q}}}^{MC} \xrightarrow{a.s.} E\left(\tilde{Q}(X)\right) = C_{\tilde{Q}}, \quad \text{as } n \to \infty.$$

□

Although, the variance of estimator $\widehat{C_{\tilde{Q}}}^{MC}$ is unknown, but we introduce a consistent estimator for it in the next theorem.

Theorem 3.3. *The estimator $\frac{\overline{\tilde{Q}(X)^2} - \overline{\tilde{Q}(X)}^2}{n}$ is a consistent estimator for the variance of estimator $\widehat{C_{\tilde{Q}}}^{MC}$.*

Proof. In the first the variance of estimator $\widehat{C_{\tilde{Q}}}^{MC}$ can be calculated by:

$$\begin{aligned} Var\left(\widehat{C_{\tilde{Q}}}^{MC}\right) &= Var\left(\overline{\tilde{Q}(X)}\right) = \frac{1}{n}Var\left(\tilde{Q}(X)\right) \\ &= \frac{1}{n}\left[E\left(\tilde{Q}(X)^2\right) - \left(E\left(\tilde{Q}(X)\right)\right)^2\right] \\ &= \frac{1}{n}\left[E\left(\tilde{Q}(X)^2\right) - C_{\tilde{Q}}^2\right]. \end{aligned} \tag{9}$$

In the second line, since using Lemma 1, $E\left(\tilde{Q}(X)\right)$ and $E\left(\tilde{Q}(X)^2\right)$ are finite and therefore by the SLLN, we conclude that $\overline{\tilde{Q}(X)} \xrightarrow{a.s.} \left[E\left(\tilde{Q}(X)\right)\right]$ and $\overline{\tilde{Q}(X)^2} \xrightarrow{a.s.} \left[E\left(\tilde{Q}(X)^2\right)\right]$ as $n \to \infty$. Hence according to the continuous mapping theorem [2], we have $\frac{\overline{\tilde{Q}(X)^2} - \overline{\tilde{Q}(X)}^2}{n} \xrightarrow{a.s.} \frac{1}{n}\left[E\left(\tilde{Q}(X)^2\right) - C_{\tilde{Q}}^2\right] = Var\left(\widehat{C_{\tilde{Q}}}^{MC}\right)$, and therefore the estimator $\frac{\overline{\tilde{Q}(X)^2} - \overline{\tilde{Q}(X)}^2}{n}$ is a consistent estimator for the variance of estimator $\widehat{C_{\tilde{Q}}}^{MC}$. □

4. Method of moments estimation for capability index

The method of moments is a classical statistical approach widely used in quality control and process capability analysis to estimate parameters by matching sample moments with population moments. While the method of moment estimation (MME) itself is well-established, its application to Yongting's fuzzy capability index is presented here for the first time. Using a random sample $X_1, ..., X_n$ from the quality characteristic, a plug-in MM-based estimator of Yongting's index can be constructed by replacing the unknown parameters with their sample moment estimates [2]. Note that such MM-based estimators may not be unique, and different formulations are possible depending on the moment equations used.

Lemma 4.1. *Under Normality assumption for the quality characteristic X with unknown mean μ and unknown variance σ^2, two MM-based estimators of Yongting's index are*

$$\widehat{C_{\tilde{Q}}}^{MM1} = \overline{\tilde{Q}(X)} = \frac{1}{n}\sum_{i=1}^{n}\tilde{Q}(X_i), \tag{10}$$

and

$$\widehat{C_{\tilde{Q}}}^{MM2} = \frac{\int_{-\infty}^{+\infty}\tilde{Q}(x)\ exp\left[-\frac{(x-\overline{X})^2}{2\left(\overline{X^2}-\overline{X}^2\right)}\right]dx}{\sqrt{2\pi\left[\overline{X^2}-\overline{X}^2\right]}}, \tag{11}$$

in which $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ *and* $\overline{X^2} = \frac{1}{n}\sum_{i=1}^{n} X_i^2$.

Proof. Substituting the random sample mean in the Eq. (6) instead of the population mean, the first MM-based estimator ($\widehat{C_{\tilde{Q}}}^{MM1}$) can be easily obtained. It must be noted that the estimator $\widehat{C_{\tilde{Q}}}^{MM1}$ dose not need the Normality assumption. For the second MM-based estimator, the Yongting's capability index can be rewritten as follows under the Normality assumption

(12)
$$C_{\tilde{Q}} = \left(2\pi \left[E(X^2) - E(X)^2\right]\right)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} \tilde{Q}(x)\ exp\left[-\frac{(x-E(X))^2}{2\left(E(X^2)-E(X)^2\right)}\right] dx.$$

The second MM-based estimator can be achieved by substituting the first and the second sample moments instead of corresponding population moments in Eq. (12). □

Lemma 4.2. *The MM-based estimators* $\widehat{C_{\tilde{Q}}}^{MM1}$ *and* $\widehat{C_{\tilde{Q}}}^{MM2}$ *are consistent for Yongting's capability index* $C_{\tilde{Q}}$*.*

Proof. We prove the consistency of each estimator separately.
1. Consistency of $\widehat{C_{\tilde{Q}}}^{MM1}$: Recall that $\widehat{C_{\tilde{Q}}}^{MM1} = \frac{1}{n}\sum_{i=1}^{n} \tilde{Q}(X_i)$. By the SLLN, since $\tilde{Q}(X_i)$ are i.i.d. with finite expectation $C_{\tilde{Q}} = E\left(\tilde{Q}(X)\right)$, we have

$$\widehat{C_{\tilde{Q}}}^{MM1} \xrightarrow{a.s.} C_{\tilde{Q}}, \quad \text{as } n \to \infty.$$

Thus, $\widehat{C_{\tilde{Q}}}^{MM1}$ is strongly consistent.
2. Consistency of $\widehat{C_{\tilde{Q}}}^{MM2}$: Recall that

$$\widehat{C_{\tilde{Q}}}^{MM2} = \frac{1}{\sqrt{2\pi\widehat{\sigma}^2}} \int_{-\infty}^{+\infty} \tilde{Q}(x) \exp\left[-\frac{(x-\widehat{\mu})^2}{2\widehat{\sigma}^2}\right] dx,$$

where $\widehat{\mu} = \bar{X}$ and $\widehat{\sigma}^2 = \overline{X^2} - \bar{X}^2$ are the method of moments estimators for μ and σ^2, respectively. By the SLLN:

$$\widehat{\mu} \xrightarrow{a.s.} \mu \quad \text{and} \quad \widehat{\sigma}^2 \xrightarrow{a.s.} \sigma^2.$$

Now, define the function

$$g(m, s^2) = \frac{1}{\sqrt{2\pi s^2}} \int_{\mathbb{R}} \tilde{Q}(x) \exp\left[-\frac{(x-m)^2}{2s^2}\right] dx,$$

for $s^2 > 0$. The unknown parameter satisfies $C_{\tilde{Q}} = g(\mu, \sigma^2)$. The function g is continuous in its arguments m and s^2 on the domain $s^2 > 0$ (this follows from the dominated convergence theorem and the boundedness of $\tilde{Q}$). Since $\widehat{\mu} \xrightarrow{a.s.} \mu$ and $\widehat{\sigma}^2 \xrightarrow{a.s.} \sigma^2 > 0$, by the continuous mapping theorem [10], we obtain

$$\widehat{C_{\tilde{Q}}}^{MM2} = g(\widehat{\mu}, \widehat{\sigma}^2) \xrightarrow{a.s.} g(\mu, \sigma^2) = C_{\tilde{Q}}.$$

Hence, $\widehat{C_{\tilde{Q}}}^{MM2}$ is also strongly consistent. □

This MM-based approach provides a simple parametric alternative for estimating Yongting's index, and its statistical properties can be systematically analyzed and compared with other estimation methods in the simulation study.

5. Maximum likelihood estimation of capability index

Maximum likelihood estimation (MLE) is a classical statistical method used to estimate the parameters of a probability distribution that best fit the observed data. While MLE itself is well-established, its application to Yongting's fuzzy capability index is introduced here for the first time. Assuming the quality characteristic X follows a Normal distribution with unknown mean μ and variance σ^2, the plug-in MLE of Yongting's index is obtained by replacing the unknown parameters with their maximum likelihood estimates [2].

Lemma 5.1. *Under Normality assumption of the quality characteristic X with unknown mean μ and unknown variance σ^2, the ML-based estimator for capabilty index $C_{\tilde{Q}}$ is equal to*

$$\widehat{C_{\tilde{Q}}}^{ML} = \frac{1}{\sqrt{2\pi}S_n}\int_{-\infty}^{+\infty}\tilde{Q}(x)\ exp\left[-\frac{(x-\bar{X})^2}{2S_n^2}\right]dx, \tag{13}$$

where $\bar{X}=\frac{1}{n}\sum_{i=1}^{n}X_i$ and $S_n^2=\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2$.

Proof. The statistics $\bar{X}$ and S_n^2 are respectively ML-based estimators of parameters μ and σ^2 under the Normality assumption for random sample $X_1,...,X_n$. Therefore, based on the invariance property of ML-based estimators [2,12], the maximum likelihood estimator for Yongting's index is $\int_{-\infty}^{+\infty}\tilde{Q}(x)\ f_{N(\bar{X},S_n^2)}(x)dx$, where $f_{N(\mu,\sigma^2)}(x)$ represents the probability density function of a Normal distribution with parameters μ and σ^2. □

Theorem 5.2. *Let $C_{\tilde{Q}}=\int_{-\infty}^{+\infty}\tilde{Q}(x)\ f_{N(\mu,\sigma^2)}(x)dx$ be Yongting's index based on a sequence of random variables $X_1,\cdots,X_n$ drawn from a Normal distribution with mean μ and standard deviation σ. Then the asymptotic distribution of the ML-based estimator $\widehat{C_{\tilde{Q}}}^{ML}$ is as follows:*

$$\sqrt{n}\left(\widehat{C_{\tilde{Q}}}^{ML}-C_{\tilde{Q}}\right)\xrightarrow{d}N\left(0,\sigma^2\left[\left(\frac{\partial C_{\tilde{Q}}}{\partial\mu}\right)^2+2\sigma^2\left(\frac{\partial C_{\tilde{Q}}}{\partial\sigma^2}\right)^2\right]\right), \tag{14}$$

in which $\frac{\partial C_{\tilde{Q}}}{\partial\mu}$ and $\frac{\partial C_{\tilde{Q}}}{\partial\sigma^2}$ represent the partial derivatives of $C_{\tilde{Q}}$ with respect to μ and σ^2, respectively.

Proof. Let $X_1,...,X_n$ be independent and identically distributed (i.i.d.) random variables from the Normal distribution $N(\mu,\sigma^2)$ and by substituting the Normal distribution into Eq. (3), $C_{\tilde{Q}}=\int_{-\infty}^{+\infty}\tilde{Q}(x)\ \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx$, where $\tilde{Q}(x)$ is the given membership function for the fuzzy quality. Let $(\widehat{\mu}, \widehat{\sigma}^2) = (\bar{X}, S_n^2)$ be the MLEs for the unknown parameters vector (μ, σ^2). Then, considering Lemma 5, the ML-based estimator for $C_{\tilde{Q}}$ is defined by $\widehat{C_{\tilde{Q}}}^{ML} = \int_{-\infty}^{+\infty} \tilde{Q}(x) \, \frac{1}{\sqrt{2\pi\widehat{\sigma}^2}} \exp\left(-\frac{(x-\widehat{\mu})^2}{2\widehat{\sigma}^2}\right) dx$. According to the asymptotic properties of the MLEs, under regularity conditions, the MLEs $\widehat{\mu}$ and $\widehat{\sigma}^2$ are asymptotically Normally distributed [11]. Specifically, we have the following result:

$$\sqrt{n}\begin{bmatrix}\widehat{\mu} - \mu \\ \widehat{\sigma}^2 - \sigma^2\end{bmatrix} \xrightarrow{d} N\left(\mathbf{0}, I^{-1}(\mu, \sigma^2)\right),$$

where

$$I^{-1}(\mu, \sigma^2) = \begin{bmatrix}\sigma^2 & 0 \\ 0 & 2\sigma^4\end{bmatrix}$$

is the asymptotic covariance matrix of the MLEs of (μ, σ^2) [20]. Now, under the Normality assumption of the random sample $X_1, ..., X_n$, the Yongting's index $C_{\tilde{Q}}$ is a differentiable function of the parameter vector (μ, σ^2). Hence, using the Delta Method [2, 11] and the asymptotic Normality property of MLEs, we can determine the asymptotic distribution of a function of the MLEs, which follows a Normal distribution:

$$\left(\widehat{C_{\tilde{Q}}}^{ML} - C_{\tilde{Q}}(\mu, \sigma^2)\right) \xrightarrow{d} N\left(0, \nabla C_{\tilde{Q}}{}^T \, \Sigma^{-1} \, \nabla C_{\tilde{Q}}\right),$$

where $\nabla C_{\tilde{Q}}{}^T = \begin{bmatrix}\frac{\partial C_{\tilde{Q}}}{\partial \mu} & \frac{\partial C_{\tilde{Q}}}{\partial \sigma^2}\end{bmatrix}$ is the transpose of the gradient vector of $C_{\tilde{Q}}$ with respect to $\left(\mu, \sigma^2\right)^T$ and $\Sigma = \begin{bmatrix}\frac{n}{\sigma^2} & 0 \\ 0 & \frac{n}{2\sigma^4}\end{bmatrix}$ is the Fisher information matrix. Finally, using the asymptotic covariance matrix of MLEs Σ^{-1}, one can conclude that

$$\nabla C_{\tilde{Q}}{}^T \, \Sigma^{-1} \, \nabla C_{\tilde{Q}} = \begin{bmatrix}\frac{\partial C_{\tilde{Q}}}{\partial \mu} & \frac{\partial C_{\tilde{Q}}}{\partial \sigma^2}\end{bmatrix} \begin{bmatrix}\frac{\sigma^2}{n} & 0 \\ 0 & \frac{2\sigma^4}{n}\end{bmatrix} \begin{bmatrix}\frac{\partial C_{\tilde{Q}}}{\partial \mu} \\ \frac{\partial C_{\tilde{Q}}}{\partial \sigma^2}\end{bmatrix} \tag{15}$$

$$= \frac{\sigma^2}{n}\left[\left(\frac{\partial C_{\tilde{Q}}}{\partial \mu}\right)^2 + 2\sigma^2\left(\frac{\partial C_{\tilde{Q}}}{\partial \sigma^2}\right)^2\right]. \tag{16}$$

Thus, the asymptotic distribution of the ML-based estimator $\widehat{C_{\tilde{Q}}}^{ML}$ can be obtained in Eq. (14). □

This ML-based estimator is a parametric plug-in estimator, which inherits the invariance property of MLE under transformations. It provides a benchmark parametric alternative for estimating Yongting's index, and its performance is evaluated and compared with other estimation approaches in the simulation study.

Remark 5.3. Under the Normality assumption for the quality characteristic, the Monte Carlo estimator (Eq. (7)) coincides with the first method-of-moments estimator (Eq. (10)), and the second method-of-moments estimator (Eq. (11))

coincides with the maximum likelihood estimator (Eq. (13)). Although numerically identical in this special case, these estimators are derived from different estimation principles (Monte Carlo, method of moments, and maximum likelihood). Presenting them separately highlights their conceptual differences and provides a clear framework for applying these general approaches to Yongting's index, particularly in Non-Normal assumption.

6. Performance evaluation of the proposed estimators based on mean square error and relative efficiency

In this section, we develop a unified performance evaluation framework for comparing the proposed estimators of Yongting's capability index $C_{\tilde{Q}}$ based on the MSE and RE criteria. The main objective of this section is twofold: (i) to assess the finite-sample behavior of the competing estimators through simulation experiments, and (ii) to provide performance-based insights that are subsequently utilized in the real data analysis presented in Section 7.

To this end, we consider m independent random samples, each of size n, generated from a Normal distribution $N(\mu, \sigma^2)$. The MSE of an estimator $\widehat{C_{\tilde{Q}}}$ is computed as

$$\text{MSE}(\widehat{C_{\tilde{Q}}}) = \frac{1}{m} \sum_{i=1}^{m} \left(\widehat{c_{\tilde{Q},i}} - C_{\tilde{Q}} \right)^2, \tag{17}$$

where $\widehat{c_{\tilde{Q},i}}$ denotes the estimate of $C_{\tilde{Q}}$ obtained from the i-th replication, and $C_{\tilde{Q}}$ is the Yongting's index, computed analytically using Eq. (3).

The MSE criterion enables a direct and interpretable comparison among the competing estimation methods. An estimator with a smaller MSE is regarded as more accurate and preferable. To facilitate reproducibility and clarify the computational procedure, Algorithm 1 summarizes the simulation steps used to evaluate the MSE of the proposed estimators.

Algorithm 1: Simulation procedure for computing MSE of Yongting's index $C_{\tilde{Q}}$ using m samples each of size n from a Normal distribution $N(\mu, \sigma^2)$.

Require:

(1) The preset fuzzy quality $\tilde{Q}$ and the desired estimator of Yongting's index $\widehat{C_{\tilde{Q}}}$.

(2) $n \geq 1$, $m \geq 1$, $\mu \in \mathbb{R}$, and $\sigma > 0$.

Ensure: MSE of the desired estimator $\widehat{C_{\tilde{Q}}}$.

for $i = 1$ `to` m **do**

`Generate independently` $X_{1,i}, \ldots, X_{n,i} \sim N(\mu, \sigma^2)$.

`Compute` $\widehat{c_{\tilde{Q},i}}$ `based on the observed sample data set` $\{x_{1,i}, \ldots, x_{n,i}\}$.

end for

`Calculate` $C_{\tilde{Q}}$ via Eq. (3).

Calculate MSE of the desired estimator $\widehat{C_{\tilde{Q}}}$ by Eq. (17).
MSE of the desired estimator $\widehat{C_{\tilde{Q}}}$.
Return MSE of the desired estimator $\widehat{C_{\tilde{Q}}}$.

Remark 6.1. The relative efficiency of two estimators $\widehat{C_{\tilde{Q}}}$ and $\widehat{C_{\tilde{Q}}}'$ is defined as

$$RE(\widehat{C_{\tilde{Q}}}, \widehat{C_{\tilde{Q}}}') = \frac{MSE(\widehat{C_{\tilde{Q}}})}{MSE(\widehat{C_{\tilde{Q}}}')} = \frac{\frac{1}{n}\sum_{i=1}^{n}(\widehat{c_{\tilde{Q}}} - C_{\tilde{Q}})^2}{\frac{1}{n}\sum_{i=1}^{n}(\widehat{c_{\tilde{Q}}}' - C_{\tilde{Q}})^2} = \frac{\sum_{i=1}^{n}(\widehat{c_{\tilde{Q}}} - C_{\tilde{Q}})^2}{\sum_{i=1}^{n}(\widehat{c_{\tilde{Q}}}' - C_{\tilde{Q}})^2}, \tag{18}$$

A relative efficiency greater than one indicates that $\widehat{C_{\tilde{Q}}}'$ is more efficient than $\widehat{C_{\tilde{Q}}}$.

The performance patterns observed in this section serve as a benchmark for interpreting the empirical results obtained from real data. In particular, the MSE-based ranking of estimators derived here is directly employed in Section 7 to identify the most suitable estimation method in a practical manufacturing application.

7. Real data application: High-tech manufacturing process

In this section, we apply the proposed estimation methods to a real-world data set from a semiconductor manufacturing process. Consistent with the evaluation framework developed in Section 6, the competing estimators are compared based on MSE-oriented performance considerations, allowing a direct identification of the most accurate method in practice.

7.1. **Illustrative example.** Photolithography is a technique used to form precise patterns on semiconductor wafers. The hard-bake process is a crucial step in photolithography, a key technique in semiconductor manufacturing. The semiconductor manufacturing process is a highly technical, involving several precise steps to convert raw materials into intricate semiconductor devices. Figure 1 indicates a semiconductor chip manufacturing process. This section delves into a presentation of a single sample consisting of 125 wafer observations has been collected when the process is believed to be in control. The time between observations is considered one hour. The flow width measurement data (in terms of microns) from this sample are shown in Table 1 [12].

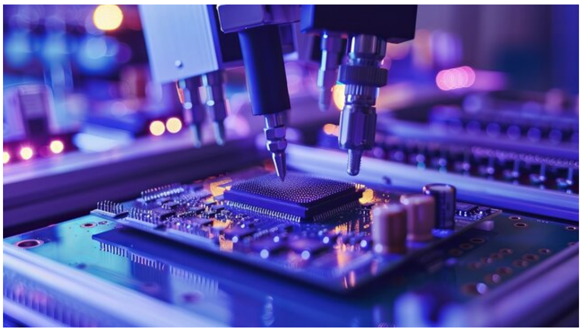

FIGURE 1. Schematic of a semiconductor chip manufacturing process.

TABLE 1. Flow width measurements (in terms of microns) for the hard-bake process.

1.3235	1.4871	1.5265	1.4177	1.3688	1.5821	1.4036	1.5171	1.6210	1.7601	1.5917	1.3663
1.4128	1.4932	1.4363	1.5144	1.4039	1.3355	1.5893	1.1839	1.5573	1.4371	1.4333	1.6240
1.6744	1.4324	1.6441	1.4190	1.6697	1.5777	1.6458	1.8662	1.5796	1.5051	1.5551	1.3732
1.4573	1.5674	1.5955	1.4303	1.5089	1.3908	1.4969	1.3680	1.4185	1.3485	1.5295	1.6887
1.6914	1.5028	1.5451	1.6637	1.4627	1.7559	1.3589	1.7269	1.6541	1.5670	1.6866	
1.4314	1.6352	1.3574	1.6067	1.5220	1.2856	1.2863	1.3957	1.5116	1.4880	1.6399	
1.3592	1.3841	1.3281	1.5519	1.4158	1.4106	1.5996	1.5014	1.7247	1.4738	1.5243	
1.6075	1.2831	1.4198	1.3884	1.7667	1.4447	1.2497	1.4449	1.7106	1.5936	1.5705	
1.4666	1.5507	1.6274	1.7277	1.4278	1.6398	1.5471	1.4163	1.4412	1.6583	1.5563	
1.6109	1.5604	1.5064	1.5355	1.5928	1.1928	1.5747	1.3864	1.2361	1.4973	1.5530	
1.4284	1.2735	1.8366	1.5176	1.4181	1.4951	1.5301	1.3057	1.3820	1.4720	1.5797	

In this work, by considering the specified membership function of the non-symmetric trapezoidal fuzzy quality

$$\tilde{Q}(x) = \begin{cases} \frac{x-1.183}{0.233} & if \quad 1.183 \le x < 1.416, \\ 1 & if \quad 1.416 \le x < 1.594, \\ \frac{1.866-x}{0.272} & if \quad 1.594 \le x < 1.866, \\ 0 & \quad elsewhere, \end{cases} \tag{19}$$

with the histogram of the observed data and the fitted density function are drawn in Figure 2. It should be noted that the function $\tilde{Q}(x)$ in Eq. (19) represents the fuzzy quality, which is defined based on the quality engineer's expertise, the specific requirements of the production process, or determined according to industry standards, regulatory guidelines, or customer expectations.

We are going to estimate the Yongting's capability index based on the following approaches: (1) Kernel density estimation, (2) Monte Carlo simulation, (3) method of moments estimation, and (4) maximum likelihood estimation.

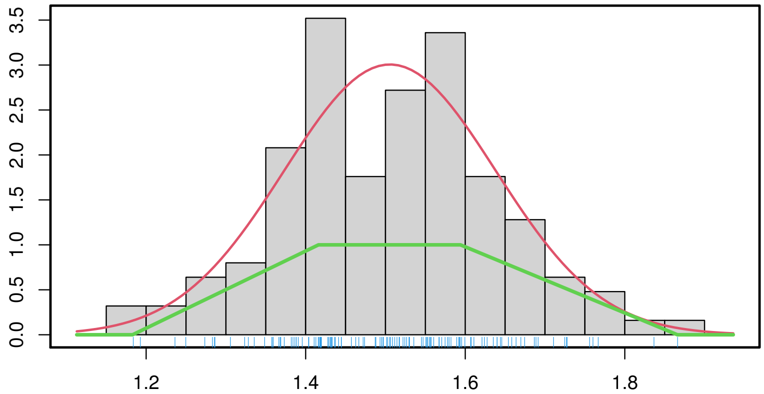

FIGURE 2. The membership function of trapezoidal fuzzy quality $\tilde{Q}$ with histogram of the observed data and the fitted density function.

Therefore, in this case study, the KD-based estimator of the Yongting capability index can be computed using triangular, rectangular, and Gaussian kernel functions with the optimal smoothing parameter $h = 0.04513$. Note that, according to Remark 2.1, for our dataset ($n = 125$, $\hat{\sigma} = 0.1332$, $IQR = 0.1765$), we approximate $\hat{h} \approx 0.04513$, ensuring appropriate smoothing over the support interval $[1, 2]$. For instance, the KD-based estimate of Yangting's capability index using the triangular kernel function is calculated as follows:

$$\begin{aligned}
\hat{c}^{KD}_{\tilde{Q}} &= \int_{-\infty}^{+\infty} \tilde{Q}(x)\, \frac{1}{125h} \sum_{i=1}^{n} K\left(\frac{x - x_i}{h}\right) dx \\
&= \frac{1}{125 \times 0.04513} \int_{1.183}^{1.416} \frac{x - 1.183}{0.233} \sum_{i=1}^{125} \max\{1 - \left|\frac{x - x_i}{0.04513}\right|, 0\} dx \\
&+ \frac{1}{125 \times 0.04513} \int_{1.416}^{1.594} \sum_{i=1}^{125} \max\{1 - \left|\frac{x - x_i}{0.04513}\right|, 0\} dx \\
&+ \frac{1}{125 \times 0.04513} \int_{1.594}^{1.866} \frac{1.866 - x}{0.272} \sum_{i=1}^{125} \max\{1 - \left|\frac{x - x_i}{0.04513}\right|, 0\}\, dx \\
&= 0.8410.
\end{aligned} \tag{20}$$

Similarly, the KD-based estimators of the Yongting's capability index, based on rectangular and Gaussian kernel functions, are calculated as 0.8375 and 0.8246, respectively. Figure 3 visually compares the three kernel density estimations (triangular, rectangular, and Gaussian) of the flow width data. This figure illustrates how each kernel function smooths the empirical data distribution, with all three methods providing consistent support for estimating Yongting's capability index. All three estimators produce smoothed density curves that closely follow the empirical distribution of the data while incorporating the considered fuzzy quality $\tilde{Q}$ in Eq. (19). The Gaussian kernel yields the smoothest curve, while the rectangular kernel produces a slightly more variable estimate. The triangular kernel offers a balance between smoothness and local adaptability. Importantly, despite these visual differences in the density estimates, all three kernel-based methods yield remarkably consistent point estimates for Yongting's index (0.8410, 0.8375, and 0.8246), demonstrating the relative robustness of the KD estimator to the choice of kernel function in this application. This consistency reinforces the practical utility of the KD-based approach.

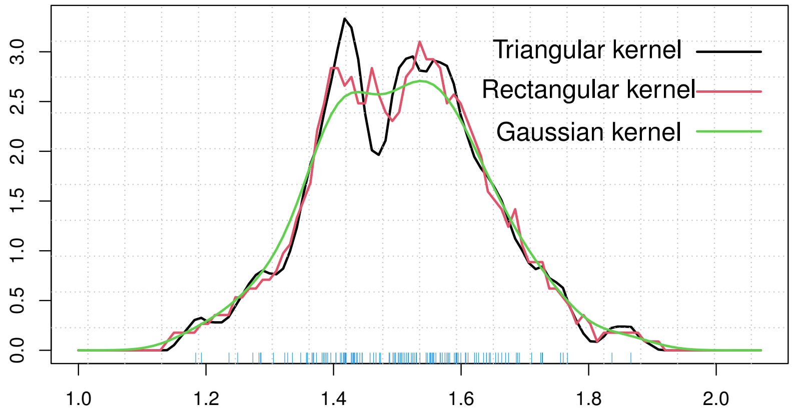

FIGURE 3. Kernel density estimation of the flow width in the case study with different kernel functions based on the trapezoidal fuzzy quality $\tilde{Q}$.

The MC-based estimation of Yongting's index is $\widehat{c_{\tilde{Q}}}^{MC} = \frac{1}{125}\sum_{i=1}^{125}\tilde{Q}(x_i) = 0.8450$, and also two different MM-based estimations of Yongting's index are

$$\widehat{c}_{\tilde{Q}}^{MM1} = \frac{1}{n}\sum_{i=1}^{n}\tilde{Q}(x_i) = 0.8450 \tag{21}$$

and

$$\widehat{c}_{\tilde{Q}}^{MM2} = \frac{\int_{1.183}^{1.866}\tilde{Q}(x)\ exp\left[-\frac{(x-\overline{x})^2}{2\left(\overline{x^2}-\overline{x}^2\right)}\right]dx}{\sqrt{2\pi\left[\overline{x^2}-\overline{x}^2\right]}} = 0.8433, \tag{22}$$

in which $\overline{x} = 1.5056$ and $\overline{x^2} = 2.2844$. Additionally, the ML-based estimation of Yongting's index is equal to

$$
\begin{aligned}
\widehat{c}_{\tilde{Q}}^{ML} &= \frac{1}{\sqrt{2\pi} s_n} \int_{-\infty}^{+\infty} \tilde{Q}(x)\ exp\Big[-\frac{(x-\overline{x})^2}{2s_n^2}\Big] dx \\
&= \frac{1}{\sqrt{2\pi} s_n} \int_{1.183}^{1.416} \frac{x-1.183}{0.233}\ exp\Big[-\frac{(x-\overline{x})^2}{2s_n^2}\Big] dx \\
&+ \frac{1}{\sqrt{2\pi} s_n} \int_{1.416}^{1.594} exp\Big[-\frac{(x-\overline{x})^2}{2s_n^2}\Big] dx \\
&+ \frac{1}{\sqrt{2\pi} s_n} \int_{1.594}^{1.866} \frac{1.866-x}{0.272}\ exp\Big[-\frac{(x-\overline{x})^2}{2s_n^2}\Big] dx \\
&= 0.8433,
\end{aligned}
\tag{23}
$$

where $\overline{x} = 1.5056$, and $s_n = 0.1332$. It should be mentioned that the first MM-based and MC-based estimators of the Yongting's index are equivalent, as well as the second MM-based and ML-based estimators of the Yongting's index in this case study.

7.2. **Discussion and comparison based on MSE.** Herein, we present a comparative analysis of the KD-based, MC-based, MM-based, and ML-based estimators. To illustrate the computational complexity, Algorithm 1 provides a step-by-step guide for calculating the MSE values of the proposed estimators $\widehat{C}_{\tilde{Q}}^{KD}$, $\widehat{C}_{\tilde{Q}}^{MC}$, $\widehat{C}_{\tilde{Q}}^{MM2}$ and $\widehat{C}_{\tilde{Q}}^{ML}$. For a given sample size n, we generate $m = 1000$ random samples, each of size n, drawn from the Normal distribution $X_{1,i}, \ldots, X_{n,i} \overset{i.i.d.}{\sim} N\left(\mu, \sigma^2\right)$, for $i = 1, \ldots, 1000$. Then, the MSE criterion is used to evaluate the KD-based estimator of Yongting's capability index (using the Gaussian kernel function) and the MC-based, MM-based, and ML-based estimators of $C_{\tilde{Q}}$, based on the sample data $\{x_{1,i}, \ldots, x_{n,i}\}$ for $i = 1, \ldots, m$. The corresponding MSEs are computed using the formulas $\frac{1}{m}\sum_{i=1}^{m}(\widehat{c}_{\tilde{Q},i}^{KD} - C_{\tilde{Q}})^2$, $\frac{1}{m}\sum_{i=1}^{m}(\widehat{c}_{\tilde{Q},i}^{MC} - C_{\tilde{Q}})^2$, $\frac{1}{m}\sum_{i=1}^{m}(\widehat{c}_{\tilde{Q},i}^{MM2} - C_{\tilde{Q}})^2$ and $\frac{1}{m}\sum_{i=1}^{m}(\widehat{c}_{\tilde{Q},i}^{ML} - C_{\tilde{Q}})^2$, where $C_{\tilde{Q}}$ is calculated using Eq. (3). Moreover, $\widehat{c}_{\tilde{Q},i}^{KD}$, $\widehat{c}_{\tilde{Q},i}^{MC}$, $\widehat{c}_{\tilde{Q},i}^{MM2}$, and $\widehat{c}_{\tilde{Q},i}^{ML}$ for $m = 1000$ are respectively computed using Eqs. (4), (7), (11) and (13). To clarify the procedure for calculating the MSE values summarized in Table 2, Algorithm 1 generates sample data sets based on the preset parameters $m = 1000$, $\mu = 1.5056$, $\sigma = 0.1332$ and the considered fuzzy quality in Eq. (19). This process is carried out for various sample sizes $n = 15, 25, \cdots, 125$ (i.e., n from 15 to 125 in steps of 10), as shown in Table 2.

For example, in the second row of Table 2, if we independently generate $m = 1000$ samples, each of size $n = 15$, from the Normal distribution $N(\mu = 1.5056, \sigma^2 = 0.1332^2)$, for $i = 1, \ldots, 1000$, the results are obtained as described. By applying Algorithm 1 using the KD-based estimator of $C_{\tilde{Q}}$, constructed with the Gaussian kernel function ($\widehat{C}_{\tilde{Q}}^{KD}$), and based on the sample

TABLE 2. MSE results for Yongting's capability index $C_{\tilde{Q}}$ based on $N(\mu = 1.5056, \sigma^2 = 0.1332^2)$ and various sample sizes, considering the fuzzy quality in Eq. (19).

n	Kernel estimation	MC estimation	MM estimation	ML estimation
15	0.005268	0.003699	0.003196	0.003196
25	0.003355	0.002242	0.001902	0.001902
35	0.002674	0.001632	0.001450	0.001450
45	0.002087	0.001270	0.001152	0.001152
55	0.001576	0.000956	0.000871	0.000871
65	0.001469	0.000887	0.000793	0.000793
75	0.001213	0.000706	0.000678	0.000678
85	0.001140	0.000632	0.000577	0.000577
95	0.000943	0.000573	0.000538	0.000538
105	0.000967	0.000559	0.000504	0.000504
115	0.000855	0.000497	0.000457	0.000457
125	0.000792	0.000469	0.000429	0.000429

data $\{x_{1,i}, \ldots, x_{14,i}, x_{15,i}\}$ for $i = 1, \ldots, 1000$, the MSE of $\widehat{C}_{\tilde{Q}}^{KD}$ is calculated using

$$MSE(\widehat{C}_{\tilde{Q}}^{KD}) = \frac{1}{1000} \sum_{i=1}^{1000} (\widehat{c}_{\tilde{Q},i}^{KD} - C_{\tilde{Q}})^2 = 0.005268,$$

where $C_{\tilde{Q}}$ is calculated by

$$\begin{aligned}
C_{\tilde{Q}} &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} \tilde{Q}(x)\ exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx \\
&= \frac{1}{0.1332\sqrt{2\pi}} \int_{1.183}^{1.416} \frac{x - 1.183}{0.233}\ exp\left[-\frac{(x-1.5056)^2}{2 \times 0.1332^2}\right] dx \\
&+ \frac{1}{\sqrt{2\pi}\sigma} \int_{1.416}^{1.594} exp\left[-\frac{(x-1.5056)^2}{2 \times 0.1332^2}\right] dx \\
&+ \frac{1}{\sqrt{2\pi}S_n} \int_{1.594}^{1.866} \frac{1.866 - x}{0.272}\ exp\left[-\frac{(x-1.5056)^2}{2 \times 0.1332^2}\right] dx \\
&= 0.8420.
\end{aligned} \tag{24}$$

The outcomes in Table 2 indicate a significant influence of sample size on the MSE values for all four estimation methods —Kernel estimation, Monte Carlo estimation, method of moments estimation, and maximum likelihood estimation— decrease as the sample size (n) increases. This is a common outcome in statistical estimation, where larger sample sizes lead to more precise estimates. A comparison of the MSE values in Table 2 reveals that the MM-based and ML estimators provide more optimal results for sample sizes $n =$

$15, 25, \cdots, 85$, while the KD-based estimators outperform others for sample sizes $n = 95, \cdots, 125$. This is attributed to the minimal impact of varying sample size quantities on the MSE for these estimators. Based on the MSE values from this simulation study, the proposed KD-based estimators are more suitable for estimation tasks with large sample sizes. Based on the MSE values from this simulation study, the proposed KD-based estimators are more suitable for estimation tasks with large sample sizes.

Additionally, in the last row of Table 2, the relative efficiency values of the four proposed estimators are computed as follows: $RE(\widehat{C}_{\tilde{Q}}^{KD}, \widehat{C}_{\tilde{Q}}^{MC}) = 1.69$, $RE(\widehat{C}_{\tilde{Q}}^{KD}, \widehat{C}_{\tilde{Q}}^{MM}) = 1.85$, $RE(\widehat{C}_{\tilde{Q}}^{KD}, \widehat{C}_{\tilde{Q}}^{ML}) = 1.85$, $RE(\widehat{C}_{\tilde{Q}}^{MC}, \widehat{C}_{\tilde{Q}}^{MM}) = 1.09$, $RE(\widehat{C}_{\tilde{Q}}^{MC}, \widehat{C}_{\tilde{Q}}^{ML}) = 1.09$, and $RE(\widehat{C}_{\tilde{Q}}^{MM}, \widehat{C}_{\tilde{Q}}^{ML}) = 1$. Based on these values, it can be concluded that the KD-based estimator is more efficient than the other estimators for a sample size of $n = 125$. The observed differences in RE values suggest opportunities to explore why the KD-based estimator performs better. Factors such as data distribution, estimator assumptions, or parameter settings could be further examined.

7.3. **Comparative advantages and theoretical justification of the methods.** The MSE results in Table 2 provide a quantitative performance ranking, but a qualitative discussion of each method's inherent strengths and weaknesses is crucial for a convincing comparison.

- **KD Estimation:** This is a non-parametric method that does not assume a specific underlying distribution (e.g., Normality). Its primary advantage is flexibility; it can adapt to the shape of the data distribution, which is beneficial when the Normal assumption is questionable or for large samples where the data's structure can be reliably estimated. However, this flexibility comes at a cost: (1) It introduces a bias-variance trade-off controlled by the bandwidth parameter h. A suboptimal h can lead to over-smoothing (high bias) or under-smoothing (high variance). (2) For small sample sizes, the variance of the KD estimator is typically higher than that of parametric methods, as confirmed by its larger MSE for $n < 95$ in our study. (3) It is computationally more intensive than simple parametric methods.
- **MC Estimation:** This method is the simplest and most direct, calculating the index as the sample average of the fuzzy membership grades. Like the KD method, it is distribution-free. Its main strength is simplicity and low computational cost. However, it is essentially a simple plug-in estimator that does not perform any density smoothing or parametric modeling. Consequently, while it performs reasonably well across all sample sizes, it is often outperformed in efficiency (lower MSE) by methods that leverage more structural information (parametric assumptions for small n, smooth density estimates for large n).

- **MM and ML Estimation:** These are parametric methods that assume the data follows a Normal distribution $N(\mu, \sigma^2)$. Their key strength is statistical efficiency. When the distributional assumption is valid —as it is in our simulation setup— they extract the maximum information from the sample using sufficient statistics ($\bar{x}$ and s_n^2). This leads to the lowest MSE for small to moderate sample sizes ($n \leq 85$), as observed. Their primary weakness is their reliance on the Normality assumption. If this assumption is violated in practice, these estimators can be biased and misleading. Computationally, they are very efficient once the distribution parameters are estimated.

The observed crossover in performance —where the KD estimator becomes superior for $n \geq 95$— can be explained by this interplay. For large samples, the KD estimator's variance reduces significantly, and its ability to model the density without restrictive parametric assumptions allows it to achieve a slightly better fit than the methods forced into a Normal shape, even when the data is truly Normal. This demonstrates the KD estimator's asymptotic consistency and its advantage in scenarios with abundant data.

This pattern aligns perfectly with the theoretical discussion above. The parametric (MM/ML) methods are most efficient when their underlying assumption holds, dominating in small to medium samples. The non-parametric KD method requires more data to reduce its variance but eventually outperforms by avoiding any potential model misspecification bias, even in this correctly specified simulation. The MC method serves as a robust baseline.

Although the value of $C_{\tilde{Q}}$ is unknown for real data, the performance ranking and theoretical rationale obtained from the simulation study in Section 7 provide valuable guidance. Specifically, the MM-based and ML-based estimators exhibited the smallest MSE values for moderate sample sizes due to their parametric efficiency, while the KD-based estimator became more competitive and ultimately superior as the sample size increased, thanks to its non-parametric flexibility and consistency.

Given that the present data set consists of $n = 125$ observations —a large sample size— the KD-based estimator is expected to provide the most accurate performance in terms of MSE. Therefore, among the competing methods, the KD-based estimator is selected as the preferred estimator for this real data application. For practical use, if the sample size were smaller (e.g., $n < 50$) and Normality tests were passed, the MM or ML estimators would be recommended for their higher precision.

8. Conclusions and future works

The paper introduced and discussed on following four different statistical estimation approaches for point estimating the Yongting's capability index: (1) kernel density estimation, (2) Monte Carlo estimation, (3) method of moments estimation, (4) maximum likelihood estimation. The article acknowledges the diversity of the proposed estimation approaches, showcasing a range of statistical techniques. This diversity is essential in catering to different data scenarios and preferences in statistical modeling. The conclusions highlight the applicability and flexibility of the proposed estimation methods. Researchers and practitioners can choose the most suitable approach based on the nature of their data and the specific requirements of their industrial processes. The paper contributes to the field by discussing several statistical estimation approaches for evaluating Yongting's capability index based on fuzzy quality and provides a comprehensive comparative study of their performance. . This enhances the methodological toolkit available for researchers and practitioners engaged in evaluating and improving industrial production processes.

8.1. **Potential applications in actuarial science and other fields.** While this study demonstrates the methodology in semiconductor manufacturing, the proposed fuzzy-process capability framework has potential applications in other domains requiring tolerance-based assessment with vague specifications. In actuarial science, for instance, it could be adapted to evaluate insurance claim processes where "acceptable" claim amounts are defined by fuzzy thresholds rather than crisp boundaries. Similarly, in finance, it could assess risk management processes against fuzzy risk tolerance levels. In healthcare quality control, it could evaluate medical device performance against fuzzy safety standards. These applications represent fruitful directions for future research. Also, investigation on confidence interval estimation for Yongting's index can be another potentiol field in future researches.

Furthermore, this article suggests several promising avenues for future research. These include: conducting comparative studies of the estimation methods across diverse industrial contexts; exploring in greater depth the impact of sample size on estimation accuracy; investigating confidence interval estimation for Yongting's index; and developing an adaptive hybrid estimator that intelligently combines the strengths of the parametric (MM/ML) and non-parametric (KD/MC) approaches based on data characteristics like sample size and distributional fit. Such an estimator would offer a robust, data-driven solution for practitioners. These directions represent fruitful opportunities to extend both the theoretical foundations and practical utility of fuzzy process capability analysis.

9. **Author Contributions**

All authors have read and agreed to the published version of the manuscript.

10. Data Availability Statement

The data that support the findings of this study are available on request from the corresponding author. The data are not publicly available due to privacy or ethical restrictions.

11. Aknowledgement

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12. Ethical considerations

The authors avoided from data fabrication and falsification.

13. Funding

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14. Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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