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EXACTNESS PRESERVATION BY FUNCTORS MAPPING MODULES TO SEMIMODULES OF VARIETIES OF SUBMODULES

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Abstract. Let R be a commutative ring with identity, and M be an R-module. We denote by $\zeta(M)$ the semimodule consisting of all varieties of submodules of M over the semiring $\zeta(R)$ of varieties of ideals of R. In this article, we introduce two functors from the category of R-modules to the category of $\zeta(R)$-semimodules and investigate conditions under which these functors preserve the short exact sequences of modules. They are the hom-functors $\mathrm{Hom}_{\zeta(R)}(\zeta(P), \zeta(-))$ and $\mathrm{Hom}_{\zeta(R)}(\zeta(-), \zeta(E))$ associated with R-modules P and E, respectively. It is shown that ζ is exact and both $\mathrm{Hom}_{\zeta(R)}(\zeta(P), \zeta(-))$ and $\mathrm{Hom}_{\zeta(R)}(\zeta(-), \zeta(E))$ are left exact on any short exact sequence $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ with M'' a radical module. In particular, we provide conditions under which the hom-functors preserve short exact sequences of modules.

Keywords: Variety of submodule, Semimodule, Semimodule homomorphism, Exact sequence, Radical submodule.
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1. Introduction

The study of exactness and homological properties in the setting of semimodules has attracted increasing attention in recent years. In this context, understanding how exact sequences of modules interact with constructions arising from prime submodules provides useful insight for the development of a homological framework adapted to semimodule theory.

Several notions of exactness for semimodules have been introduced by different authors (see, for example, [1,5,6,8]). In this paper, we consider Patchkoria's notion of exactness for semimodules [5] and study a construction that assigns to each R-module M a semimodule $\zeta(M)$ consisting of the varieties of submodules of M. In particular, when $M = R$, this construction yields a commutative semiring $\zeta(R)$.

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Moreover, any R-module homomorphism $f : M \to M'$ naturally induces a $\zeta(R)$-semimodule homomorphism

$$\zeta(f) : \zeta(M) \longrightarrow \zeta(M'),$$

defined by sending the variety of a submodule N of M to the variety of $f(N)$. In this way, the assignments $M \mapsto \zeta(M)$ and $f \mapsto \zeta(f)$ become a functor from the category of R-modules to the category of $\zeta(R)$-semimodules.

Our first main result shows that if

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

is a short exact sequence of R-modules, then the induced sequence

$$0 \longrightarrow \zeta(M') \xrightarrow{\zeta(f)} \zeta(M) \xrightarrow{\zeta(g)} \zeta(M'') \longrightarrow 0$$

is exact whenever M'' is a radical module (Lemma 4.1). Furthermore, for R-modules P and E, we show that the assignments $\mathrm{Hom}_{\zeta(R)}(\zeta(P), \zeta(-))$ and $\mathrm{Hom}_{\zeta(R)}(\zeta(-), \zeta(E))$ define covariant and contravariant functors, respectively. These functors are left exact on any short exact sequence

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

provided that M' and M'' are radical R-modules (Theorems 4.4 and 4.8).

We also determine conditions under which these functors are exact. For example, if V is a vector space, then $\mathrm{Hom}_{\zeta(R)}(\zeta(V), \zeta(-))$ preserves short exact sequences of vector spaces (Corollary 4.5). If R is a field, then $\mathrm{Hom}_{\zeta(R)}(\zeta(-), \zeta(R))$ preserves short exact sequences of R-vector spaces (Corollary 4.11).

2. **Preliminaries**

In this section, we recall basic notions concerning semirings, semimodules, prime submodules, and varieties.

Semirings and Semimodules. A commutative semiring $(R; +, \cdot, 0_R, 1_R)$ consists of two commutative monoids $(R; +, 0_R)$ and $(R; \cdot, 1_R)$ such that multiplication distributes over addition and $r \cdot 0_R = 0_R$ for all $r \in R$.

A commutative monoid $(M; +, 0_M)$ is called an R-semimodule if R acts on M as in the case of modules and, in addition, $r \cdot 0_M = 0_R \cdot m = 0_M$ for all $r \in R$ and $m \in M$.

A subset N of an R-semimodule M is called a *subsemimodule* if it is closed under addition and scalar multiplication. A subsemimodule N is called *subtractive* if $m + n \in N$ with $n \in N$ implies $m \in N$.

For any subset $X \subseteq M$, the *subtractive closure* of X is

$$\overline{X} = \{m \in M \mid m + x_1 = x_2 \text{ for some } x_1, x_2 \in X\}.$$

A subsemimodule N is subtractive if and only if $N = \overline{N}$.

Let M and M' be R-semimodules. A semigroup homomorphism $f : M \to M'$ is called an R-semimodule homomorphism if $f(rm) = rf(m)$ for all $r \in R$

and $m \in M$. Its kernel is $\operatorname{Ker} f = f^{-1}(0_{M'})$, which is a subtractive subsemimodule of M, and its image $\operatorname{Im} f$ is a (not necessarily subtractive) subsemimodule of M'.
An R-semimodule homomorphism is an R-monomorphism(resp.R-epimorphism) in the usual categorical sense; it is an R-isomorphism if it is both. A surjective homomorphism with zero kernel is called an R-semiisomorphism, denoted by $M \simeq_R M'$. Every R-isomorphism is an R-semiisomorphism, but not conversely. Moreover, an R-homomorphism is injective (resp. surjective) if and only if it is an R-monomorphism (resp. an R-epimorphism whose image is subtractive) [2, Proposition 15.15].

An R-semimodule homomorphism $f : M \to M'$ is called *steady* if $f(m_1) = f(m_2)$ implies that $m_1 + k_1 = m_2 + k_2$ for some $k_1, k_2 \in \operatorname{Ker} f$. Every module homomorphism is steady.

Each subsemimodule N of M induces an R-congruence relation $\equiv_N$ on M, called the *Bourne relation*, defined by $m_1 \equiv_N m_2$ if $m_1 + n_1 = m_2 + n_2$ for some $n_1, n_2 \in N$. The corresponding quotient semimodule is denoted by M/N (see [2]).

A sequence $M' \xrightarrow{f} M \xrightarrow{g} M''$ of semimodules is exact at M if $\operatorname{Im} f = \operatorname{Ker} g$ [5]. A sequence is exact if it is exact at each junction.

Prime Submodules and Varieties. Let R be a commutative ring with identity and, M a unitary R-module. A proper submodule P of M is called a *prime submodule* if $rx \in P$ implies that $rM \subseteq P$ or $x \in P$. The set of all prime submodules of M is denoted by $\operatorname{Spec}(M)$.

The radical of a submodule N, denoted $\operatorname{rad} N$, is the intersection of all prime submodules of M containing N. A submodule N is called *radical* if $\operatorname{rad}(N) = N$, and M is called a *radical module* if $\operatorname{rad}(0) = 0$. We denote by $\mathcal{R}(M)$ the set of all radical submodules of M.

For a submodule N of M, its *variety* is $V(N) = \{P \in \operatorname{Spec}(M) \mid N \subseteq P\}$.

We define $\zeta(M) = \{V(N) \mid N\text{is a submodule of } M\}$ and $\zeta(R) = \{V(I) \mid I\text{is an ideal of } R\}$.

It is known that $\zeta(R)$ becomes a commutative semiring under the operations

$$V(I) + V(J) = V(I + J), \qquad V(I) \cdot V(J) = V(IJ),$$

where $I + J$ and IJ are the usual sum and product of ideals.

Moreover, $\zeta(M)$ becomes a $\zeta(R)$-semimodule with operations

$$V(N) + V(L) = V(N + L), \qquad V(I) \cdot V(N) = V(IN),$$

and zero element $0_{\zeta(M)} = V(0)$.

If $f : M \to M'$ is an R-module homomorphism, then $\zeta(f) : \zeta(M) \to \zeta(M')$ defined by $\zeta(f)(V(N)) = V(f(N))$ is a $\zeta(R)$-semimodule homomorphism [4]. In particular, if f is an epimorphism, then $\operatorname{Ker} \zeta(f) = \overline{\{V(\operatorname{Ker} f)\}}$ and $\zeta(M)/\overline{\{V(\operatorname{Ker} f)\}} \cong \zeta(M')$ [4, Theorem 14]. In particular, ζ preserves surjective homomorphisms and isomorphisms, but not necessarily injective homomorphisms (see [4, Lemma 8] and the example following it).

3. On semimodule of varieties of submodules

Let R be a ring and M be an R-module. Recall that for any submodule N of M, the variety $V(N)$ of N is the collection of all prime submodules of M containing N, and $\zeta(M)$ is the set of all varieties of submodules of M. As shown in [4, Theorem 6], if $f : M \to M'$ is an R-module homomorphism, then $\zeta(f) : \zeta(M) \to \zeta(M')$ defined by $\zeta(f)(V(N)) = V(f(N))$ is a $\zeta(R)$-semimodule homomorphism. With this definition, for any two R-homomorphisms $M' \xrightarrow{f} M$ and $M \xrightarrow{g} M''$, we have $\zeta(g \circ f) = \zeta(g) \circ \zeta(f)$ and $\zeta(1_M) = 1_{\zeta(M)}$, i.e., ζ is a covariant functor from the category of R-modules to the category of $\zeta(R)$-semimodules.

Lemma 3.1. *Let $f : M \to M'$ be a surjective R-module homomorphism and N be a submodule of M such that* $\operatorname{Ker} f \subseteq N$. *Then*

$$V(f(N)) = \{f(P) \mid P \in \operatorname{Spec}(M) \text{ and } P \supseteq N\}.$$

Proof. We first observe that the inclusion $\subseteq$ is immediate, since for any $P' \in V(f(N)$, if we let $P := f^{-1}(P')$, then P is a prime submodule of M containing N. Moreover, the surjectivity of f yields tat $P' = f(f^{-1}(P')) = f(P)$, so P' belongs to the right-hand set. For the reverse inclusion, let P be a prime submodule containing N. We will show that $f(P)$ is a prime submodule of M'. First, note that $f(P) \neq M'$, for otherwise, for every $m \in M$, there exists $n \in P$ such that $m - n \in \operatorname{Ker} f$. By assumption, this implies $m \in P$, and hence $M = P$, which is a contradiction. Now, let $rf(m) \in f(P)$ for some $r \in R$ and $m \in M$. Then there exists $n \in P$ such that $rm - n \in \operatorname{Ker} f$, and hence, by assumption, $rm \in P$. Since $P \in \operatorname{Spec}(M)$, it follows that either $rM \subseteq P$ or $m \in P$. Therefore, either $rM' \subseteq f(P)$ or $f(m) \in f(P)$, which shows that $f(P) \in \operatorname{Spec}(M')$, as required. □

Note that, by [4, Lemma 8], ζ preserves the surjectivity and bijectivity of R-homomorphisms, but it does not necessarily preserve the injectivity (see [4, p. 1362]). Nevertheless, the following result holds.

Lemma 3.2. *Let $f : M \to M'$ be an R-module injective homomorphism such that* $f(\operatorname{Spec}(M)) \subseteq \operatorname{Spec}(M')$. *Then $\zeta(f) : \zeta(M) \to \zeta(M')$ is injective.*

Proof. Let N and L be submodules of M and $\zeta(f)(V(N)) = \zeta(f)(V(L))$. Then $V(f(N)) = V(f(L))$. We assume that $P \in \operatorname{Spec}(M)$ and $N \subseteq P$. Hence, by the hypothesis, $f(P)$ is a prime submodule of M' containing $f(N)$. Thus $f(P) \in V(f(L))$, in other words , $f(L) \subseteq f(P)$. Now, since f is injective, we have $L \subseteq P$. Therefore $V(N) \subseteq V(L)$. Similarly, we have $V(L) \subseteq V(N)$. We are done. □

The following examples illustrate Lemma 3.2.

Example 3.3. *Let* V *and* V' *be two* F*-vector spaces, and let* $f : V \to V'$ *be an injective linear transformation. Since the prime spectrum of any vector space is the set of all its proper submodules, we have* $f(\mathrm{Spec}(V)) \subseteq \mathrm{Spec}(V')$*. Moreover, for any* $P \in V(N) \setminus V(L)$*, we have* $f(P) \in V(f(N)) \setminus V(f(L))$*, which shows that* $\zeta(f)$ *is injective.*

Example 3.4. *It should be noted that* Spec *may or may not preserve any inclusion of* R*-modules. In the case where it does not, consider the ring of integers* $\mathbb{Z}$ *and the abelian group* $\mathbb{Q}$ *(the rationals). Here,* $\mathrm{Spec}(\mathbb{Z}) \supset \mathrm{Spec}(\mathbb{Q})$*, because*

$$\mathrm{Spec}(\mathbb{Z}) = \{(0)\} \cup \{p\mathbb{Z} \mid p \textit{ is a prime number}\},$$

and $\mathrm{Spec}(\mathbb{Q}) = \{(0)\}$*.*

However, in the case where Spec *preserves the inclusion, consider* $M = \mathbb{Z}(p^\infty)$ *and* $M' = \mathbb{Q}/\mathbb{Z}$*, which are well-known primeless* $\mathbb{Z}$*-modules. In this case,* Spec *preserves the inclusion* $i : M \to M'$*, and* $\zeta(i) : \zeta(M) \to \zeta(M')$ *is clearly injective.*

Example 3.5. *Consider the* $\mathbb{Z}$*-module* $\mathbb{Q} \oplus \mathbb{Z}$*, and the injection*

$$\iota : \mathbb{Q} \to \mathbb{Q} \oplus \mathbb{Z}, \quad \iota(x) = (0, x).$$

Since $\mathrm{Spec}(\mathbb{Q}) = \{(0)\}$*, we have*

$$\mathrm{Spec}(\mathbb{Q} \oplus \mathbb{Z}) = \{(0) \oplus \mathbb{Z}, \mathbb{Q} \oplus (0)\} \cup \{\mathbb{Q} \oplus (p) \mid p \textit{ is a prime number}\}.$$

Thus,

$$\iota(\mathrm{Spec}(\mathbb{Q})) = \iota(\{0\}) = \{(0) \oplus \mathbb{Z}\} \subseteq \mathrm{Spec}(\mathbb{Q} \oplus \mathbb{Z}).$$

Now, since $\zeta(\mathbb{Q}) = \{\emptyset, (0)\}$*, we have*

$$\begin{aligned}\zeta(\iota)(\emptyset) &= \zeta(\iota)(V(\mathbb{Q})) = V(\iota(\mathbb{Q})) = V(\mathbb{Q} \oplus (0)) \\ &= \{\mathbb{Q} \oplus (0)\} \cup \{\mathbb{Q} \oplus (p) \mid p \textit{ is a prime number}\},\end{aligned}$$

and

$$\zeta(\iota)((0)) = \zeta(\iota(V(0))) = V(\iota(0)) = V((0) \oplus (0)) = \mathrm{Spec}(\mathbb{Q} \oplus \mathbb{Z}).$$

Therefore, we find that $\zeta(\iota)$ *is injective.*

Lemma 3.6. *Let* $f : M \to M'$ *be an* R*-module homomorphism. Then* $\zeta(f) : \zeta(M) \to \zeta(M')$ *is injective if and only if* $\mathrm{Ker}\,\zeta(f) = \{V(0)\}$ *and* $\zeta(f)$ *is steady.*

Proof. $\Rightarrow$) Clear.
$\Leftarrow$) Let $\zeta(f)(V(N)) = \zeta(f)(V(L))$ for submodules N and L of M. Since $\zeta(f)$ is steady, there exist $V(T_1), V(T_2) \in \mathrm{Ker}\,\zeta(f)$ such that $V(N) + V(T_1) = V(L) + V(T_2)$. Thus by hypothesis $V(T_1) = V(T_2) = V(0)$, and so $V(N) = V(L)$ which shows $\zeta(f)$ is injective. □

Theorem 3.7. *Let* $f : M \to M'$ *be an* R*-module homomorphism. Then*

$$\overline{\{\mathrm{Im}\zeta(\mathrm{f})\}} = \{V(0)/\,\mathrm{Im}\zeta(\mathrm{f})\}.$$

Proof. Let $V(N) \in \{V(0)/\operatorname{Im}\zeta(\mathrm{f})\}$. Then $V(N)$ is equivalent to $V(0)$ modulo $Im\zeta(f)$ under the Bourne relation. Thus there exist $V(N'), V(L') \in \operatorname{Im}\zeta(\mathrm{f})$ such that $V(N)+V(N') = V(0)+V(L') = V(L')$. Hence $V(N) \in \overline{\{\operatorname{Im}\zeta(\mathrm{f})\}}$ i.e $\{V(0)/\operatorname{Im}\zeta(\mathrm{f})\} \subseteq \overline{\{\operatorname{Im}\zeta(\mathrm{f})\}}$. The reverse inclusion $\overline{\{\operatorname{Im}\zeta(\mathrm{f})\}} \subseteq \{V(0)/\operatorname{Im}\zeta(\mathrm{f})\}$ is shown similarly. □

As a direct result of [4, Lemma 13 and Theorem 14], for any R-module homomorphism $f : M \to M'$, we have $\operatorname{Ker}\zeta(f) = \overline{\{V(\operatorname{Ker} f)\}}$ and obtain $\zeta(M)/\overline{\{V(\operatorname{Ker} f)\}} \cong \zeta(M')$ as a $\zeta(R)$-semimodule isomorphism. In particular, if $f : M \to M'$ is an R-module isomorphism, we can conclude again $\zeta(f)$ is an $\zeta(R)$-semimodule isomorphism. In fact, if f is injective, then $\overline{\{V(\operatorname{Ker} f)\}} = \overline{\{V(0)\}} = V(0)$. But $\zeta(M)/V(0) \cong \zeta(M')$. Therefore $\zeta(M) \cong \zeta(M')$.

Let M be an R-module and N a submodule of M. As we know, every prime submodule of M/N has the form K/N where K is a prime submodule of M containing N. Using this fact and considering the natural projection $\pi : M \to M/N$, we get that $\operatorname{Im}\zeta(\pi) = \zeta(M/N)$. Also, as remarked in [2, Example 15.3], if N is a subsemimodule of a R-semimodule M, then $(0)/N$ is a subtractive subsemimodule of M and, indeed, is the subtractive closure of N in M. Therefore $\overline{\{\operatorname{Im}\zeta(\pi)\}} = V(N)/\zeta(M/N)$ (where $V(N)$ is the zero element of $\zeta(M/N)$). Hence, by Theorems 12 and 14, we have the following result.

Corollary 3.8. *Let N be an R-submodule of M. Then*

$$\zeta(M)/\overline{\{V(N)\}} \cong \zeta(M/N)(= V(N)/\zeta(M/N)).$$

4. On preservation of exact sequences by ζ

As mentioned earlier, with the aim of introducing a consistent homological algebra for semimodules, various definitions of the exactness of semimodule sequences have been given, which by restricting to the subcategory of modules, all coincide with the usual definition of the exactness of module sequences. Indeed, a sequence $\cdots \to M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} M_{i+2} \to \dots$ of R-semimodules is called exact if for each $i, \operatorname{Im} f_i = \operatorname{Ker} f_{i+1}$. In the sequel, it will be understood that the symbol 0 in any sequence is the trivial R-module or $\zeta(R)$-semimodule. As immediate observations, the sequence $M' \xrightarrow{f} M \to 0$ of R-semimodules is exact (where the symbol 0 is understood as the trivial semimodule) if and only if f is a surjective R-homomorphism of semimodules, and also the sequence $0 \to M' \xrightarrow{f} M$ of R-semimodules is exact if and only if $\operatorname{Ker} f = (0)$. Moreover, $0 \to M' \xrightarrow{f} M \to 0$ is a short exact sequence of semimodules if and only if M' is semimodule isomorphic to M.

Lemma 4.1. *Let M, M' and M'' be R-modules. Then*

(1) *If $0 \to M' \xrightarrow{f} M$ is exact and M' a radical module, then* $\operatorname{Ker}\zeta(f) = \{V(0)\}$.

(2) *If* $M' \xrightarrow{f} M \to 0$ *is exact, then* $\zeta(M') \xrightarrow{\zeta(f)} \zeta(M) \to 0$ *is exact and* $\zeta(f)$ *is steady.*

(3) *If* $0 \to M' \xrightarrow{f} M \to 0$ *is exact, then* $\zeta(f)$ *is a* $\zeta(R)$*-isomorphism.*

(4) *If* $M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ *is exact and* M'' *a radical module, then*

$$\zeta(M') \xrightarrow{\zeta(f)} \zeta(M) \xrightarrow{\zeta(g)} \zeta(M'') \to 0$$

is exact.

(5) *If* $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ *is a short exact sequence of* R*-modules and* M' *and* M'' *radical modules, then* $0 \to \zeta(M') \xrightarrow{\zeta(f)} \zeta(M) \xrightarrow{\zeta(g)} \zeta(M'') \to 0$ *is a short exact sequence of* $\zeta(R)$*-semimodules.*

(6) *If* $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ *is a short exact sequence of* R*-modules and* M'' *a radical module, then* $\zeta(M)/\overline{\{V(\operatorname{Im} f)\}} \cong \zeta(M'')$.

Proof. (1) Let $V(N) \in \operatorname{Ker}\zeta(f)$). Then $\zeta(f)(V(N)) = V(0)$, that is $V(f(N)) = V(0)$. Thus $\operatorname{rad} f(N) = \operatorname{rad}(0)$, and since M' is a radical module, $f(N) = 0$, and so by the injectivity of f, we have $N = (0)$. Therefore, $\operatorname{Ker}\zeta(f) = \{V(0)\}$.
(2) and (3) follow from [4, Lemma 8].
(4) Suppose that $V(N) \in \operatorname{Im}\zeta(f)$. Hence there exists $V(N') \in \zeta(M')$ such that $\zeta(f)(V(N')) = V(N)$. So $V(f(N')) = V(N)$, and then $\zeta(g)(V(f(N')) = \zeta(g)(V(N))$. Now, since

$$V(0) = V(g(f(N'))) = \zeta(g)(V(f(N')) = \zeta(g)(V(N)),$$

which shows $V(N) \in \operatorname{Ker}\zeta(g)$. Hence $\operatorname{Im}\zeta(f) \subseteq \operatorname{Ker}\zeta(g)$. For the reverse inclusion, we assume that $V(N) \in \operatorname{Ker}\zeta(g)$, that is $\zeta(g)(V(N)) = V(0)$, and so $V(g(N)) = V(0)$. Thus we have $g(N) \subseteq \operatorname{rad}(0) = 0$, which implies that $N \subseteq \operatorname{Ker} g = \operatorname{Im} f$. Hence there exists a submodule L of M' such that $f(L) = N$. Therefore $\zeta(f)(V(L)) = V(f(L)) = V(N)$, which yields that $V(N) \in \operatorname{Im}\zeta(f)$. Consequently $\operatorname{Ker}\zeta(g) \subseteq \operatorname{Im}\zeta(f)$, as required. The surjectivity of $\zeta(g)$ follows from (2).
(5) Apply (1) and (4).
(6) By (5), the sequence $0 \to \zeta(M') \xrightarrow{\zeta(f)} \zeta(M) \xrightarrow{\zeta(g)} \zeta(M'') \to 0$ is short exact, and so we have $\operatorname{Ker}\zeta(g) = \operatorname{Im}\zeta(f)$. Now, since $\operatorname{Ker}\zeta(g)$ is a subtractive $\zeta(R)$-subsemimodule, we have $\operatorname{Ker}\zeta(g) = \operatorname{Im}\zeta(f)$. Hence by [4, Lemma 13 and Theorem 14], we are done. □

Let M, M' be R-semimodules. Then the set of all R-homomorphisms from M to M', denoted by $\operatorname{Hom}_R(M, M')$, forms an R-semimodule with the natural addition and scalar multiplication (see [2, P. 159]). We are going to examine here how the exactness of module sequences is preserved by the hom-functors $\operatorname{Hom}_{\zeta(R)}(\zeta(M), \zeta(-))$ and $\operatorname{Hom}_{\zeta(R)}(\zeta(-), \zeta(M'))$. But before that, let us determine some home-sets $\operatorname{Hom}_{\zeta(R)}(\zeta(M), \zeta(M'))$.

Note that for any R-module M,

$$\operatorname{Hom}_{\zeta(R)}(\zeta(R), \zeta(M)) \cong \zeta(M) \cong \zeta(\operatorname{Hom}_R(R, M)),$$

where the first isomorphism is well-known in the context of semimodules, and the second one follows from Lemma 3.2. These isomorphisms help us to determine some home-sets $\mathrm{Hom}_{\zeta(R)}(\zeta(M), \zeta(M'))$.

Example 4.2. *Let $\mathbb{Z}$ be the ring of integers, and the set of rational numbers $\mathbb{Q}$ be its field of fractions. Then we have*
(1) Since $\mathrm{Hom}_{\zeta(\mathbb{Z})}(\zeta(\mathbb{Z}), \zeta(\mathbb{Z}))$ *is $\zeta(\mathbb{Z})$-isomorphic to*

$$\begin{aligned}\zeta(\mathbb{Z}) = \{\{p_1\mathbb{Z}, p_2\mathbb{Z}, \ldots, p_n\mathbb{Z}\} \mid p_1, \ldots, p_n \textit{ are distinct prime numbers}, n \in \mathbb{N}\}\\ \cup \{\emptyset\} \cup \{\mathrm{Spec}(\mathbb{Z})\},\end{aligned}$$

we see that $\mathrm{Hom}_{\zeta(\mathbb{Z})}(\zeta(\mathbb{Z}), \zeta(\mathbb{Z}))$ *is an infinite $\zeta(\mathbb{Z})$-semimodule.*
(2) Since $\mathrm{Hom}_{\zeta(\mathbb{Z})}(\zeta(\mathbb{Z}), \zeta(\mathbb{Q})) \cong \zeta(\mathbb{Q}) = \{\{0\}, \emptyset\}$, $\mathrm{Hom}_{\zeta(\mathbb{Z})}(\zeta(\mathbb{Z}), \zeta(\mathbb{Q}))$ *contains only two $\zeta(\mathbb{Z})$-homomorphisms that one of them maps $\zeta(\mathbb{Z})$ to $\{0\}$ and the other maps $\zeta(\mathbb{Z})$ to $\mathbb{Q}$. Note that, as stated in [10, p,99], is the only prime $\mathbb{Z}$-submodule of $\mathbb{Q}$.*
(3) As shown in [10, Lemma 1.3], $\mathbb{Q}/\mathbb{Z}$ has no prime $\mathbb{Z}$-submodule. It follows that

$$\mathrm{Hom}_{\zeta(\mathbb{Z})}(\zeta(\mathbb{Z}), \zeta(\mathbb{Q}/\mathbb{Z})) \cong \zeta(\mathbb{Q}/\mathbb{Z}) = \{\emptyset.$$

Let M be an R-module. Then for any R-module homomorphism $f : N' \to N$, we consider the natural map

$$\mathrm{Hom}_{\zeta(R)}(\zeta(M), \zeta(f)) : \mathrm{Hom}_{\zeta(R)}(\zeta(M), \zeta(N')) \to \mathrm{Hom}_{\zeta(R)}(\zeta(M), \zeta(N))$$

defined by $\zeta(f)^*(\phi) = \zeta(f) \circ \phi$.

In the next result, we see that $\mathrm{Hom}_{\zeta(R)}(\zeta(M), \zeta(-)) : R$-**Mod** $\to \zeta(R)$-**Semod** is a covariant functor. We will denote it by $\zeta(-)^*$ for short.

Note the following lemma.

Lemma 4.3. *Let M be an R-module. Then $\zeta(-)^* : R$-**Mod** $\to \zeta(R)$-**Semod** which maps any R-module N to $\zeta(R)$-semimodule* $\mathrm{Hom}_{\zeta(R)}(\zeta(M), \zeta(N)))$*, and any R-module homomorphism f to $\zeta(f)^*$ is a covariant functor.*

Proof. We first show that $\zeta(f)^*$ is a $\zeta(R)$-semimodule homomorphism. Clearly, $\zeta(f)^*$ is a semigroup homomorphism. Now, let $\phi \in \mathrm{Hom}_{\zeta(R)}(\zeta(M), \zeta(N'))$ and $V(I) \in \zeta(R)$. Then for any $V(L) \in \zeta(M)$, we have

$$\begin{aligned}(\zeta(f)^*(V(I)\phi))(V(L)) &= (\zeta(f) \circ (V(I)\phi))(V(L)) = \zeta(f)((V(I)\phi))(V(L)))\\ &= V(I)(\zeta(f)(\phi((V(L)))) = V(I)(\zeta(f) \circ \phi)(V(L))\\ &= V(I)(\zeta(f)^*(\phi))(V(L)).\end{aligned}$$

Thus $\zeta(f)^*(V(I)\phi) = V(I)(\zeta(f)^*(\phi)$.
Secondly, we assume that $\phi \in \mathrm{Hom}_{\zeta(R)}(\zeta(M), \zeta(N'))$ and $V(L) \in \zeta(M)$. Then by the functoriality of ζ, for any two R-homomorphisms $f : N' \to N$ and

$g : N \to N''$, we have

$$\begin{aligned}((\zeta(g) \circ \zeta(f))^*(\phi))(V(L)) &= ((\zeta(g \circ f))^*(\phi))(V(L)) = (\zeta(g \circ f) \circ (\phi))(V(L)) \\ &= (\zeta(g) \circ \zeta(f) \circ (\phi))(V(L)) = (\zeta(g)^*(\zeta(f) \circ (\phi)))(V(L)) \\ &= (\zeta(g)^* \circ \zeta(f)^*)(\phi)))(V(L)).\end{aligned}$$

Finally, by an easy verification, we observe that $\zeta(1_N)^* = 1_{\zeta(N)}$. □

We now examine the preservation of exact module sequences by

$$\mathrm{Hom}_{\zeta(R)}(\zeta(M), \zeta(-)).$$

Note that $\mathrm{Hom}_{\zeta(R)}(\zeta(M), \zeta(-))$ preserves some exact module sequences. For example, since for any R-module M, we have $\mathrm{Hom}_{\zeta(R)}(\zeta(R), \zeta(M)) \cong \zeta(M)$, then by Lemma 4.1(5), $\mathrm{Hom}_{\zeta(R)}(\zeta(R), \zeta(-))$ preserves any exact sequence of radical modules. As another example, if M is a torsion divisible R-module (such as the $\mathbb{Z}$-modules $\mathbb{Z}_{(p^\infty)}$ and $\mathbb{Q} = \mathbb{Z}$ and $\mathbb{Q}/\mathbb{Z}$), then $\zeta(M) = \{\emptyset\}$, and so $\mathrm{Hom}_{\zeta(R)}(\zeta(M), \zeta(-)) = \{0\}$ which shows $\mathrm{Hom}_{\zeta(R)}(\zeta(M), \zeta(-))$ is an exact functor.

Theorem 4.4. *Let $0 \to N' \xrightarrow{f} N \xrightarrow{g} N'' \to 0$ be a short exact sequence of R-modules such that $f(\mathrm{Spec}(N')) \subseteq \mathrm{Spec}(N)$ and N'' is a radical modules. Then*

(1) *For any R-module M, the sequence*

$$0 \to \mathrm{Hom}_{\zeta(R)}(\zeta(M), \zeta(N')) \xrightarrow{\zeta(f)^*} \mathrm{Hom}_{\zeta(R)}(\zeta(M), \zeta(N)) \xrightarrow{\zeta(g)^*} \mathrm{Hom}_{\zeta(R)}(\zeta(M), \zeta(N''))$$

of $\zeta(R)$-semimodules is exact.

(2) *$\zeta(g)^*$ is a surjective monoid homomorphism.*

(3) *If for any ideal I of R with $V(I) \neq V(0)$, we have $V(I\,\mathrm{Ker}\, g) = V(\mathrm{Ker}\, g)$, then $\zeta(g)^*$ is a surjective $\zeta(R)$-semimodule homomorphism. In particular, for any R-module M, the sequence*

$$0 \to \mathrm{Hom}_{\zeta(R)}(\zeta(M), \zeta(N')) \xrightarrow{\zeta(f)^*} \mathrm{Hom}_{\zeta(R)}(\zeta(M), \zeta(N)) \xrightarrow{\zeta(g)^*} \mathrm{Hom}_{\zeta(R)}(\zeta(M), \zeta(N'')) \to 0$$

of $\zeta(R)$-semimodules is exact.

Proof. (1) Let $0 \to N' \xrightarrow{f} N \xrightarrow{g} N'' \to 0$ be an exact sequence of R modules, Then by Lemma 3.2 and Lemma 4.1(4), the sequence

$$0 \to \zeta(N') \xrightarrow{\zeta(f)} \zeta(N) \xrightarrow{\zeta(g)} \zeta(N'') \to 0$$

is a short exact sequence of $\zeta(R)$-semimodules. So, we have the hom-sequence

$$0 \to \mathrm{Hom}_{\zeta(R)}(\zeta(M), \zeta(N')) \xrightarrow{\zeta(f)^*} \mathrm{Hom}_{\zeta(R)}(\zeta(M), \zeta(N)) \xrightarrow{\zeta(g)^*} \mathrm{Hom}_{\zeta(R)}(\zeta(M), \zeta(N''))$$

of $\zeta(R)$-semimodules. To show the exactness of this sequence, we first note that $\mathrm{Ker}\, \zeta(f)^* = (0)$, because if for $\phi \in \mathrm{Hom}_{\zeta(R)}(\zeta(M), \zeta(N'))$, $\zeta(f)^*(\phi) = 0$, then for each $V(K) \in \zeta(M)$, we have $\zeta(f)(\phi(V(K)) = 0_{\zeta(N)}$ which yields that $\phi = 0$, by Lemma 3.2. Moreover, by the functoriality of ζ (Lemma 4.3), we have

$$\zeta(g)^* \circ \zeta(f)^* = (\zeta(g) \circ \zeta(f))^* = (\zeta(g \circ f))^* = (\zeta(0))^* = 0,$$

which shows that $\operatorname{Im}\zeta(f)^* \subseteq \operatorname{Ker}\zeta(g)^*$. For the reverse inclusion, we let $\phi \in \operatorname{Ker}\zeta(g)^*$. Then $\zeta(g)\circ\phi = \zeta(g)^*(\phi) = 0$. Therefore $\operatorname{Im}\phi \subseteq \operatorname{Ker}\zeta(g)$, and so $\operatorname{Im}\phi \subseteq \operatorname{Im}\zeta(f)$. It should be noted that since $f(\operatorname{Spec}(N')) \subseteq \operatorname{Spec}(N)$, $\zeta(f)$ is injective, by Lemma 3.2. Now, by considering $\zeta(f)^{-1} : \operatorname{Im}\zeta(f) \to \zeta(N')$ as the inverse map of $\zeta(f)$, and letting $\lambda := \zeta(f)^{-1}\circ\phi \in \operatorname{Hom}_{\zeta(R)}(\zeta(M), \zeta(N'))$, we have $\zeta(f)^*(\lambda) = \zeta(f)\circ\lambda = \phi$ which shows that $\phi \in \operatorname{Im}\zeta(f)^*$. Therefore $\operatorname{Ker}\zeta(g)^* \subseteq \operatorname{Im}\zeta(f)^*$, as required.
(2) To show that $\zeta(g)^*$ is surjective as a monoid homomorphism, we assume that $\phi \in \operatorname{Hom}_{\zeta(R)}(\zeta(M), \zeta(N''))$, and let $V(0) \neq V(X) \in \zeta(M)$. Then, since $\zeta(g)$ is surjective, there exists $V(K_X) \in \zeta(N)$ such that $\zeta(g)(V(K_X)) = \phi(V(X))$. We define $\psi : \zeta(M) \to \zeta(N)$ by $\psi(V(X)) = V(K_X + \operatorname{Ker} g)$ and $\psi(V(0)) = V(0)$. To show that ψ is well-defined, we let $V(K_X), V(K'_X) \in \zeta(M)$ such that $\zeta(g)(V(K_X)) = \zeta(g)(V(K'_X)) = \phi(V(X))$. Thus, by adding $\zeta(g)(V(Kerg))$ on both sides of the first equality, we get that $\zeta(g)(V(K_X + \operatorname{Ker} g)) = \zeta(g)(V(K'_X + \operatorname{Ker} g)))$. So by Lemma 4.1(2), since $\zeta(g)$ is steady, there are $V(T_1), V(T_2) \in \operatorname{Ker}\zeta(g)$ such that $V(K_X + \operatorname{Ker} g) + V(T_1) = V(K'_X + \operatorname{Ker} g) + V(T_2)$. Now, since M'' is a radical module, we have $V(K_X + \operatorname{Ker} g) = V(K'_X + \operatorname{Ker} g)$, as required. It remains to show that ψ preserves "+". For this, we assume that for $V(X), V(Y) \in \zeta(M) \setminus V(0)$. Thus there are $V(K_X), V(K_Y) \in \zeta(M)$ such that $\zeta(g)(V(K_X)) = \phi(V(X))$ and $\zeta(g)(V(K_Y)) = \phi(V(Y))$. Therefore

$$\begin{aligned}\zeta(g)(V(K_X + K_Y)) &= \zeta(g)(V(K_X) + V(K_Y)) = \phi(V(X)) + \phi(V(Y))\\ &= \phi(V(X) + V(Y)) = \phi(V(X + Y)).\end{aligned}$$

It follows that

$$\begin{aligned}\psi(V(X) + V(Y)) &= \psi(V(X + Y)) = V(K_X + K_Y + \operatorname{Ker} g)\\ &= V(K_X + \operatorname{Ker} g) + V(K_Y + \operatorname{Ker} g)\\ &= \psi(V(X)) + \psi(V(Y)).\end{aligned}$$

Also, for the case that $V(X) = V(0)$ or $V(Y) = V(0)$, it is easily seen that $\psi(V(X) + V(Y)) = \psi(V(X)) + \psi(V(Y))$.
(3) By (2), It is enough to show that the monoid homomorphism ψ in the proof of (2) is an $\zeta(R)$-semimodule homomorphism. For this, let $V(I) \in \zeta(R)$ and $V(X) \in \zeta(M)$. If $V(I) = V(0)$, we are done. Suppose that $V(I) \neq V(0)$. Then

$$\begin{aligned}\phi(V(IX) &= \phi(V(I)\cdot V(X)) = V(I)\cdot\phi(V(X)) = V(I)\cdot(\zeta(g)(V(K_X)))\\ &= \zeta(g)(V(I)\cdot V(K_X)) = \zeta(g)(V(IK_X)).\end{aligned}$$

It implies that $\psi(V(IX)) = V(IK_X + Kerg)$. Consequently, by hypothesis, we get that

$$\begin{aligned}V(I)\cdot\psi(V(X)) &= V(I)\cdot(V(K_X + \operatorname{Ker} g)) = V(I((K_X + \operatorname{Ker} g))\\ &= V(IK_X + I\operatorname{Ker} g)) = V(IK_X) + V(I\operatorname{Ker} g)\\ &= V(IK_X) + V(\operatorname{Ker} g) = V(IK_X + \operatorname{Ker} g)\\ &= \psi(V(IX)) = \psi(V(I).V(X)).\end{aligned}$$

□

Before stating the next result, it is necessary to point out the well-known fact that every subspace of a vector space is a prime submodule of it.

Corollary 4.5. *Let F be a field and $0 \to V' \xrightarrow{f} V \xrightarrow{g} V'' \to 0$ be a short exact sequence of F-vector spaces. Then for any F-vector space M, the sequence*

$$0 \to \mathrm{Hom}_{\zeta(F)}(\zeta(M), \zeta(V')) \xrightarrow{\zeta(f)^*} \mathrm{Hom}_{\zeta(F)}(\zeta(M), \zeta(V)) \xrightarrow{\zeta(g)^*} \mathrm{Hom}_{\zeta(F)}(\zeta(M), \zeta(V'')) \to 0$$

is a short exact sequence of $\zeta(F)$-semimodules.

Proof. Firstly, we note that $f(\mathrm{Spec}(V')) \subseteq \mathrm{Spec}(V)$. Now, since $\zeta(F) = \{V(0), V(F)\} = \{\{0\}, \emptyset\}$, and for any surjective linear transformation $g : N \to N''$, $V(F \operatorname{Ker} g) = V(\operatorname{Ker} g)$, the result follows from Theorem 4.4. □

It is well-known that if R is an integral domain with the quotient field K, then (0) is the only prime R-submodule of K (see [3, p,99]). This fact is used in the next result.

Proposition 4.6. *Let R be an integral domain with the quotient field K. If*

$$0 \to N' \xrightarrow{f} N \xrightarrow{g} N'' \to 0$$

is a short exact sequence of R modules with M'' a non-faithful radical module, then

$$0 \to \mathrm{Hom}_{\zeta(R)}(\zeta(K), \zeta(N')) \xrightarrow{\zeta(f)^*} \mathrm{Hom}_{\zeta(R)}(\zeta(K), \zeta(N)) \xrightarrow{\zeta(g)^*} \mathrm{Hom}_{\zeta(R)}(\zeta(K), \zeta(N'')) \to 0$$

is a short exact sequence of $\zeta(R)$-semimodules.

Proof. Since $\mathrm{Ann}(N'') \neq 0$ and M'' is a radical module, by considering $I := \mathrm{Ann}(N'')$, we find that $V(\sqrt{I} \cdot L) = V(IL) = V(0)$ for every submodule L of M''. So $\mathrm{Hom}_{\zeta(R)}(\zeta(K), \zeta(N'')) = (0)$, for otherwise if there exists $0 \neq \phi \in \mathrm{Hom}_{\zeta(R)}(\zeta(K), \zeta(N''))$, then $\phi(V(K)) = V(L) \neq V(0)$ for some submodule L of N''. It follows that $\phi(V(\sqrt{I} \cdot K)) = \phi(V(IK)) = \phi(V(K)) = V(L)$ one hand, and $V(I)\phi(V(K)) = V(I) \cdot V(L) = V(IL) = V(0)$ on the other hand. Thus we have $V(N) = V(0)$, a contradiction. Therefore the given Hom-sequence is short exact by Theorem 4.3. □

Let N be an R-module. Then for any R-module homomorphism $f : M' \to M$, we have the map $\zeta(f)_* : \mathrm{Hom}_{\zeta(R)}(\zeta(M), \zeta(N)) \to \mathrm{Hom}(\zeta(M'), \zeta(N)))$ given by $\zeta(f)_*(\alpha) = \alpha \circ \zeta(f)$. In the next result, we see that

$$\mathrm{Hom}_{\zeta(R)}(\zeta(-), \zeta(N)) : R\textbf{-Mod} \to \zeta(R)\textbf{-Semod}$$

is a contravariant functor. We will denote $\mathrm{Hom}_{\zeta(R)}(\zeta(f), \zeta(N))$ by $\zeta(f)_*$ for short.

Lemma 4.7. *Let $f : M' \to M$ be an R-homomorphisms. Then*

1. *For any R-module N, $\zeta(f)_* : \mathrm{Hom}_{\zeta(R)}(\zeta(M), \zeta(N)) \to \mathrm{Hom}(\zeta(M'), \zeta(N)))$ defined by $\zeta(f)_*(\alpha) = \alpha \circ \zeta(f)$, $\alpha \in \mathrm{Hom}(\zeta(M), \zeta(N))$ is an $\zeta(R)$-semimodule homomorphism.*
2. *Let $f : M' \to M$ be an R-module homomorphism. Then for any R-module homomorphism $g : M \to M''$*

$$(\zeta(g) \circ \zeta(f))_* = \zeta(f)_* \circ \zeta(g)_* = (\zeta(g \circ f))_*.$$

(3) $\zeta(1_M)_* = 1_{\zeta(M)}$.

Proof. (1) Let N be an R-module. Clearly $\zeta(f)_*$ is a semigroup homomorphism. Now, let $\alpha \in \operatorname{Hom}_{\zeta(R)}(\zeta(M), \zeta(N))$ and $V(I) \in \zeta(R)$. Then for any $V(L) \in \zeta(M)$, we have

$$\begin{aligned}(\zeta(f)_*(V(I)\alpha))(V(L)) &= ((V(I)\alpha) \circ \zeta(f))(V(L)) = ((V(I)\alpha(\zeta(f)(V(L))) \\ &= \alpha(V(I) \cdot \zeta(f)(V(L))) = (\alpha \circ \zeta(f))(V(I) \cdot V(L)) \\ &= (\zeta(f)_*(\alpha))(V(I) \cdot V(L)) = (V(I)(\zeta(f)_*(\alpha))(V(L)).\end{aligned}$$

Thus $\zeta(f)_*(V(I)\alpha) = V(I)(\zeta(f))_*(\alpha)$.
(2) Let $\alpha \in \operatorname{Hom}_{\zeta(R)}(\zeta(M'), \zeta(M))$ and $V(L) \in \zeta(N)$. By the functoriality of ζ, we have

$$\begin{aligned}((\zeta(g) \circ \zeta(f))_*(\alpha))(V(L)) &= (\zeta(g \circ f))_*(\alpha))(V(L)) = (\alpha \circ \zeta(g \circ f))(V(L)) \\ &= (\alpha \circ \zeta(g) \circ \zeta(f))(V(L)) = \zeta(f)_*(\alpha \circ \zeta(g))(V(L)) \\ &= (\zeta(f)_* \circ \zeta(g)_*)(\alpha)(V(L)).\end{aligned}$$

(3) Clear. □

The following theorem shows that the contravariant functor $\zeta(-)_*$ is left exact.

Theorem 4.8. *Let* $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ *be a short exact sequence of* R*-modules with* M'' *a radical module. Then for any* R*-module* N*, the sequence*

$$0 \to \operatorname{Hom}_{\zeta(R)}(\zeta M''), \zeta(N)) \xrightarrow{\zeta(g)_*} \operatorname{Hom}_{\zeta(R)}(\zeta(M), \zeta(N)) \xrightarrow{\zeta(f)_*} \operatorname{Hom}_{\zeta(R)}(\zeta(M'), \zeta(N))$$

of $\zeta(R)$*-semimodules is exact.*

Proof. Let $\alpha \in \operatorname{Ker}\zeta(g)_*$. Then $\alpha \circ \zeta(g) = \zeta(g)_*(\alpha) = 0$. Therefore $\zeta(M'') = \operatorname{Im}\zeta(g) \subseteq \operatorname{Ker}\alpha$. Thus $\alpha = 0$, and so $\operatorname{Ker}\zeta(g)_* = \{0\}$. Now, we show $\operatorname{Im}\zeta(g)_* = \operatorname{Ker}\zeta(f)_*$. we have

$$\zeta(f)_* \circ \zeta(g)_* = (\zeta(g) \circ \zeta(f))_* = (\zeta(g \circ f))_* = (\zeta(0))_* = 0,$$

which shows $\operatorname{Im}\zeta(g)_* \subseteq \operatorname{Ker}\zeta(f)_*$. For the reverse inclusion, let $\alpha \in \operatorname{Ker}\zeta(f)_*$. Then $\alpha \circ \zeta(f) = \zeta(f)_*(\alpha) = 0$. Therefore $\operatorname{Im}\zeta(f) \subseteq \operatorname{Ker}\alpha$, or equivalently, $\operatorname{Ker}\zeta(g) \subseteq \operatorname{Ker}\alpha$. So the mapping $\beta : \zeta(M)/\operatorname{Ker}\zeta(g) \to \zeta(N)$ defined by $\beta(V(L)/\operatorname{Ker}\zeta(g)) = \alpha(V(L))$ is a $\zeta(R)$-semimodule homomorphism. Let $t = \beta \circ \alpha^{-1} \in \operatorname{Hom}_{\zeta(R)}(\zeta(M''), \zeta(N))$, where $\alpha : \zeta(M)/\operatorname{Ker}\zeta(g) \to \zeta(M'')$ defined by $\alpha(V(L')/\operatorname{Ker}\zeta(g)) = \zeta(g)(V(L'))$ is the remarked $\zeta(R)$-semimodule isomorphism after Theorem 3.7. It is easy to verify $\zeta(g)_*(t) = \alpha$, and so $\alpha \in \operatorname{Im}\zeta(g)_*$, as desired.

□

An R-module M is called a distributive module, if the usual lattice of submodules of M is distributive. In [7, Lemma 1.4], it has been shown that M is a distributive R-module if and only if for every R-module M' and R-module homomorphism $f : M' \to M$, we have $f^{-1}(L + L') = f^{-1}(L) + f^{-1}(L')$ for all submodules L and L' of M.

Theorem 4.9. *Let* $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ *be a short exact sequence of* R*-modules with* M *a distributive* R*-module and* M'' *a radical* R*-module . If for any ideal* I *of* R *and any submodule* L *of* M *we have,* $f^{-1}(\mathrm{rad}(IL)) = \mathrm{rad}(If^{-1}(L))$*, then for any* R*-module* N*, the sequence*

$$0 \to \mathrm{Hom}_{\zeta(R)}(\zeta M''), \zeta(N)) \xrightarrow{\zeta(g)_*} \mathrm{Hom}_{\zeta(R)}(\zeta(M), \zeta(N)) \xrightarrow{\zeta(f)_*} \mathrm{Hom}_{\zeta(R)}(\zeta(M'), \zeta(N)) \to 0$$

of $\zeta(R)$*-semimodules is short exact.*

Proof. By Theorem 4.8, it suffices to show that $\zeta(f)_*$ is a surjective $\zeta(R)$-module homomorphism. Let $\alpha \in \mathrm{Hom}_{\zeta(R)}(\zeta(M'), \zeta(N))$, and define $\beta : \zeta(M) \to \zeta(N)$ by $\beta(V(L)) = \alpha(V(f^{-1}(\mathrm{radL})))$. Clearly β is well-defined. Moreover, by the hypothesis,

$$\begin{aligned}(\zeta(f)_*(\beta)(V(L)) &= \beta(\zeta(f)(V(L))) = \beta(V(f(L))) = \alpha(V(f^{-1}(\mathrm{rad}(f(L))))) \\ &= \alpha(V(\mathrm{rad}(f^{-1}(f(L))))) = \alpha(V(L)).\end{aligned}$$

It remains to show that β is an $\zeta(R)$-semimodule homomorphism. By hypothesis,

$$\begin{aligned}\beta(V(L)+V(L')) &= \beta(V(L + L')) = \alpha(V(\mathrm{rad}(f^{-1}(L + L')))) \\ &= \alpha(V(\mathrm{rad}(f^{-1}(L) + f^{-1}(L')))) \\ &= \alpha(V(\mathrm{rad}(\mathrm{rad}(f^{-1}(L)) + \mathrm{rad}(f^{-1}(L'))))) \\ &= \alpha(V(\mathrm{rad}(f^{-1}(L))+V(\mathrm{rad}(f^{-1}(L'))) \\ &= \alpha(V(f^{-1}(\mathrm{rad}(L)))+V(f^{-1}(\mathrm{rad}(L'))) \\ &= \alpha(V(f^{-1}(\mathrm{rad}(L)))+\alpha(V(f^{-1}(\mathrm{rad}(L'))) \\ &= \beta(V(L))+\beta(V(L')),\end{aligned}$$

and

$$\begin{aligned}\beta(V(I) \cdot V(L)) &= \beta(V(IL)) = \alpha(V(f^{-1}(\mathrm{rad}(IL)))) \\ &= \alpha(V(\mathrm{rad}(If^{-1}(L)))) \\ &= \alpha(V(If^{-1}(L))) \\ &= \alpha(V(I) \cdot V(\mathrm{rad}(f^{-1}(L))) \\ &= \alpha(V(I) \cdot V(f^{-1}(\mathrm{rad}(L)))) \\ &= V(I)\alpha(V(f^{-1}(\mathrm{rad}(L)))) \\ &= V(I)\beta(V(L)).\end{aligned}$$

We are done. □

We recall that if K is the quotient field an integral domain R, then $\zeta(K) = \{V(0), V(K)\} = \{\{0\}, \emptyset\}$. This fact is used in the following result.

Theorem 4.10. *Let R be an integral domain with the quotient field K. If*

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

is an exact sequence of R modules, with M'' a radical module and $f(\mathrm{Spec}(M')) \subseteq \mathrm{Spec}(M)$, then

$$0 \to \mathrm{Hom}_{\zeta(R)}(\zeta(M''), \zeta(K)) \xrightarrow{\zeta(g)_*} \mathrm{Hom}_{\zeta(R)}(\zeta(M), \zeta(K)) \xrightarrow{\zeta(f)_*} \mathrm{Hom}_{\zeta(R)}(\zeta(M'), \zeta(K))$$

is a short exact sequence of $\zeta(R)$-semimodules and $\zeta(f)_*$ is a surjective monoid homomorphism.

Proof. By Theorem 4.8, it is enough to show that $\zeta(f)_*$ is surjective. For this, we assume that $\alpha \in \mathrm{Hom}_{\zeta(R)}(\zeta(M'), \zeta(K))$. By Lemma 3.2, we can let $\lambda := \alpha \circ \zeta(f)^{-1} : \mathrm{Im}\,\zeta(f) \to \zeta(M')$. Now, we define $\beta : \zeta(M) \to \zeta(K)$ by setting $\beta(V(L)) = \{\{0\}\}$ if there exists $V(X) \in \zeta(M)$ such that $V(L + X) \in \mathrm{Ker}\,\lambda$; otherwise $\beta(V(L)) = \{\emptyset\}$. We show that β preserves the addition. For this, let $V(L), V(L') \in \zeta(M)$. If $\beta(V(L + L')) = \{\{0\}\}$, then there exists $V(X) \in \zeta(M)$ such that $V((L + L') + X) \in \mathrm{Ker}\,\lambda$. Then $V(L + (L' + X)), V(L' + (L + X)) \in \mathrm{Ker}\,\lambda$, and so

$$\beta(V(L)) + \beta(V(L')) = \{\{0\}\} + \{\{0\}\} = \{\{0\}\} = \beta(V(L) + V(L')).$$

In the other case, if $\beta(V(L) + V(L'))) = \{\emptyset\}$, then either $\beta(V(L)) = \{\emptyset\}$ or $\beta(V(L')) = \{\emptyset\}$, for otherwise if both of these are equal to $\{\{0\}\}$, then there are $V(X_1), V(X_2) \in \zeta(M)$ satisfying $V(L + X_1), V(L' + X_2) \in \mathrm{Ker}\,\lambda$. Thus $V(L + L' + X_1 + X_2) \in \mathrm{Ker}\,\lambda$ which implies that $\beta(V(L) + V(L'))) = \{\{0\}\}$, a contradiction. Therefore, in any case, $\beta(V(L) + V(L')) = \beta(V(L)) + \beta(V(L'))$. To complete the proof, we show that $\zeta(f)_*(\beta) = \alpha$, or in other words, $\beta(\zeta(f)) = \alpha$. For this, we first assume that $\zeta(f)_*(\beta)(V(L)) = V(0)$, or in other words, $\beta(\zeta(f))(V(L)) = \{\{0\}\}$.
Then there exists $V(X) \in \zeta(M)$ such that $(\zeta(f)(V(L)) + V(X)) \in \mathrm{Ker}\,\lambda$. So

$$(\alpha \circ \zeta(f)^{-1})(\zeta(f)((V(L)) + V(X)) = \lambda(\zeta(f)(V(L)) + V(X)) = V(0) = \{\{0\}\}$$

which follows that $\alpha(V(L)) = V(0) = \{\{0\}\}$. Thus in this case, $\zeta(f)_*(\beta) = \alpha$. In the other case, let $\beta(\zeta(f)(V(L))) = V(K) = \{\emptyset\}$. Suppose the contrary that $\alpha(V(L)) = V(0)$. Now since $V(L) \in \mathrm{Ker}\,\alpha$, there exists $V(L') \in \zeta(M)$ such that $\zeta(f)(V(L)) = V(L')$. So

$$\begin{aligned} \lambda(\zeta(f)(V(L))) &= \lambda(V(L')) = \alpha \circ \zeta(f)^{-1}(V(L')) \\ &= \alpha(\zeta(f)^{-1}(V(L'))) \\ &= \alpha(V(L)) = \{0\}. \end{aligned}$$

Thus $\zeta(f)((V(L)) \in \mathrm{Ker}\,\lambda$, and so $\beta(\zeta(f)(V(L))) = \{0\}$, a contradiction. Therefore, in any case, $\zeta(f)_*(\beta) = \alpha$. □

Finally, if we consider the special case that R is a field, then the monoid homomorphism in the proof of Theorem 4.10 is an $\zeta(R)$-semimodule homomorphism. Hence, given that the conditions of Theorem 4.10 are met, we conclude the following result.

Corollary 4.11. *Let Let F be a field and $0 \to V' \xrightarrow{f} V \xrightarrow{g} V'' \to 0$ be a short exact sequence of F-vector spaces. Then For any F-module N, the sequence*

$$0 \to \mathrm{Hom}_{\zeta(F)}(\zeta(V'), \zeta(N)) \xrightarrow{\zeta(g)_*} \mathrm{Hom}_{\zeta(F)}(\zeta(V), \zeta(N)) \xrightarrow{\zeta(f)_*} \mathrm{Hom}_{\zeta(F)}(\zeta(V''), \zeta(N)) \to 0$$

is a short exact sequence of $\zeta(F)$-semimodules.

Finally, note that in the trivial case that N has no prime submodule, $\zeta(N) = \{\emptyset\}$ and hence $\mathrm{Hom}_{\zeta(R)}(\zeta(M), \zeta(N)) = \{0\}$ which implies $\mathrm{Hom}_{\zeta(R)}(\zeta(-), \zeta(N))$ is obviously an exact functor.

5. Conclusion

In this paper, we investigated the behavior of the functor ζ from the category of R-modules to the category of $\zeta(R)$-semimodules which assigns to each R-module M the $\zeta(R)$-semimodule $\zeta(M)$ of varieties of its submodules. This construction provides a natural bridge between classical module theory over a commutative ring and the theory of semimodules over the semiring $\zeta(R)$. Our first main result establishes that ζ preserves short exact sequences whenever the cokernels are radical modules. We further studied the hom-functors $\mathrm{Hom}\,\zeta(R)(\zeta(P), \zeta(-))$ and $\mathrm{Hom}\,\zeta(R)(\zeta(-), \zeta(E))$, showing that, for these functors to be left exact on a short exact sequence, it is necessary that the last term be a radical module. This identifies radicality as a key condition ensuring stability of homological behavior after applying the variety construction. In addition, we determined situations in which these hom-functors preserve short exact sequences entirely. In particular, for vector spaces, the corresponding hom-functors become exact. Overall, our results demonstrate that the functor ζ and its associated hom-functors provide a meaningful framework for transferring homological questions from modules to semimodules of varieties. The role of radical modules appears to be central in ensuring exactness properties. This suggests several directions for further research, including extensions to broader classes of semirings and semimodules, and the development of derived homological constructions in the setting of $\zeta(R)$-semimodules.

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