

## SH-GRAPH AUTOMATA WITH APPLICATIONS

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**ABSTRACT.** In this paper, because semihoops are the most basic residuated structure that contain at logical algebras based on Galois connections, therefore we introduce a new type fuzzy graph and its complement based on semihoops, denoted by *SH*-graph  $G$  and  $G'$ , respectively. Then, the *SH*-graph automata related to the *SH*-graph  $G$  and a minimum zero forcing set are introduced, denoted by  $A(Z(G))$ . After that, the concepts of isomorphism between two *SH*-graphs  $G_1$  and  $G_2$  and isomorphism between two *SH*-graph automata  $A(Z(G_1))$  and  $A(Z(G_2))$  are introduced. Then, we prove that if two *SH*-graphs  $G_1$  and  $G_2$  are isomorphic, then two *SH*-graph automata  $A(Z(G_1))$  and  $A(Z(G_2))$  are isomorphic; otherwise, it is not true. In addition, the concept of equivalence of *SH*-graph automata is proposed. Moreover, we prove that *SH*-graph automata obtained from the same *SH*-graphs are equivalent under different zero forcing sets in some special *SH*-graphs. And then, we know that *SH*-graph automata  $A(Z(G_1))$  and  $A(Z(G_2))$  are equivalent can not be characterized by *SH*-graphs  $G_1$  and  $G_2$  being isomorphic. Finally, we introduce several simple practical applications of *SH*-graph and *SH*-graph automata.

*Keywords:* Semihoop, *SH*-graph, *SH*-graph automaton, Isomorphism, Equivalence

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### 1. Introduction

In 1736, Leonhard Euler presented graph theory to solve the Konigsberg bridge problem. Since then, graph theory has been widely applied to describe and analyze a large number of practical problems or events in multiple fields such as operations researches, computer sciences and social science. In addition, it is also applied to urban transportation roads, city maps, etc. An innovation across the times was that Zadeh [28] introduced fuzzy sets to solve uncertain events in 1965. After that, many authors have surveyed its applications. Kaufman [10] first proposed the definition of a fuzzy graph, which is more realistic and useful in practice. Subsequently, Rosenfeld [21] laid the foundation for the application of fuzzy graphs in many fields, including data

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mining, communication, clustering, scheduling theory and planning. In addition, fuzzy graphs have applications in many fields, such as cryptography and decision problems. In 2022, Zahedi and his team [20] proposed the concept of  $L$ -graphs and discussed their properties, and then applied them to practical problems.

One of the concepts proposed in graph theory was the imposed zero set in 2008. The zero forcing process is employed in [1] to set a limit on the minimum rank or the maximum nullity of a graph  $G$ . Assume  $G$  is a graph and  $V$  is the vertex set of graph  $G$ . Vertices of graph  $G$  are in either black or white. Suppose  $Z$  represents the set of black vertices of graph  $G$ . The color change rule is that when a black vertex has only one white vertex as its neighbor, the color of the white vertex is converted from white to black. If all vertices of  $G$  are converted into black after a finite number of color-change rules, it is said that set  $Z$  is a zero forcing set of  $G$ . The above process responsible for turning all white vertices into black is called zero forcing process [1]. As a model of graph propagation process, it is a topic often mentioned in computer science and mathematics as referenced in [11]. Various graph processes find utility in modeling social or technical phenomena, such as social network analysis [8] and physics [2]. A comprehensive survey of different applications and models is available in [23, 27].

In the late 1960s, the theory of fuzzy automata was first proposed by scholars such as Wee [26] and Santos [22]. There are many applications for fuzzy finite automata in various areas, such as pattern recognition, learning system, database theory, and neural networks. After that, automata theory based on residuated lattices was studied by a number of authors such as Ciric and his co-authors [4, 5, 9], Qiu [13, 14], and Tiwari and his coauthors [24]. Moreover, Qiu [16-18] first proposed and established the automata theory based on complete residuated lattices valued logic, and then Qiu further developed this subject systemically. In addition, Tiwari and Sharan [24] introduced the notion of fuzzy automata based on lattice-ordered monoid,  $L$ -fuzzy automata, as the fuzzy automata generalization [13, 14]. They indicated that the results of such  $L$ -fuzzy automata are dependent on the fact that the associated monoid has or does not have zero divisors. An important application of fuzzy finite automata is fuzzy discrete event system. Therefore, Qiu first proposed and established supervisory control theory of fuzzy discrete event system [18, 19]. Zahedi and his team [20] proposed theory of  $L$ -graphs and then studied theory of  $L$ -graph fuzzy automata based on theory of  $L$ -graphs in 2022.

The  $L$ -graph and  $L$ -graph automata based on residuated lattices introduced by Zahedi and his team have been applied in various areas. However, in practical applications, many models do not have a join operation, then many problems cannot be solved by  $L$ -graph and  $L$ -graph-automata based on residuated lattices. Semihoops are the most basic residuated structure that contain all logical algebras based on Galois connections. Therefore, we attempt to construct fuzzy graph and fuzzy automata based on semihoops. Not only has the

range of the fuzzy automata been extended, but also the basic characteristics of the fuzzy automata have been preserved, and can solve problems that other fuzzy automata cannot solve.

In this paper, we first attempt to construct *SH*-graph based on semihoops. Using the same method as constructing *SH*-graph, we establish *SH*-graph automata based on *SH*-graph. Then, the notions of isomorphism of *SH*-graphs and *SH*-graph automata are introduced. After that, we discuss that (i) the relationship between two *SH*-graphs being isomorphic and the complement of two *SH*-graphs being isomorphic; (ii) the relationship between two *SH*-graphs being isomorphic and two *SH*-graph automata related to two *SH*-graphs being isomorphic; (iii) the relationship between two *SH*-graphs being isomorphic and two *SH*-graph automata related to the complement of two *SH*-graphs being isomorphic. In addition, the concept of equivalence of *SH*-graph automata is given. Then, we prove that some special *SH*-graph automata are equivalent under different zero forcing sets. Also, we obtain that the relationship between two *SH*-graphs being isomorphic and two *SH*-graph automata related to two *SH*-graphs being equivalent. Moreover, the concept of the direct product of *SH*-graph automata is introduced. Then, we research the relationship between direct product and component *SH*-graph automata. Finally, some applications of *SH*-graphs and *SH*-graph automata are introduced.

## 2. Preliminaries

In this section, we first recall some basic notions and propositions, which will be essential for the other sections.

**Definition 2.1.** [27] A path is a simple graph whose vertices can be ordered such that two vertices are adjacent iff they are consecutive in the list. In a path, if the first vertex and the last vertex are connected, we call it cycle.

**Definition 2.2.** [27] A graph  $G$  is connected if each pair of vertices in  $G$  belongs to a path.

**Definition 2.3.** [27] A complete graph  $G$  is a simple graph in which every pair of distinct vertices is connected by a unique edge.

**Definition 2.4.** [27] A graph  $G = (V, E)$  is called bipartite graph if  $V$  can be divided into two classes so that the vertices of any edge belong to different classes (vertices in the same class are not adjacent). In the bipartite graph, if any two vertices in different classes are connected, then we call such a graph a complete bipartite graph.

**Definition 2.5.** [27] A homomorphism from a simple graph  $G = (V_G, E_G)$  to a simple graph  $H = (V_H, E_H)$  is a surjection  $f : V_G \rightarrow V_H$  such that  $uv \in E_G$  iff  $f(u)f(v) \in E_H$ .  $f$  is called an isomorphism iff  $f$  is a homomorphism that is one-one.

**Definition 2.6.** [7] An algebra  $(S, \odot, \rightarrow, \wedge, 1)$  of type  $(2, 2, 2, 0)$  is called a semihoop if it satisfies:

- (S1)  $(S, \wedge, 1)$  is a  $\wedge$ -semilattice and it has a upper bound 1;
- (S2)  $(S, \odot, 1)$  is a commutative monoid;
- (S3)  $(\alpha \odot \beta) \rightarrow \theta = \alpha \rightarrow (\beta \rightarrow \theta)$ , for any  $\alpha, \beta, \theta \in S$ .

In a semihoop  $(S, \odot, \rightarrow, \wedge, 1)$ , we define  $\alpha \leq \beta$  if and only if  $\alpha \rightarrow \beta = 1$ , for any  $\alpha, \beta \in S$ . It is easy to check that  $\leq$  is a partial order relation on  $S$  and we get  $\alpha \leq 1$ , for all  $\alpha \in S$ .

**Proposition 2.7.** [7] Let  $S$  be a semihoop. Then the following properties hold:

- (1)  $\alpha \odot \beta \leq \theta$  if and only if  $\alpha \leq \beta \rightarrow \theta$ , for every  $\alpha, \beta, \theta \in S$ ;
- (2)  $\alpha \odot \beta \leq \alpha, \beta$ , for any  $\alpha, \beta \in S$ ;
- (3)  $1 \rightarrow \alpha = \alpha, \alpha \rightarrow 1 = 1$ , for all  $\alpha \in S$ ;
- (4)  $\alpha^n \leq \alpha$ , for every  $\alpha \in S, n \in \mathbb{N}^+$ ;
- (5)  $\alpha \odot (\alpha \rightarrow \beta) \leq \beta$ , for any  $\alpha, \beta \in S$ ;
- (6)  $\alpha \leq \beta$  implies  $\alpha \odot \theta \leq \beta \odot \theta, \beta \rightarrow \theta \leq \alpha \rightarrow \theta$  and  $\theta \rightarrow \alpha \leq \theta \rightarrow \beta$ , for every  $\alpha, \beta, \theta \in S$ ;
- (7)  $\alpha \leq (\alpha \rightarrow \beta) \rightarrow \beta$ , for any  $\alpha, \beta \in S$ ;
- (8)  $\alpha \rightarrow (\beta \rightarrow \theta) = \beta \rightarrow (\alpha \rightarrow \theta)$ , for every  $\alpha, \beta, \theta \in S$ ;
- (9)  $\alpha' = \alpha \rightarrow 0$ .
- (10)  $\alpha \leq \alpha''$ .

**Definition 2.8.** [12] Let  $L$  be a lattice. An  $L$ -fuzzy automaton  $A$  is a 5 tuple  $(Q, X, \mu, F, \sigma)$ , where

- (L1)  $Q$  is a finite nonempty set (states),
- (L2)  $X$  is a finite nonempty set (inputs),
- (L3)  $\mu$  is a function from  $Q \times X \times Q$  into  $L$  (transition function),
- (L4)  $F$  is a subset of  $Q$  (final states),
- (L5)  $\sigma$  is a function from  $Q$  into  $L$  (initial distribution).

Elements of  $X$  are called inputs symbols. Finite sequences of input symbols are called input strings, words or tapes. Remember that the empty tape  $\Lambda$  has the property that  $x\Lambda = x = \Lambda x$  for every tape  $x$ . The collection of all inputs tapes is denoted by  $X^*$ . So, the expanded function  $\mu$  is defined as

$$\mu^*(q, \Lambda, q') = \begin{cases} 1 & \text{if } q = q', \\ 0 & \text{if } q \neq q', \end{cases}$$

$$\mu^*(q, xu, q') = \bigvee_{q'' \in Q} \mu^*(q, x, q'') \odot \mu(q'', u, q').$$

**Definition 2.9.** [12] A response function of an  $L$ -fuzzy automaton  $A$  is defined as

$$r_\mu(x, q) = \bigvee_{q' \in Q} \sigma(q') \odot \mu^*(q', x, q).$$

**Definition 2.10.** [12] A  $k$ -behavior of an  $L$ -fuzzy automaton  $A$  with threshold  $c$  is the set

$$B_k(A, c) = \{x \in X^* \mid \bigvee_{q \in F} r_\mu(x, q) > c \text{ and } |x| \leq k\},$$

**Definition 2.11.** [12] Let  $A_1 = (Q_1, X, \mu_1, F_1, \sigma_1)$  and  $A_2 = (Q_2, X, \mu_2, F_2, \sigma_2)$  be two  $L$ -fuzzy automata. We claim that  $A_1$  and  $A_2$  are equivalent iff  $B_k(A_1, c) = B_k(A_2, c)$ , for every  $k = 0, 1, 2, \dots$  and for every  $c \in L \setminus 1$ .

**Definition 2.12.** [1] Color-change rule:

If  $G$  is a graph with each vertex colored either white or black,  $u$  is a black vertex of  $G$ , and exactly one neighbor  $v$  of  $u$  is white, then change the color of  $v$  to black.

Given a coloring of  $G$ , the derived coloring is the result of applying the color-change rule until no more changes are possible.

A zero forcing set for a graph  $G$  is a subset of vertices  $Z$  such that if initially the vertices in  $Z$  are colored black and the remaining vertices are colored white, the derived coloring of  $G$  is all black.

$Z(G)$  is the minimum of  $|Z|$  over all zero forcing sets  $Z \subseteq V(G)$ .

For example, an endpoint of a path is a zero forcing set for the path. In a cycle, any set of two adjacent vertices is a zero forcing set.

**Definition 2.13.** [20]  $G = (\alpha, \beta)$  is called an  $L$ -graph on  $G^* = (V, E)$  if  $\alpha : V \rightarrow L$  and  $\beta : E \rightarrow L$  are functions ( $L$  is a residuated lattice), with  $\beta(q_i q_j) \leq \alpha(q_i) \odot \alpha(q_j)$  for every  $q_i q_j \in E$ . Besides, if  $G^*$  is a path (cycle, bipartite, complete, complete bipartite) graph, then  $G$  is called a path (cycle, bipartite, complete, complete bipartite)  $L$ -graph on  $G^*$ .

**Example 2.14.** [20] Suppose  $L = ([0, 1], \wedge, \vee, \odot, \rightarrow, 0, 1)$ , where  $a \odot b = \begin{cases} a + b - 1 & \text{if } a + b \geq 1, \\ 0 & \text{if } a + b < 1, \end{cases}$  and  $a \rightarrow b = \begin{cases} 1 & \text{if } b - a \geq 0, \\ 1 - a + b & \text{if } b - a < 0. \end{cases}$  Then,  $G = (\alpha, \beta)$  is a cycle  $L$ -graph on  $G^*$ , where  $V = \{q_1, q_2, q_3, q_4\}$ ,  $E = \{q_1 q_2, q_2 q_3, q_3 q_4, q_4 q_1\}$ ,  $\beta(q_i q_j) = \alpha(q_i) \odot \alpha(q_j)$  for every  $q_i q_j \in E$ ,  $\alpha(q_1) = 0.8$ ,  $\alpha(q_2) = 0.3$ ,  $\alpha(q_3) = 0.1$ ,  $\alpha(q_4) = 1$ ,  $\beta(q_1 q_2) = 0.1$ ,  $\beta(q_2 q_3) = 0$ ,  $\beta(q_3 q_4) = 0.1$ ,  $\beta(q_4 q_1) = 0.8$ .

**Definition 2.15.** [20] The complement of  $L$ -graph  $G = (\alpha, \beta)$  on  $G^* = (V, E)$  is denoted by  $G' = (\alpha', \beta')$  on  $(G^*)' = (V, E')$  that is the complement of  $G^*$ , where  $\alpha' = \alpha$ ,  $\beta'(q_i q_j) = \alpha(q_i) \odot \alpha(q_j)$ , for every  $q_i q_j \notin E$ .

**Example 2.16.** [20] Consider the  $L$ -graph  $G$  in Example 2.11. It is observed  $G' = (\alpha', \beta')$  on  $(G^*)'$ , where  $V = \{q_1, q_2, q_3, q_4\}$ ,  $E = \{q_1q_3, q_2q_4\}$ ,  $\alpha'(q_1) = 0.8$ ,  $\alpha'(q_2) = 0.3$ ,  $\alpha'(q_3) = 0.1$ ,  $\alpha'(q_4) = 1$ ,  $\beta'(q_1q_3) = 0$  and  $\beta'(q_2q_4) = 0.3$ .

### 3. $SH$ -graphs and $SH$ -graph automata

In this section, we introduce a  $SH$ -graph based on semihoops. Then, a  $SH$ -graph automaton which is constructed by the  $SH$ -graph and the minimum zero forcing set will be introduced and relevant results will be obtained.

In this paper, we assume that  $S$  is a semihoop, and  $G^*$  is a graph  $(V, E)$ .

**Definition 3.1.**  $G = (\alpha, \beta)$  is called a  $SH$ -graph on  $G^* = (V, E)$  if  $\alpha : V \rightarrow S$  and  $\beta : E \rightarrow S$  are functions, with  $\beta(q_iq_j) \leq \alpha(q_i) \odot \alpha(q_j)$  for every  $q_iq_j \in E$ . Besides, if  $G^*$  is a path (cycle, bipartite, complete, complete bipartite) graph, then  $G$  is called a path (cycle, bipartite, complete, complete bipartite)  $SH$ -graph on  $G^*$ .

**Example 3.2.** Let  $S = (\{0, a, e, p, q, r, m, 1\}, \odot, \rightarrow, \wedge, 1)$  and  $0 < a < e < 1, 0 < p < q < m < 1, 0 < r < m < 1$ , where  $e$  and  $p$  are incomparable,  $q$  and  $r$  are incomparable. We define  $\odot$  and  $\rightarrow$  on  $S$  as follows:

$\odot$	0	a	e	p	q	r	m	1	$\rightarrow$	0	a	e	p	q	r	m	1
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
a	0	0	0	0	0	0	0	a	a	m	1	1	1	1	1	1	1
e	0	0	e	0	e	0	0	e	e	r	r	1	r	1	r	1	1
p	0	0	0	0	0	p	p	p	p	q	q	q	1	1	1	1	1
q	0	0	e	0	e	p	q	q	q	p	p	q	r	1	r	1	1
r	0	0	0	p	p	r	r	r	r	e	e	e	q	q	1	1	1
m	0	0	e	p	q	r	m	m	m	a	a	e	p	q	r	1	1
1	0	a	e	p	q	r	m	1	1	0	a	e	p	q	r	m	1

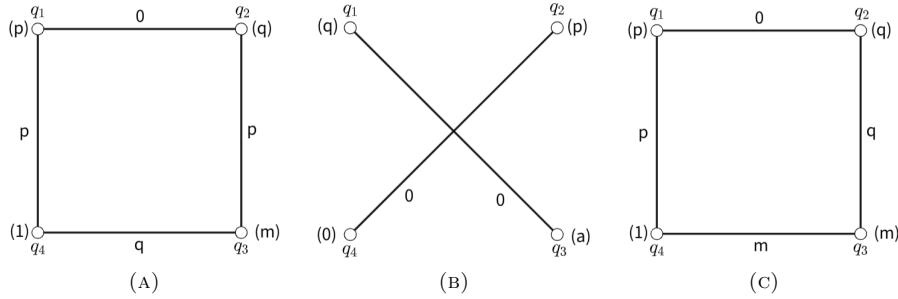
In [25], we can know that  $S$  is a semihoop. Then,  $G = (\alpha, \beta)$  is a cycle  $SH$ -graph on  $G^*$ , as in Fig. 1, where  $V = \{q_1, q_2, q_3, q_4\}$ ,  $E = \{q_1q_2, q_2q_3, q_3q_4, q_4q_1\}$  for every  $q_iq_j \in E$ ,  $\alpha(q_1) = p$ ,  $\alpha(q_2) = q$ ,  $\alpha(q_3) = m$ ,  $\alpha(q_4) = 1$ ,  $\beta(q_1q_2) = 0$ ,  $\beta(q_2q_3) = p$ ,  $\beta(q_3q_4) = q$ ,  $\beta(q_4q_1) = p$ .

**Definition 3.3.** The complement of  $SH$ -graph  $G = (\alpha, \beta)$  on  $G^* = (V, E)$  is denoted by  $G' = (\alpha', \beta')$  on  $(G^*)' = (V, E')$  that is the complement of  $G^*$ , where  $\alpha'(q_i) = [\alpha(q_i)]' = [\alpha(q_i)] \rightarrow 0$ ,  $\beta'(q_iq_j) = [\alpha(q_i)]' \odot [\alpha(q_j)]'$ .

**Example 3.4.** Consider the  $SH$ -graph  $G$  in Example 3.2. It is observed that  $G' = (\alpha', \beta')$  on  $(G^*)'$ , as in Fig. 2, where  $V = \{q_1, q_2, q_3, q_4\}$ ,  $E' = \{q_1q_3, q_2q_4\}$ ,  $\alpha'(q_1) = p \rightarrow 0 = q$ ,  $\alpha'(q_2) = q \rightarrow 0 = p$ ,  $\alpha'(q_3) = m \rightarrow 0 = a$ ,  $\alpha'(q_4) = 1 \rightarrow 0 = 0$ ,  $\beta'(q_1q_3) = \alpha'(q_1) \odot \alpha'(q_3) = q \odot 0 = 0$ ,  $\beta'(q_2q_4) = \alpha'(q_2) \odot \alpha'(q_4) = p \odot 0 = 0$ .

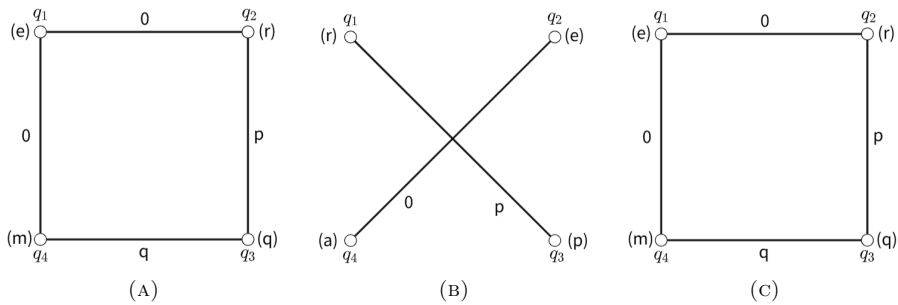
**Example 3.5.** Consider the  $SH$ -graph  $G'$  in Example 3.4. It is observed that  $G'' = (\alpha'', \beta'')$  on  $(G^*)''$ , as in Fig. 3, where  $V = \{q_1, q_2, q_3, q_4\}$ ,  $E'' =$

$\{q_1q_2, q_2q_3, q_3q_4, q_4q_1\}$ ,  $\alpha''(q_1) = q \rightarrow 0 = p$ ,  $\alpha''(q_2) = p \rightarrow 0 = q$ ,  $\alpha''(q_3) = a \rightarrow 0 = m$ ,  $\alpha''(q_4) = 0 \rightarrow 0 = 1$ ,  $\beta''(q_1q_2) = \alpha''(q_1) \odot \alpha''(q_2) = p \odot q = 0$ ,  $\beta''(q_2q_3) = \alpha''(q_2) \odot \alpha''(q_3) = q \odot m = q$ ,  $\beta''(q_3q_4) = \alpha''(q_3) \odot \alpha''(q_4) = m \odot 1 = m$ ,  $\beta''(q_4q_1) = \alpha''(q_4) \odot \alpha''(q_1) = 1 \odot p = p$ . Therefore, we can know that SH-graph  $G$  and SH-graph  $G''$  are not same.



**FIGURE 1.** The SH-graph  $G$  on  $G^*$ ; **FIGURE 2.** The complement of SH-graph  $G'$ ; **FIGURE 3.** The complement of SH-graph  $G''$ .

**Example 3.6.** Consider the semihoop in Example 3.2. Then,  $G = (\alpha, \beta)$  is a cycle SH-graph on  $G^*$ , as in Fig. 4, where  $V = \{q_1, q_2, q_3, q_4\}$ ,  $E = \{q_1q_2, q_2q_3, q_3q_4, q_4q_1\}$  for every  $q_iq_j \in E$ ,  $\alpha(q_1) = e$ ,  $\alpha(q_2) = r$ ,  $\alpha(q_3) = q$ ,  $\alpha(q_4) = m$ ,  $\beta(q_1q_2) = \alpha(q_1) \odot \alpha(q_2) = 0$ ,  $\beta(q_2q_3) = \alpha(q_2) \odot \alpha(q_3) = p$ ,  $\beta(q_3q_4) = \alpha(q_3) \odot \alpha(q_4) = q$ ,  $\beta(q_4q_1) = \alpha(q_4) \odot \alpha(q_1) = 0$ .



**FIGURE 4.** The SH-graph  $G$  on  $G^*$ ; **FIGURE 5.** The complement of SH-graph  $G'$ ; **FIGURE 6.** The complement of SH-graph  $G''$ .

It is observed that  $G' = (\alpha', \beta')$  on  $(G^*)'$ , as in Fig. 5, where  $V = \{q_1, q_2, q_3, q_4\}$ ,  $E' = \{q_1q_3, q_2q_4\}$ ,  $\alpha'(q_1) = e \rightarrow 0 = r$ ,  $\alpha'(q_2) = r \rightarrow 0 = e$ ,  $\alpha'(q_3) = q \rightarrow 0 = p$ ,  $\alpha'(q_4) = m \rightarrow 0 = a$ ,  $\beta'(q_1q_3) = \alpha'(q_1) \odot \alpha'(q_3) = r \odot p = p$ ,  $\beta'(q_2q_4) = \alpha'(q_2) \odot \alpha'(q_4) = e \odot a = 0$ .

It is observed that  $G'' = (\alpha'', \beta'')$  on  $(G^*)''$ , as in Fig. 6, where  $V = \{q_1, q_2, q_3, q_4\}$ ,  $E'' = \{q_1q_2, q_2q_3, q_3q_4, q_4q_1\}$ ,  $\alpha''(q_1) = r \rightarrow 0 = e$ ,  $\alpha''(q_2) = e \rightarrow 0 = r$ ,  $\alpha''(q_3) = p \rightarrow 0 = q$ ,  $\alpha''(q_4) = a \rightarrow 0 = m$ ,  $\beta''(q_1q_2) = \alpha''(q_1) \odot \alpha''(q_2) = e \odot r = 0$ ,  $\beta''(q_2q_3) = \alpha''(q_2) \odot \alpha''(q_3) = r \odot q = p$ ,  $\beta''(q_3q_4) = \alpha''(q_3) \odot \alpha''(q_4) = q \odot m = q$ ,  $\beta''(q_4q_1) = \alpha''(q_4) \odot \alpha''(q_1) = m \odot e = 0$ .

Therefore, we can know that  $SH$ -graph  $G$  and  $SH$ -graph  $G''$  are same.

**Theorem 3.7.** If  $\beta(q_iq_j) = \alpha(q_i) \odot \alpha(q_j)$  and  $\alpha(q_i) = \alpha''(q_i)$  in  $SH$ -graph  $G = (\alpha, \beta)$  and  $SH$ -graph  $G'' = (\alpha'', \beta'')$ , then  $SH$ -graph  $G$  and  $SH$ -graph  $G''$  are same.

*Proof.* Because  $\beta(q_iq_j) = \alpha(q_i) \odot \alpha(q_j)$ ,  $\beta''(q_iq_j) = \alpha''(q_i) \odot \alpha''(q_j)$ ,  $\alpha(q_i) = \alpha''(q_i)$ , then  $\beta(q_iq_j) = \beta''(q_iq_j)$ . So,  $SH$ -graph  $G$  and  $SH$ -graph  $G''$  are same.  $\square$

In the following contents, we introduce  $SH$ -graph automata based on  $SH$ -graph using zero forcing sets and provide an example.

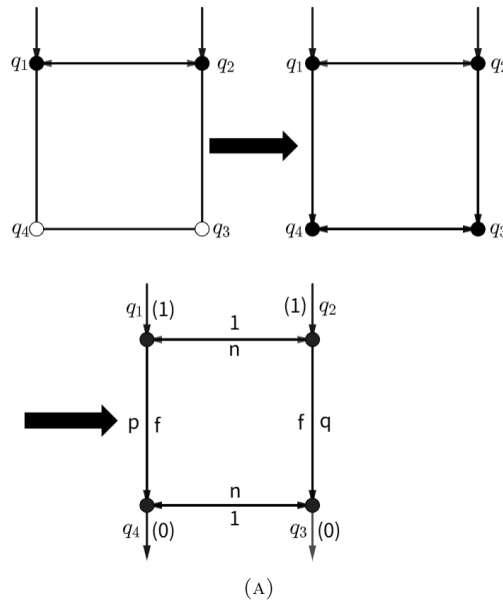


FIGURE 7. The related  $SH$ -graph automaton  $A(Z(G))$ .

**Definition 3.8.** Let  $G = (\alpha, \beta)$  be a SH-graph on  $G^*$  and  $Z(G)$  be a zero forcing set of  $G^*$ . A SH-graph automaton  $A(Z(G))$  related to  $G$  is denoted by 5-tuple  $(Q, X, \mu, F, \sigma)$ , where

- (1)  $Q = V$  is the finite nonempty set,
- (2)  $X = \{f, n\}$  is the set of alphabet letters,
- (3)  $\mu : V \times X \times V \rightarrow S$  is the transition function;
- (3)  $\mu(q_i, u, q_j) = \begin{cases} \beta(q_i q_j) & \text{if } u = f, \\ 1 & \text{if } u = n, \end{cases}$
- (4)  $F$  is the set of final states, where  $q \in F$  iff  $q$  does not force any vertice;
- (5)  $\sigma : V \rightarrow S$  is the initial distribution;
- (5)  $\sigma(q_i) = \begin{cases} 1 & \text{if } q_i \in Z(G), \\ \text{least element in semihoops} & \text{otherwise,} \end{cases}$  for every  $q_i \in Q$ .

Furthermore, we can prove that a new set  $Z(A(Z(G))) = \{q_i \in Q \mid \sigma(q_i) = 1\} = \{q_i \in Q \mid q_i \in Z(G)\}$ .

**Example 3.9.** Consider the SH-graph  $G$  in Example 3.2, SH-graph automaton  $A(Z(G)) = (Q, X, \mu, F, \sigma)$  related to  $G$  can be determined with respect to  $Z(G) = (q_1, q_2)$ , as in Fig. 7, where  $Q = V$ ,  $X = \{n, f\}$ ,  $F = \{q_3, q_4\}$ ,  $\mu(q_1, n, q_2) = \mu(q_2, n, q_1) = 1$ ,  $\mu(q_3, n, q_4) = \mu(q_4, n, q_3) = 1$ ,  $\mu(q_1, f, q_4) = p$ ,  $\mu(q_2, f, q_3) = q$ ,  $\sigma(q_1) = \sigma(q_2) = 1$  and  $\sigma(q_3) = \sigma(q_4) = 0$ .

In the following section, the isomorphic concepts of SH-graphs and SH-graph automata are first introduced. And, related examples are given. Then, we discuss the relationship between two SH-graphs being isomorphic and their related SH-graph automata being isomorphic.

**Definition 3.10.** Let  $G_1 = (\alpha_1, \beta_1)$  and  $G_2 = (\alpha_2, \beta_2)$  be two SH-graphs on  $G_1^* = (V_1, E_1)$  and  $G_2^* = (V_2, E_2)$ , respectively, and  $c \in SH \setminus \{1\}$ .  $G_1$  and  $G_2$  are called to be isomorphic with threshold  $c$ , denoted by  $G_1 \cong_c G_2$  if there exists a bijection  $h: V_1 \rightarrow V_2$  such that the following conditions hold for all  $q_i, q_j \in V_1$ :

- (I1)  $q_i q_j \in E_1$  if and only if  $h(q_i)h(q_j) \in E_2$ ,
- (I2)  $\alpha_1(q_i) > c$  if and only if  $\alpha_2(h(q_i)) > c$ ,
- (I3)  $\beta_1(q_i q_j) > c$  if and only if  $\beta_2(h(q_i)h(q_j)) > c$ .

**Example 3.11.** Let  $S = (\{0, s, t, p, m, a, b, 1\}, \odot, \rightarrow, \wedge, 1)$  and  $0 < s, t < p < m < a < b < 1$ . We define  $\odot$  and  $\rightarrow$  on  $S$  as follows:

$\odot$	0	s	t	p	m	a	b	1	$\rightarrow$	0	s	t	p	m	a	b	1
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
s	0	s	0	s	s	s	s	s	s	t	1	t	1	1	1	1	1
t	0	0	t	t	t	t	t	t	t	s	s	1	1	1	1	1	1
p	0	s	t	p	p	p	p	p	p	0	s	t	1	1	1	1	1
m	0	s	t	p	m	m	m	m	m	0	s	t	p	1	1	1	1
a	0	s	t	p	m	a	a	a	a	0	s	t	p	m	1	1	1
b	0	s	t	p	m	a	b	b	b	0	s	t	p	m	a	1	1
1	0	s	t	p	m	a	b	1	1	0	s	t	p	m	a	b	1

Through proof, we can know that  $S$  is a semihoop. Let  $G_1 = (\alpha_1, \beta_1)$  and  $G_2 = (\alpha_2, \beta_2)$  be two SH-graphs on  $G_1^* = (V_1, E_1)$  and  $G_2^* = (V_2, E_2)$ , respectively, as in Fig. 8, where  $V_1 = \{q_1, q_2, q_3, q_4\}$ ,  $E_1 = \{q_1q_2, q_2q_3, q_3q_4, q_4q_1, q_1q_3, q_2q_4\}$ ,  $\alpha_1(q_1) = s, \alpha_1(q_2) = p, \alpha_1(q_3) = m, \alpha_1(q_4) = 1, \beta_1(q_1q_2) = s, \beta_1(q_2q_3) = p, \beta_1(q_3q_4) = m, \beta_1(q_4q_1) = 0, \beta_1(q_1q_3) = 0, \beta_1(q_2q_4) = t, V_2 = \{q'_1, q'_2, q'_3, q'_4\}$ ,  $E_2 = \{q'_1q'_2, q'_2q'_3, q'_3q'_4, q'_4q'_1, q'_1q'_3, q'_2q'_4\}$ ,  $\alpha_2(q'_1) = 1, \alpha_2(q'_2) = s, \alpha_2(q'_3) = p, \alpha_2(q'_4) = m, \beta_2(q'_1q'_2) = 0, \beta_2(q'_2q'_3) = s, \beta_2(q'_3q'_4) = p, \beta_2(q'_4q'_1) = m, \beta_2(q'_1q'_3) = t, \beta_2(q'_2q'_4) = 0$ . Additionally, let  $h$  be a bijection from  $G_1$  into  $G_2$ , where  $h(q_1) = q'_2, h(q_2) = q'_3, h(q_3) = q'_4, h(q_4) = q'_1$ . Then,  $G_1$  and  $G_2$  are isomorphic SH-graphs.

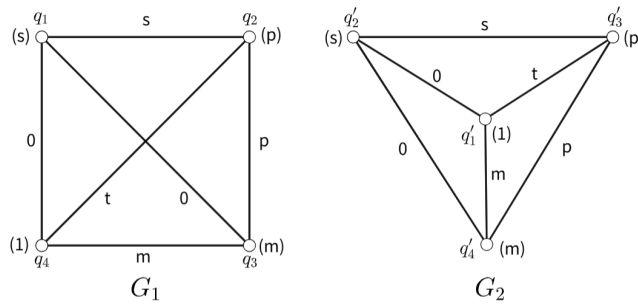


FIGURE 8. Two SH-graphs  $G_1$  and  $G_2$  .

**Example 3.12.** Consider the semihoop in Example 3.11,  $\beta(q_iq_j) = \alpha(q_i) \odot \alpha(q_j)$ , for every  $q_iq_j \in E$ . Let  $G_1 = (\alpha_1, \beta_1)$  and  $G_2 = (\alpha_2, \beta_2)$  be two SH-graphs on  $G_1^* = (V_1, E_1)$  and  $G_2^* = (V_2, E_2)$ , respectively, as in Fig. 9, where  $V_1 = \{q_1, q_2, q_3, q_4\}$ ,  $E_1 = \{q_1q_2, q_2q_3, q_3q_4, q_4q_1\}$ ,  $\alpha_1(q_1) = s, \alpha_1(q_2) = p, \alpha_1(q_3) = s, \alpha_1(q_4) = t, \beta_1(q_1q_2) = s, \beta_1(q_2q_3) = s, \beta_1(q_3q_4) = 0, \beta_1(q_4q_1) = 0, V_2 = \{q'_1, q'_2, q'_3, q'_4\}$ ,  $E_2 = \{q'_1q'_2, q'_2q'_3, q'_3q'_4, q'_4q'_1\}$ ,  $\alpha_2(q'_1) = p, \alpha_2(q'_2) = s, \alpha_2(q'_3) = t, \alpha_2(q'_4) = s, \beta_2(q'_1q'_2) = s, \beta_2(q'_2q'_3) = 0, \beta_2(q'_3q'_4) = 0, \beta_2(q'_4q'_1) = s$ . Additionally, let  $h$  be a function from  $G_1$  into  $G_2$ , where  $h(q_1) = q'_2, h(q_2) = q'_1, h(q_3) = q'_4, h(q_4) = q'_3$ . Then,  $G_1$  and  $G_2$  are isomorphic SH-graphs. It is observed that  $G'_1 = (\alpha'_1, \beta'_1)$  on  $(G_1^*)' = (V_1, E'_1)$  and  $G'_2 = (\alpha'_2, \beta'_2)$  on  $(G_2^*)' = (V_2, E'_2)$ , respectively, as in Fig. 10, where  $V_1 = \{q_1, q_2, q_3, q_4\}$ ,  $E'_1 =$

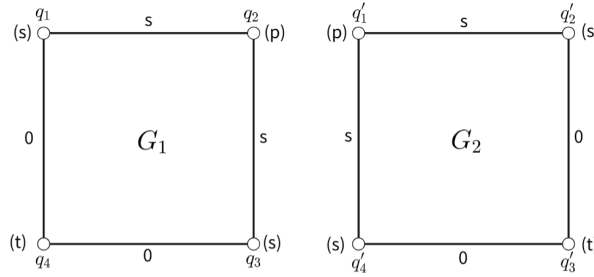


FIGURE 9.  $G_1$  and  $G_2$  are two SH-graphs on  $G_1^*$  and  $G_2^*$ .

$\{q_1q_3, q_2q_4\}$ ,  $\alpha'_1(q_1) = s \rightarrow 0 = t$ ,  $\alpha'_1(q_2) = p \rightarrow 0 = 0$ ,  $\alpha'_1(q_3) = s \rightarrow 0 = t$ ,  $\alpha'_1(q_4) = t \rightarrow 0 = s$ ,  $\beta'_1(q_1q_3) = \alpha'_1(q_1) \odot \alpha'_1(q_3) = t$ ,  $\beta'_1(q_2q_4) = \alpha'_1(q_2) \odot \alpha'_1(q_4) = 0$ ,  $V_2 = \{q'_1, q'_2, q'_3, q'_4\}$ ,  $E'_2 = \{q'_1q'_3, q'_2q'_4\}$ ,  $\alpha'_2(q'_1) = p \rightarrow 0 = 0$ ,  $\alpha'_2(q'_2) = s \rightarrow 0 = t$ ,  $\alpha'_2(q'_3) = t \rightarrow 0 = s$ ,  $\alpha'_2(q'_4) = s \rightarrow 0 = t$ ,  $\beta'_2(q'_1q'_3) = \alpha'_2(q'_1) \odot \alpha'_2(q'_3) = 0$ ,  $\beta'_2(q'_2q'_4) = \alpha'_2(q'_2) \odot \alpha'_2(q'_4) = t$ . Then,  $G'_1$  and  $G'_2$  are isomorphic SH-graphs.

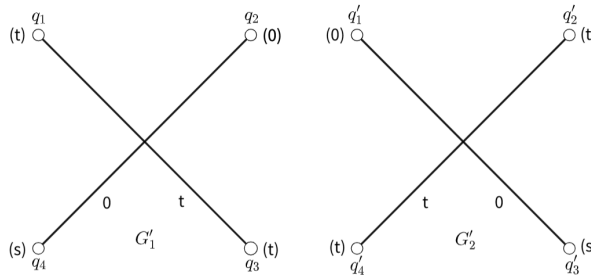


FIGURE 10.  $G'_1$  and  $G'_2$  are two complement of SH-graphs  $G_1$  and  $G_2$ .

**Example 3.13.** Let  $S = (\{0, s, t, p, m, 1\}, \odot, \rightarrow, \wedge, 1)$  and  $0 < s, t, p, m < 1$ . We define  $\odot$  and  $\rightarrow$  on  $S$  as follows:

$\odot$	0	s	t	p	m	1
0	0	0	0	0	0	0
s	0	s	t	m	m	s
t	0	t	t	0	0	t
p	0	m	0	m	m	p
m	0	m	0	m	m	m
1	0	s	t	p	m	1

$\rightarrow$	0	s	t	p	m	1
0	1	1	1	1	1	1
s	0	1	t	p	p	1
t	p	s	1	p	p	1
p	t	s	t	1	s	1
m	t	s	t	s	1	1
1	0	s	t	p	m	1

In [3], we can know that  $S$  is a semihoop. Let  $\beta(q_iq_j) = \alpha(q_i) \odot \alpha(q_j)$ , for every  $q_iq_j \in E$ . Let  $G_1 = (\alpha_1, \beta_1)$  and  $G_2 = (\alpha_2, \beta_2)$  be two SH-graphs

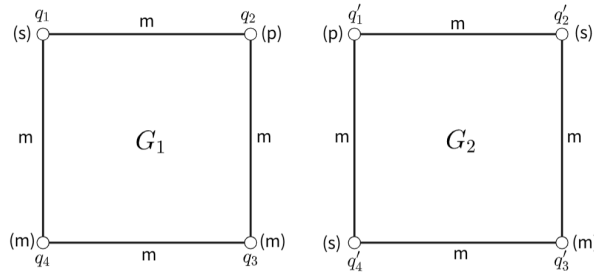


FIGURE 11.  $G_1$  and  $G_2$  are two  $SH$ -graphs on  $G_1^*$  and  $G_2^*$ .

on  $G_1^* = (V_1, E_1)$  and  $G_2^* = (V_2, E_2)$ , respectively, as in Fig. 11, where  $V_1 = \{q_1, q_2, q_3, q_4\}$ ,  $E_1 = \{q_1q_2, q_2q_3, q_3q_4, q_4q_1\}$ ,  $\alpha_1(q_1) = s$ ,  $\alpha_1(q_2) = p$ ,  $\alpha_1(q_3) = m$ ,  $\alpha_1(q_4) = m$ ,  $\beta_1(q_1q_2) = m$ ,  $\beta_1(q_2q_3) = m$ ,  $\beta_1(q_3q_4) = m$ ,  $\beta_1(q_4q_1) = m$ ,  $V_2 = \{q'_1, q'_2, q'_3, q'_4\}$ ,  $E_2 = \{q'_1q'_2, q'_2q'_3, q'_3q'_4, q'_4q'_1\}$ ,  $\alpha_2(q'_1) = p$ ,  $\alpha_2(q'_2) = s$ ,  $\alpha_2(q'_3) = m$ ,  $\alpha_2(q'_4) = s$ ,  $\beta_2(q'_1q'_2) = m$ ,  $\beta_2(q'_2q'_3) = m$ ,  $\beta_2(q'_3q'_4) = m$ ,  $\beta_2(q'_4q'_1) = m$ . Additionally, let  $h$  be a function from  $G_1$  into  $G_2$ , where  $h(q_i) = q'_i$ . Then,  $G_1$  and  $G_2$  are isomorphic  $SH$ -graphs.

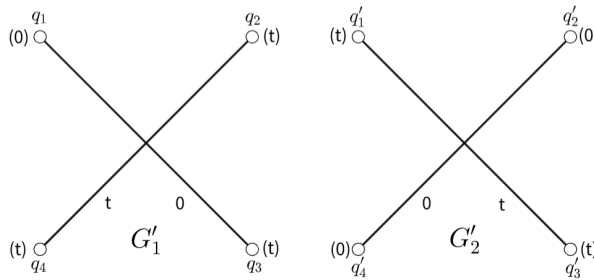


FIGURE 12.  $G'_1$  and  $G'_2$  are two complement of  $SH$ -graphs  $G_1$  and  $G_2$ .

It is observed that  $G'_1 = (\alpha'_1, \beta'_1)$  on  $(G_1^*)' = (V_1, E'_1)$  and  $G'_2 = (\alpha'_2, \beta'_2)$  on  $(G_2^*)' = (V_2, E'_2)$ , respectively, as in Fig. 12, where  $V_1 = \{q_1, q_2, q_3, q_4\}$ ,  $E'_1 = \{q_1q_3, q_2q_4\}$ ,  $\alpha'_1(q_1) = s \rightarrow 0 = 0$ ,  $\alpha'_1(q_2) = p \rightarrow 0 = t$ ,  $\alpha'_1(q_3) = m \rightarrow 0 = t$ ,  $\alpha'_1(q_4) = m \rightarrow 0 = t$ ,  $\beta'_1(q_1q_3) = \alpha'_1(q_1) \odot \alpha'_1(q_3) = 0$ ,  $\beta'_1(q_2q_4) = \alpha'_1(q_2) \odot \alpha'_1(q_4) = t$ ,  $V_2 = \{q'_1, q'_2, q'_3, q'_4\}$ ,  $E'_2 = \{q'_1q'_3, q'_2q'_4\}$ ,  $\alpha'_2(q'_1) = p \rightarrow 0 = t$ ,  $\alpha'_2(q'_2) = s \rightarrow 0 = 0$ ,  $\alpha'_2(q'_3) = m \rightarrow 0 = t$ ,  $\alpha'_2(q'_4) = s \rightarrow 0 = 0$ ,  $\beta'_2(q'_1q'_3) = \alpha'_2(q'_1) \odot \alpha'_2(q'_3) = t$ ,  $\beta'_2(q'_2q'_4) = \alpha'_2(q'_2) \odot \alpha'_2(q'_4) = 0$ . Then,  $G'_1$  and  $G'_2$  are not isomorphic  $SH$ -graphs.

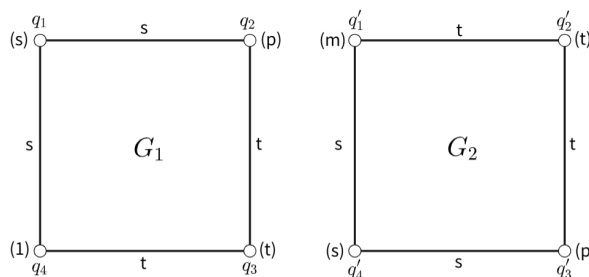


FIGURE 13.  $G_1$  and  $G_2$  are two SH-graphs on  $G_1^*$  and  $G_2^*$ .

**Example 3.14.** Consider the semihoop in Example 3.11,  $\beta(q_i q_j) = \alpha(q_i) \odot \alpha(q_j)$ , for every  $q_i q_j \in E$ . Let  $G_1 = (\alpha_1, \beta_1)$  and  $G_2 = (\alpha_2, \beta_2)$  be two SH-graphs on  $G_1^* = (V_1, E_1)$  and  $G_2^* = (V_2, E_2)$ , respectively, as in Fig. 13, where  $V_1 = \{q_1, q_2, q_3, q_4\}$ ,  $E_1 = \{q_1 q_2, q_2 q_3, q_3 q_4, q_4 q_1\}$ ,  $\alpha_1(q_1) = s$ ,  $\alpha_1(q_2) = p$ ,  $\alpha_1(q_3) = t$ ,  $\alpha_1(q_4) = 1$ ,  $\beta_1(q_1 q_2) = s$ ,  $\beta_1(q_2 q_3) = t$ ,  $\beta_1(q_3 q_4) = t$ ,  $\beta_1(q_4 q_1) = s$ ,  $V_2 = \{q'_1, q'_2, q'_3, q'_4\}$ ,  $E_2 = \{q'_1 q'_2, q'_2 q'_3, q'_3 q'_4, q'_4 q'_1\}$ ,  $\alpha_2(q'_1) = m$ ,  $\alpha_2(q'_2) = t$ ,  $\alpha_2(q'_3) = p$ ,  $\alpha_2(q'_4) = s$ ,  $\beta_2(q'_1 q'_2) = t$ ,  $\beta_2(q'_2 q'_3) = t$ ,  $\beta_2(q'_3 q'_4) = s$ ,  $\beta_2(q'_4 q'_1) = s$ . Additionally, let  $h$  be a function from  $G_1$  into  $G_2$ , where  $h(q_1) = q'_2$ ,  $h(q_2) = q'_1$ ,  $h(q_3) = q'_4$ ,  $h(q_4) = q'_3$ . Then,  $G_1$  and  $G_2$  are not isomorphic SH-graphs.

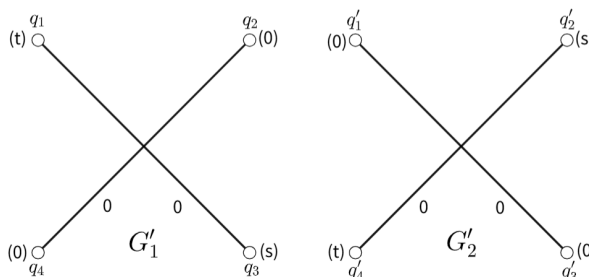


FIGURE 14.  $G'_1$  and  $G'_2$  are two complement of SH-graphs  $G_1$  and  $G_2$ .

It is observed that  $G'_1 = (\alpha'_1, \beta'_1)$  on  $(G_1^*)' = (V_1, E'_1)$  and  $G'_2 = (\alpha'_2, \beta'_2)$  on  $(G_2^*)' = (V_2, E'_2)$ , respectively, as in Fig. 14, where  $V_1 = \{q_1, q_2, q_3, q_4\}$ ,  $E'_1 = \{q_1 q_3, q_2 q_4\}$ ,  $\alpha'_1(q_1) = s \rightarrow 0 = t$ ,  $\alpha'_1(q_2) = p \rightarrow 0 = 0$ ,  $\alpha'_1(q_3) = t \rightarrow 0 = s$ ,  $\alpha'_1(q_4) = 1 \rightarrow 0 = 0$ ,  $\beta'_1(q_1 q_3) = \alpha'_1(q_1) \odot \alpha'_1(q_3) = 0$ ,  $\beta'_1(q_2 q_4) = \alpha'_1(q_2) \odot \alpha'_1(q_4) = 0$ ,  $V_2 = \{q'_1, q'_2, q'_3, q'_4\}$ ,  $E'_2 = \{q'_1 q'_3, q'_2 q'_4\}$ ,  $\alpha'_2(q'_1) = m \rightarrow 0 = 0$ ,  $\alpha'_2(q'_2) = t \rightarrow 0 = s$ ,  $\alpha'_2(q'_3) = p \rightarrow 0 = 0$ ,  $\alpha'_2(q'_4) = s \rightarrow 0 = t$ ,  $\beta'_2(q'_1 q'_3) = \alpha'_2(q'_1) \odot \alpha'_2(q'_3) = 0$ ,  $\beta'_2(q'_2 q'_4) = \alpha'_2(q'_2) \odot \alpha'_2(q'_4) = 0$ . Then,  $G'_1$  and  $G'_2$  are isomorphic SH-graphs.

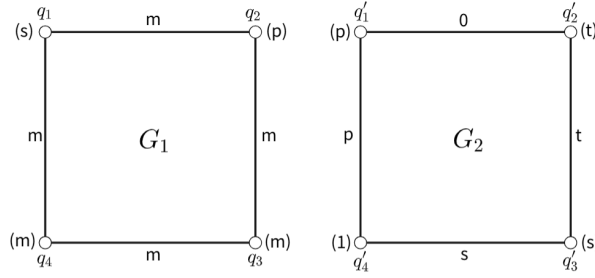


FIGURE 15.  $G_1$  and  $G_2$  are two SH-graphs on  $G_1^*$  and  $G_2^*$ .

**Example 3.15.** Consider the semihoop in Example 3.13, and  $\beta(q_i q_j) = \alpha(q_i) \odot \alpha(q_j)$ , for every  $q_i q_j \in E$ . Let  $G_1 = (\alpha_1, \beta_1)$  and  $G_2 = (\alpha_2, \beta_2)$  be two SH-graphs on  $G_1^* = (V_1, E_1)$  and  $G_2^* = (V_2, E_2)$ , respectively, as in Fig. 15, where  $V_1 = \{q_1, q_2, q_3, q_4\}$ ,  $E_1 = \{q_1 q_2, q_2 q_3, q_3 q_4, q_4 q_1\}$ ,  $\alpha_1(q_1) = s$ ,  $\alpha_1(q_2) = p$ ,  $\alpha_1(q_3) = m$ ,  $\alpha_1(q_4) = m$ ,  $\beta_1(q_1 q_2) = m$ ,  $\beta_1(q_2 q_3) = m$ ,  $\beta_1(q_3 q_4) = m$ ,  $\beta_1(q_4 q_1) = m$ ,  $V_2 = \{q'_1, q'_2, q'_3, q'_4\}$ ,  $E_2 = \{q'_1 q'_2, q'_2 q'_3, q'_3 q'_4, q'_4 q'_1\}$ ,  $\alpha_2(q'_1) = p$ ,  $\alpha_2(q'_2) = t$ ,  $\alpha_2(q'_3) = s$ ,  $\alpha_2(q'_4) = 1$ ,  $\beta_2(q'_1 q'_2) = 0$ ,  $\beta_2(q'_2 q'_3) = t$ ,  $\beta_2(q'_3 q'_4) = s$ ,  $\beta_2(q'_4 q'_1) = p$ . Additionally, let  $h$  be a function from  $G_1$  into  $G_2$ , where  $h(q_i) = q'_i$ . Then,  $G_1$  and  $G_2$  are not isomorphic SH-graphs.

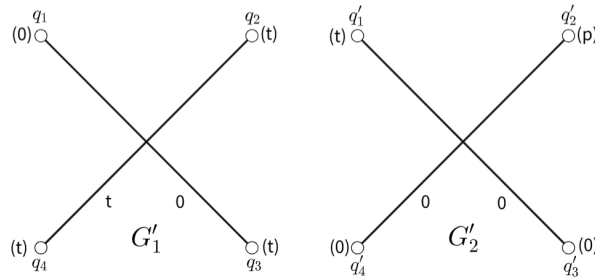
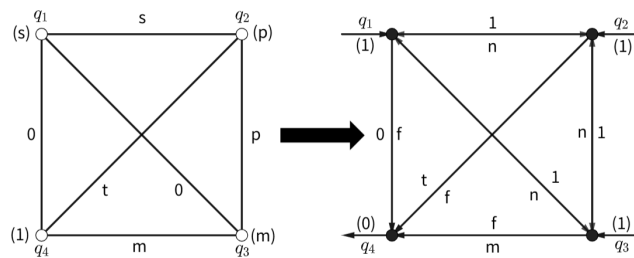


FIGURE 16.  $G'_1$  and  $G'_2$  are two complement of SH-graphs  $G_1$  and  $G_2$ .

It is observed that  $G'_1 = (\alpha'_1, \beta'_1)$  on  $(G_1^*)' = (V_1, E'_1)$  and  $G'_2 = (\alpha'_2, \beta'_2)$  on  $(G_2^*)' = (V_2, E'_2)$ , respectively, as in Fig. 16, where  $V_1 = \{q_1, q_2, q_3, q_4\}$ ,  $E'_1 = \{q_1 q_3, q_2 q_4\}$ ,  $\alpha'_1(q_1) = s \rightarrow 0 = 0$ ,  $\alpha'_1(q_2) = p \rightarrow 0 = t$ ,  $\alpha'_1(q_3) = m \rightarrow 0 = t$ ,  $\alpha'_1(q_4) = m \rightarrow 0 = t$ ,  $\beta'_1(q_1 q_3) = \alpha'_1(q_1) \odot \alpha'_1(q_3) = 0$ ,  $\beta'_1(q_2 q_4) = \alpha'_1(q_2) \odot \alpha'_1(q_4) = t$ ,  $V_2 = \{q'_1, q'_2, q'_3, q'_4\}$ ,  $E'_2 = \{q'_1 q'_3, q'_2 q'_4\}$ ,  $\alpha'_2(q'_1) = p \rightarrow 0 = t$ ,  $\alpha'_2(q'_2) = t \rightarrow 0 = p$ ,  $\alpha'_2(q'_3) = s \rightarrow 0 = 0$ ,  $\alpha'_2(q'_4) = 1 \rightarrow 0 = 0$ ,  $\beta'_2(q'_1 q'_3) = \alpha'_2(q'_1) \odot \alpha'_2(q'_3) = 0$ ,  $\beta'_2(q'_2 q'_4) = \alpha'_2(q'_2) \odot \alpha'_2(q'_4) = 0$ . Then,  $G'_1$  and  $G'_2$  are not isomorphic SH-graphs.

*Remark 3.16.* The above four examples express that SH-graphs  $G'_1$  and  $G'_2$  are isomorphic can not be characterized by SH-graphs  $G_1$  and  $G_2$  being isomorphic.

**Definition 3.17.** Let  $A(Z(G_1)) = (Q_1, X, \mu_1, F_1, \sigma_1)$  and  $A(Z(G_2)) = (Q_2, X, \mu_2, F_2, \sigma_2)$  be two SH-graph automata related to  $G_1$  and  $G_2$ , respectively, and  $c \in SH \setminus \{1\}$ . If a function  $g: Q_1 \rightarrow Q_2$  is bijection and satisfy the followings:  
 (I1)  $\mu_1(q_i, u, q_j) > c$  if and only if  $\mu_2(g(q_i), u, g(q_j)) > c$ ;  
 (I2)  $\sigma_1(q_i) > c$  if and only if  $\sigma_2(g(q_i)) > c$ ;  
 (I3)  $q \in F_1$  if and only if  $g(q) \in F_2$ ;  
 for every  $q_i, q_j \in Q_1$  and  $u \in X$ , then  $g$  is called an isomorphism from  $A(Z(G_1))$  onto  $A(Z(G_2))$  with threshold  $c$ , denoted by  $A(Z(G_1)) \simeq A(Z(G_2))$ .



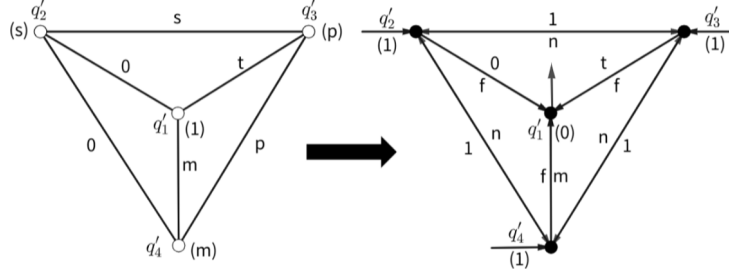
**FIGURE 17.** SH-graph  $G_1$  and related SH-graph automaton  $A(Z(G_1))$ .

**Example 3.18.** Let  $G_1$  and  $G_2$  be two SH-graphs, as in Example 3.11. Consider  $Z(G_1) = \{q_1, q_2, q_3\}$  and  $Z(G_2) = \{q'_2, q'_3, q'_4\}$ , we have SH-graph automata  $A(Z(G_1)) = (Q_1, X, \mu_1, F_1, \sigma_1)$ , as in Fig. 17, and  $A(Z(G_2)) = (Q_2, X, \mu_2, F_2, \sigma_2)$ , as in Fig. 18, where  $Q_1 = \{q_1, q_2, q_3, q_4\}$ ,  $\mu_1(q_1, n, q_2) = \mu_1(q_2, n, q_1) = 1$ ,  $\mu_1(q_1, n, q_3) = \mu_1(q_3, n, q_1) = 1$ ,  $\mu_1(q_2, n, q_3) = \mu_1(q_3, n, q_2) = 1$ ,  $\mu_1(q_1, f, q_4) = 0$ ,  $\mu_1(q_2, f, q_4) = t$ ,  $\mu_1(q_3, f, q_4) = m$ ,  $F_1 = \{q_4\}$ ,  $\sigma_1(q_1) = \sigma_1(q_2) = \sigma_1(q_3) = 1$ ,  $\sigma_1(q_4) = 0$ ,  $Q_2 = \{q'_1, q'_2, q'_3, q'_4\}$ ,  $\mu_2(q'_2, n, q'_3) = \mu_2(q'_3, n, q'_2) = 1$ ,  $\mu_2(q'_3, n, q'_4) = \mu_2(q'_4, n, q'_3) = 1$ ,  $\mu_2(q'_2, n, q'_4) = \mu_2(q'_4, n, q'_2) = 1$ ,  $\mu_2(q'_2, f, q'_1) = 0$ ,  $\mu_2(q'_3, f, q'_1) = t$ ,  $\mu_2(q'_4, f, q'_1) = m$ ,  $F_2 = \{q'_1\}$ ,  $\sigma_2(q'_2) = \sigma_2(q'_3) = \sigma_2(q'_4) = 1$ ,  $\sigma_2(q'_1) = 0$ . Hence,

$$A(Z(G_1)) \simeq A(Z(G_2)).$$

**Theorem 3.19.** Let  $G_1 = (\alpha_1, \beta_1)$  and  $G_2 = (\alpha_2, \beta_2)$  be two SH-graphs on  $G_1^* = (V_1, E_1)$  and  $G_2^* = (V_2, E_2)$ , respectively. Let  $G_1 \cong G_2$ . Then  $A(Z(G_1)) \simeq A(Z(G_2))$ .

*Proof.* Let  $h : V_1 \rightarrow V_2$  be an isomorphism with threshold  $c$  and  $Z(G_1)$  be a zero forcing set. Then,



**FIGURE 18.**  $SH$ -graph  $G_2$  and related  $SH$ -graph automaton  $A(Z(G_2))$ .

$$Z(A(Z(G_2))) = \{j \in Q_2 \mid \exists i \in Z(A(Z(G_1))); h(i) = j\} = \{j \in Q_2 \mid \exists i \in Z(G_1); h(i) = j\}.$$

If  $F(A(Z(G_1)))$  is a set of final states of  $A(Z(G_1))$ , then  $F(A(Z(G_2))) = \{j \in Q_2 \mid \exists i \in F(A(Z(G_1))); h(i) = j\}$  is the final state set of  $A(Z(G_2))$ .

Case 1.  $u = f \in X$ , for every  $q_i, q_j \in Q_1$ . Then,

$$\mu_1(q_i, f, q_j) > c \Leftrightarrow \beta_1(q_i q_j) > c \Leftrightarrow \beta_2(h(q_i)h(q_j)) > c \Leftrightarrow \mu_2(h(q_i), f, h(q_j)) > c.$$

Case 2.  $u = n \in X$ . Then,  $\mu_1(q_i, n, q_j) = \mu_2(h(q_i), n, h(q_j)) = 1$ , for every  $q_i, q_j \in Q_1$ . For every  $q_i \in Q_1$

$$\sigma_1(q_i) = 1 > c \Leftrightarrow q_i \in Z(A(Z(G_1))) \Leftrightarrow h(q_i) \in Z(A(Z(G_2))) \Leftrightarrow h(q_i) \in Z(G_2) \Leftrightarrow \sigma(h(q_i)) = 1 > c.$$

For every  $q \in F(A(Z(G_1))) \Leftrightarrow h(q) \in F(A(Z(G_2)))$ , as mentioned above, we have three conditions of isomorphism with threshold  $c$ . Then,  $A(Z(G_1))$  and  $A(Z(G_2))$  are isomorphic  $SH$ -graph automata with threshold  $c$ .  $\square$

**Example 3.20.** Let  $G_1$  and  $G_2$  be two isomorphic  $SH$ -graphs in Example 3.11. Then, we can know that  $A(Z(G_1))$  and  $A(Z(G_2))$  are isomorphic  $SH$ -graph automata.

**Example 3.21.** Consider the semihoop in Example 3.11, let  $G_1 = (\alpha_1, \beta_1)$  and  $G_2 = (\alpha_2, \beta_2)$  be two  $SH$ -graphs on  $G_1^* = (V_1, E_1)$  and  $G_2^* = (V_2, E_2)$ , respectively, as in Fig. 19, where  $V_1 = \{q_1, q_2, q_3, q_4\}$ ,  $E_1 = \{q_1 q_2, q_2 q_3, q_3 q_4, q_4 q_1\}$ ,  $\alpha_1(q_1) = p$ ,  $\alpha_1(q_2) = s$ ,  $\alpha_1(q_3) = m$ ,  $\alpha_1(q_4) = 1$ ,  $\beta_1(q_1 q_2) = s$ ,  $\beta_1(q_2 q_3) = s$ ,  $\beta_1(q_3 q_4) = p$ ,  $\beta_1(q_4 q_1) = t$ ,  $V_2 = \{q'_1, q'_2, q'_3, q'_4\}$ ,  $E_2 = \{q'_1 q'_2, q'_2 q'_3, q'_3 q'_4, q'_4 q'_1\}$ ,  $\alpha_2(q'_1) = p$ ,  $\alpha_2(q'_2) = m$ ,  $\alpha_2(q'_3) = 1$ ,  $\alpha_2(q'_4) = p$ ,  $\beta_2(q'_1 q'_2) = p$ ,  $\beta_2(q'_2 q'_3) = t$ ,  $\beta_2(q'_3 q'_4) = 0$ ,  $\beta_2(q'_4 q'_1) = s$ . Additionally, let  $h$  be a bijection from  $G_1$  into  $G_2$ , where  $h(q_1) = q'_2$ ,  $h(q_2) = q'_1$ ,  $h(q_3) = q'_4$ ,  $h(q_4) = q'_3$ . Therefore,  $G_1$

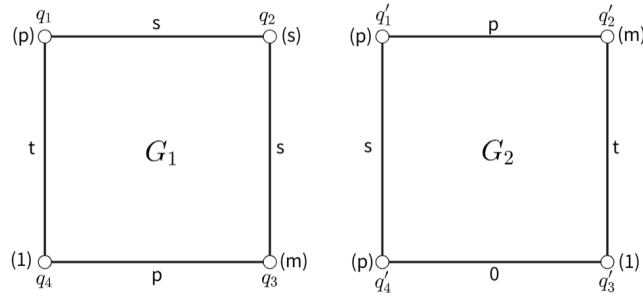


FIGURE 19.  $G_1$  and  $G_2$  are two SH-graphs on  $G_1^*$  and  $G_2^*$ .

and  $G_2$  are not isomorphic SH-graphs. But, we can know that  $A(Z(G_1)) = (Q_1, X, \mu_1, F_1, \sigma_1)$  and  $A(Z(G_2)) = (Q_2, X, \mu_2, F_2, \sigma_2)$  are isomorphic SH-graph automata, respectively, as in Fig. 20, where  $Q_1 = \{q_1, q_2, q_3, q_4\}$ ,  $Z(A(Z(G_1))) = \{q_1, q_2\}$ ,  $F_1 = \{q_3, q_4\}$ ,  $\mu_1(q_1, n, q_2) = \mu_1(q_2, n, q_1) = 1$ ,  $\mu_1(q_3, n, q_4) = \mu_1(q_4, n, q_3) = 1$ ,  $\mu_1(q_1, f, q_4) = t$ ,  $\mu_1(q_2, f, q_3) = s$ ,  $\sigma_1(q_1) = \sigma_1(q_2) = 1$  and  $\sigma_1(q_3) = \sigma_1(q_4) = 0$ ,  $Q_2 = \{q'_1, q'_2, q'_3, q'_4\}$ ,  $Z(A(Z(G_2))) = \{q'_1, q'_2\}$ ,  $F_2 = \{q'_3, q'_4\}$ ,  $\mu_2(q'_1, n, q'_2) = \mu_2(q'_2, n, q'_1) = 1$ ,  $\mu_2(q'_3, n, q'_4) = \mu_2(q'_4, n, q'_3) = 1$ ,  $\mu_2(q'_1, f, q'_4) = s$ ,  $\mu_2(q'_2, f, q'_3) = t$ ,  $\sigma_2(q'_1) = \sigma_2(q'_2) = 1$  and  $\sigma_2(q'_3) = \sigma_2(q'_4) = 0$ .

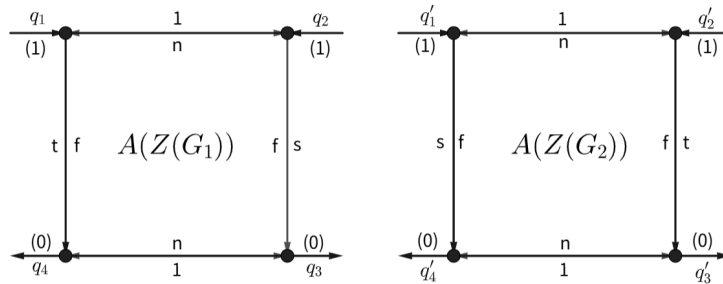
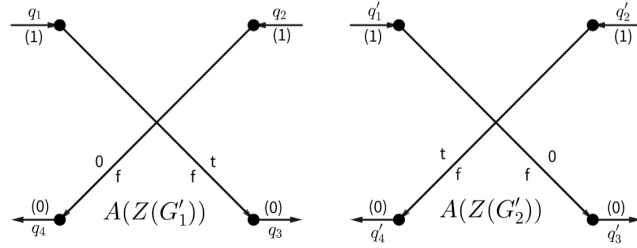


FIGURE 20.  $A(Z(G_1))$  and  $A(Z(G_2))$  are two SH-graph automata related to  $G_1$  and  $G_2$ .

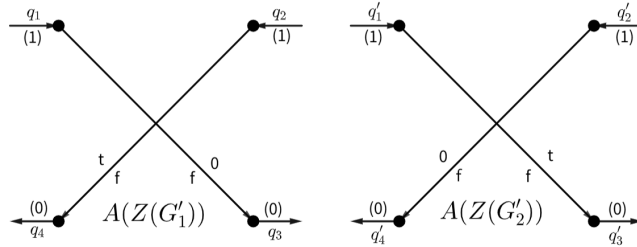
*Remark 3.22.* The above example expresses that if two SH-graphs  $G_1$  and  $G_2$  are not isomorphic, but the related SH-graph automata may be isomorphic.

**Example 3.23.** Consider Example 3.12, we know that  $G_1 \cong G_2$ . Let  $A(Z(G'_1)) = (Q_1, X, \mu_1, F_1, \sigma_1)$  and  $A(Z(G'_2)) = (Q_2, X, \mu_2, F_2, \sigma_2)$  are two SH automata related  $G'_1$  and  $G'_2$ , respectively, as in Fig. 21, where  $Q_1 = \{q_1, q_2, q_3, q_4\}$ ,  $Z(G'_1) = \{q_1, q_2\}$ ,  $F_1 = \{q_3, q_4\}$ ,  $\mu_1(q_1, f, q_3) = t$ ,  $\mu_1(q_2, f, q_4) = 0$ ,  $\sigma_1(q_1) = \sigma_1(q_2) = 1$ ,  $\sigma_1(q_3) = \sigma_1(q_4) = 0$ ,  $Q_2 = \{q'_1, q'_2, q'_3, q'_4\}$ ,  $Z(G'_2) = \{q'_1, q'_2\}$ ,



**FIGURE 21.**  $A(Z(G'_1))$  and  $A(Z(G'_2))$  are two  $SH$ -graph automata related to  $G'_1$  and  $G'_2$ .

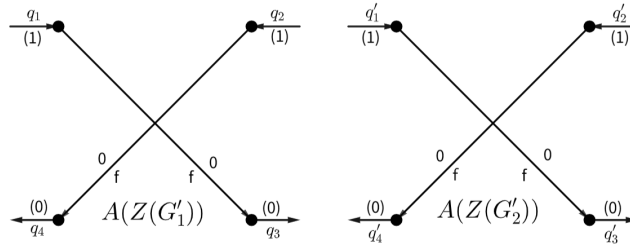
$F_2 = \{q'_3, q'_4\}$ ,  $\mu_2(q'_1, f, q'_3) = 0$ ,  $\mu_2(q'_2, f, q'_4) = t$ ,  $\sigma_2(q'_1) = \sigma_2(q'_2) = 1$ ,  $\sigma_2(q'_3) = \sigma_2(q'_4) = 0$ . Then,  $A(Z(G'_1))$  and  $A(Z(G'_2))$  are isomorphic  $SH$ -graph automata.



**FIGURE 22.**  $A(Z(G'_1))$  and  $A(Z(G'_2))$  are two  $SH$ -graph automata related to  $G'_1$  and  $G'_2$ .

**Example 3.24.** Consider Example 3.13, we know that  $G_1 \cong G_2$ . Let  $A(Z(G'_1)) = (Q_1, X, \mu_1, F_1, \sigma_1)$  and  $A(Z(G'_2)) = (Q_2, X, \mu_2, F_2, \sigma_2)$  are two  $SH$  automata related  $G'_1$  and  $G'_2$ , respectively, as in Fig. 22, where  $Q_1 = \{q_1, q_2, q_3, q_4\}$ ,  $Z(G'_1) = \{q_1, q_2\}$ ,  $F_1 = \{q_3, q_4\}$ ,  $\mu_1(q_1, f, q_3) = 0$ ,  $\mu_1(q_2, f, q_4) = t$ ,  $\sigma_1(q_1) = \sigma_1(q_2) = 1$ ,  $\sigma_1(q_3) = \sigma_1(q_4) = 0$ ,  $Q_2 = \{q'_1, q'_2, q'_3, q'_4\}$ ,  $Z(G'_2) = \{q'_1, q'_2\}$ ,  $F_2 = \{q'_3, q'_4\}$ ,  $\mu_2(q'_1, f, q'_3) = t$ ,  $\mu_2(q'_2, f, q'_4) = 0$ ,  $\sigma_2(q'_1) = \sigma_2(q'_2) = 1$ ,  $\sigma_2(q'_3) = \sigma_2(q'_4) = 0$ . Then,  $A(Z(G'_1))$  and  $A(Z(G'_2))$  are not isomorphic  $SH$ -graph automata.

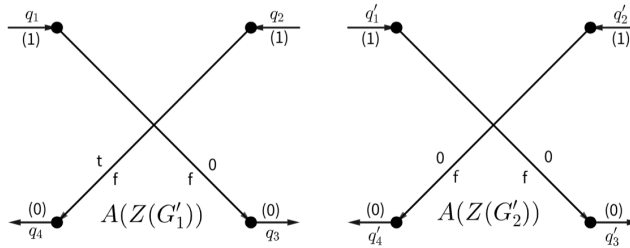
**Example 3.25.** Consider Example 3.14, we know that  $G_1 \not\cong G_2$ . Let  $A(Z(G'_1)) = (Q_1, X, \mu_1, F_1, \sigma_1)$  and  $A(Z(G'_2)) = (Q_2, X, \mu_2, F_2, \sigma_2)$  are two  $SH$  automata related  $G'_1$  and  $G'_2$ , respectively, as in Fig. 23, where  $Q_1 = \{q_1, q_2, q_3, q_4\}$ ,  $Z(G'_1) = \{q_1, q_2\}$ ,  $F_1 = \{q_3, q_4\}$ ,  $\mu_1(q_1, f, q_3) = 0$ ,  $\mu_1(q_2, f, q_4) = 0$ ,  $\sigma_1(q_1) = \sigma_1(q_2) = 1$ ,  $\sigma_1(q_3) = \sigma_1(q_4) = 0$ ,  $Q_2 = \{q'_1, q'_2, q'_3, q'_4\}$ ,  $Z(G'_2) = \{q'_1, q'_2\}$ ,  $F_2 = \{q'_3, q'_4\}$ ,  $\mu_2(q'_1, f, q'_3) = 0$ ,  $\mu_2(q'_2, f, q'_4) = 0$ ,  $\sigma_2(q'_1) = \sigma_2(q'_2) = 1$ ,



**FIGURE 23.**  $A(Z(G'_1))$  and  $A(Z(G'_2))$  are two SH-graph automata related to  $G'_1$  and  $G'_2$ .

$\sigma_2(q'_3) = \sigma_2(q'_4) = 0$ . Then,  $A(Z(G'_1))$  and  $A(Z(G'_2))$  are isomorphic SH-graph automata.

**Example 3.26.** Consider Example 3.15, we know that  $G_1 \not\cong G_2$ . Let  $A(Z(G'_1)) = (Q_1, X, \mu_1, F_1, \sigma_1)$  and  $A(Z(G'_2)) = (Q_2, X, \mu_2, F_2, \sigma_2)$  are two SH automata related  $G'_1$  and  $G'_2$ , respectively, as in Fig. 24, where  $Q_1 = \{q_1, q_2, q_3, q_4\}$ ,  $Z(G'_1) = \{q_1, q_2\}$ ,  $F_1 = \{q_3, q_4\}$ ,  $\mu_1(q_1, f, q_3) = 0$ ,  $\mu_1(q_2, f, q_4) = t$ ,  $\sigma_1(q_1) = \sigma_1(q_2) = 1$ ,  $\sigma_1(q_3) = \sigma_1(q_4) = 0$ ,  $Q_2 = \{q'_1, q'_2, q'_3, q'_4\}$ ,  $Z(G'_2) = \{q'_1, q'_2\}$ ,  $F_2 = \{q'_3, q'_4\}$ ,  $\mu_2(q'_1, f, q'_3) = 0$ ,  $\mu_2(q'_2, f, q'_4) = 0$ ,  $\sigma_2(q'_1) = \sigma_2(q'_2) = 1$ ,  $\sigma_2(q'_3) = \sigma_2(q'_4) = 0$ . Then,  $A(Z(G'_1))$  and  $A(Z(G'_2))$  are not isomorphic SH-graph automata.



**FIGURE 24.**  $A(Z(G'_1))$  and  $A(Z(G'_2))$  are two SH-graph automata related to  $G'_1$  and  $G'_2$ .

*Remark 3.27.* The above four examples express that SH-graph automata  $A(Z(G'_1))$  and  $A(Z(G'_2))$  are isomorphic can not be characterized by SH-graphs  $G_1$  and  $G_2$  being isomorphic.

In this study, each zero forcing set on SH-graph is built on various SH-graph automata related to it. However, they may not be equivalent. Therefore, we prove several theorems about equivalence and provide corresponding examples to clarify them.

*Remark 3.28.* [7] Let  $S$  be a semihoop and for any  $\alpha, \beta \in S$ , we define  $\alpha \vee \beta = ((\alpha \rightarrow \beta) \rightarrow \beta) \wedge ((\beta \rightarrow \alpha) \rightarrow \alpha)$ .

In the following content, we will use the above operation to study the equivalence of  $SH$ -graph automata by Definition 2.8, Definition 2.9, Definition 2.10, Definition 2.11 and Remark 3.28.

**Theorem 3.29.** *Let  $G$  a path  $SH$ -graph. Then,*

(i)  $A(Z_1(G))$  and  $A(Z_2(G))$  are equivalent  $SH$ -graph automata related to  $G$ , where  $Z_i(G)$ ;  $i = 1, 2$  are different zero forcing sets.

(ii)  $\bigvee_{q \in F} r_\mu(x, q) = \mu^*(q_1, x, q)$ ;  $q_1 \in Z(G)$ , where  $Z(G)$  is a zero forcing set.

*Proof.* (i) We know that a path  $SH$ -graph is symmetric. Let  $A(Z_1(G)) = (Q, X, \mu_1, F_1, \sigma_1)$ , and  $A(Z_2(G)) = (Q, X, \mu_2, F_2, \sigma_2)$ . Firstly, suppose that  $Z_1(G) = \{p_1\} = F_2$  and  $Z_2(G) = \{p_n\} = F_1$ . Then,

$$\mu_1(p_i, f, p_j) = \mu_2(p_j, f, p_i) = \beta(p_i p_j), \forall p_i \text{ and } p_j \text{ are adjacent, } 1 \leq i < j \leq n.$$

We know that  $F_1$  and  $F_2$  are singletons. Thus,

$$\begin{aligned} r_{\mu_1}(f^{n-1}, p_n) &= \mu_1^*(p_1, f^{n-1}, p_n) = \mu_1(p_1, f, p_2) \odot \cdots \odot \mu_1(p_{n-1}, f, p_n) = \\ &= \mu_2(p_n, f, p_{n-1}) \odot \cdots \odot \mu_2(p_2, f, p_1) = \mu_2^*(p_n, f^{n-1}, p_1) = r_{\mu_2}(f^{n-1}, p_1). \end{aligned}$$

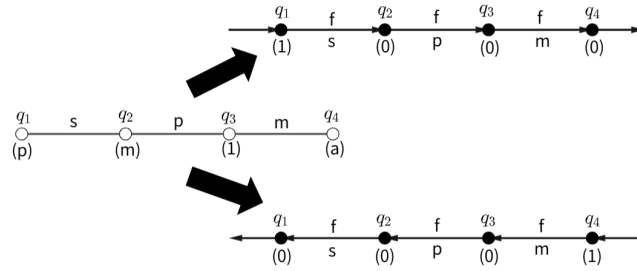
Therefore,  $A(Z_1(G))$  and  $A(Z_2(G))$  are equivalent.

(ii) The proof is clear by (i).  $\square$

**Example 3.30.** *Consider the semihoop in Example 3.11, let  $G = (\alpha, \beta)$  be the path  $SH$ -graph on  $G^* = (V, E)$ , as in Fig. 25, where  $V = \{q_1, q_2, q_3, q_4\}$ ,  $E = \{q_1 q_2, q_2 q_3, q_3 q_4\}$ ,  $\alpha(q_1) = p$ ,  $\alpha(q_2) = m$ ,  $\alpha(q_3) = 1$ ,  $\alpha(q_4) = a$ ,  $\beta(q_1 q_2) = s$ ,  $\beta(q_2 q_3) = p$ ,  $\beta(q_3 q_4) = m$ . Then,  $A(Z_1(G)) = (Q, X, \mu_1, F_1, \sigma_1)$  is the  $SH$ -graph automaton related to  $G$ , as in Fig. 25, where  $Z_1(G) = \{q_1\}$ ,  $F_1 = \{q_4\}$ ,  $\mu_1(q_1, f, q_2) = s$ ,  $\mu_1(q_2, f, q_3) = p$ ,  $\mu_1(q_3, f, q_4) = m$ ,  $\sigma_1(q_1) = 1$ ,  $\sigma_1(q_2) = \sigma_1(q_3) = \sigma_1(q_4) = 0$  and  $A(Z_2(G)) = (Q, X, \mu_2, F_2, \sigma_2)$  is the  $SH$ -graph automaton related to  $G$ , as in Fig. 25, where  $Z_2(G) = \{q_4\}$ ,  $F_2 = \{q_1\}$ ,  $\mu_2(q_4, f, q_3) = m$ ,  $\mu_2(q_3, f, q_2) = p$ ,  $\mu_2(q_2, f, q_1) = s$ ,  $\sigma_2(q_4) = 1$ ,  $\sigma_2(q_1) = \sigma_2(q_2) = \sigma_2(q_3) = 0$ . Moreover,  $r_{\mu_1}(f^3, q_4) = s \odot p \odot m = s$  and  $r_{\mu_2}(f^3, q_1) = m \odot p \odot s = s$ . Therefore,  $A(Z_1(G))$  and  $A(Z_2(G))$  are equivalent.*

**Definition 3.31.** If two zero forcing sets are symmetric about a symmetric  $SH$ -graph, then the two zero forcing sets are called symmetric zero forcing sets.

**Theorem 3.32.** *Consider a cycle  $SH$ -graph  $G$  on  $2k$  vertices and let  $Z_1(G)$  and  $Z_2(G)$  be symmetric zero forcing sets. Then the  $SH$ -graph automata  $A(Z_1(G))$  and  $A(Z_2(G))$  are equivalent.*



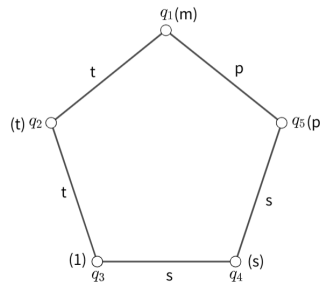
**FIGURE 25.** The path SH-graph  $G$ , and two related SH-graph automata  $A(Z_1(G))$  and  $A(Z_2(G))$ .

*Proof.* We know that the cycle SH-graph with even edges is symmetric. Let  $A(Z_1(G)) = (Q, X, \mu_1, F_1, \sigma_1)$  and  $A(Z_2(G)) = (Q, X, \mu_2, F_2, \sigma_2)$ . Without loss of generality, suppose  $Z_1(G) = F_2 = \{q_1, q_2\}$  and  $Z_2(G) = F_1 = \{q_{k+1}, q_{k+2}\}$ . Then  $\mu_1(q_1, n, q_2) = \mu_1(q_2, n, q_1) = 1$ ,  $\mu_1(q_{k+1}, n, q_{k+2}) = \mu_1(q_{k+2}, n, q_{k+1}) = 1$ , and  $\mu_2(q_1, n, q_2) = \mu_2(q_2, n, q_1) = 1$ ,  $\mu_2(q_{k+1}, n, q_{k+2}) = \mu_2(q_{k+2}, n, q_{k+1}) = 1$ ,  $\mu_1(q_i, f, q_j) = \beta(q_i q_j) = \beta(q_j q_i) = \mu_2(q_j, f, q_i)$ , for every  $q_i q_j \in E$  such that  $q_i q_j \neq q_1 q_2, q_{k+1} q_{k+2}$ .

$$x \in B(A(Z_1(G)), c) \iff \bigvee_{q \in F_1} r_{\mu_1}(x, q) > c = \bigvee_{q \in F_1} \bigvee_{q' \in Z_1(G)} \mu_1^*(q', x, q) > c$$

$$\iff \bigvee_{q \in Z_2(G)} \bigvee_{q' \in F_2} \mu_2^*(q, x, q') > c = \bigvee_{q' \in F_2} r_{\mu_2}(x, q') > c \iff x \in B(A(Z_2(G)), c),$$

where  $c \in SH \setminus \{1\}$ . Hence, the SH-graph automata  $A(Z_1(G))$  and  $A(Z_2(G))$  are equivalent.  $\square$



**FIGURE 26.** The SH-graph  $G$ .

**Example 3.33.** Consider the semihoop in Example 3.11,  $\beta(q_i q_j) = \alpha(q_i) \odot \alpha(q_j)$ , for every  $q_i q_j \in E$ . Let  $G = (\alpha, \beta)$  be a SH-graph on  $G^* = (V, E)$ , as in Fig. 26, where  $V = \{q_1, q_2, q_3, q_4, q_5\}$ ,  $E = \{q_1 q_2, q_2 q_3, q_3 q_4, q_4 q_5, q_5 q_1\}$ ,  $\alpha(q_1) = m$ ,  $\alpha(q_2) = t$ ,  $\alpha(q_3) = 1$ ,  $\alpha(q_4) = s$ ,  $\alpha(q_5) = p$ ,  $\beta(q_1 q_2) = t$ ,  $\beta(q_2 q_3) =$

$t, \beta(q_3q_4) = s, \beta(q_4q_5) = s, \beta(q_5q_1) = p$ . Then,  $A(Z_1(G)) = (Q, X, \mu_1, F_1, \sigma_1)$  is the  $SH$ -graph automaton related to  $G$ , as in Fig. 27, where  $Q = V$ ,  $Z_1(G) = \{q_2, q_3\}$ ,  $F_1 = \{q_5\}$ ,  $\mu_1(q_2, n, q_3) = \mu_1(q_3, f, q_2) = 1, \mu_1(q_2, f, q_1) = t, \mu_1(q_1, f, q_5) = p, \mu_1(q_3, f, q_4) = s, \mu_1(q_4, f, q_5) = s, \sigma_1(q_2) = \sigma_1(q_3) = 1, \sigma_1(q_1) = \sigma_1(q_4) = \sigma_1(q_5) = 0$ .

$$\bigvee_{q \in F_1} r_{\mu_1}(n^* f^2, q) = r_{\mu_1}(n^* f^2, q_5) = \bigvee_{q' \in Z_1(G)} \mu_1^*(q', n^* f^2, q_5) = [\mu_1(q_2, f, q_1) \odot \mu_1(q_1, f, q_5)] \vee [\mu_1(q_3, f, q_4) \odot \mu_1(q_4, f, q_5)] = (t \odot p) \vee (s \odot s) = t \vee s = [(t \rightarrow s) \rightarrow s] \wedge [(s \rightarrow t) \rightarrow t] = (s \rightarrow s) \wedge (t \rightarrow t) = 1 \wedge 1 = 1.$$

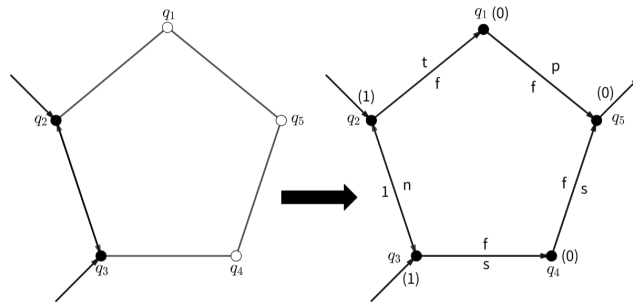


FIGURE 27. The related  $SH$ -graph automaton  $A(Z_1(G))$ .

Now, by considering  $Z_2(G) = \{q_4, q_5\}$ . Then,  $A(Z_2(G)) = (Q, X, \mu_2, F_2, \sigma_2)$  is the  $SH$ -graph automaton related to  $G$ , as in Fig. 28, where  $Q = V$ ,  $F_2 = \{q_2\}$ ,  $\mu_2(q_4, n, q_5) = \mu_2(q_5, n, q_4) = 1, \mu_2(q_4, f, q_3) = s, \mu_2(q_3, f, q_2) = t, \mu_2(q_5, f, q_1) = p, \mu_2(q_1, f, q_2) = t, \sigma_2(q_4) = \sigma_2(q_5) = 1, \sigma_2(q_1) = \sigma_2(q_2) = \sigma_2(q_3) = 0$ .

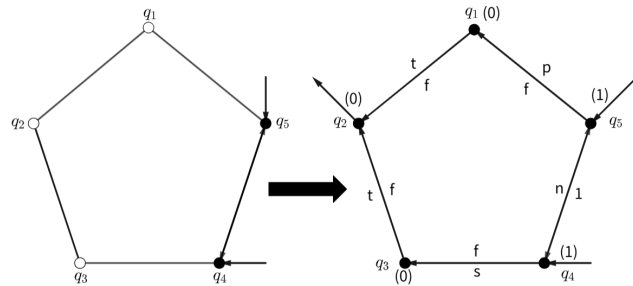


FIGURE 28. The related  $SH$ -graph automaton  $A(Z_2(G))$ .

$$\bigvee_{q \in F_2} r_{\mu_2}(n^* f^2, q) = r_{\mu_2}(n^* f^2, q_2) = \bigvee_{q' \in Z_2(G)} \mu_2^*(q', n^* f^2, q_2) = [\mu_2(q_4, f, q_3) \odot \mu_2(q_3, f, q_2)] \vee [\mu_2(q_5, f, q_1) \odot \mu_2(q_1, f, q_2)] = (s \odot t) \vee (p \odot t) = 0 \vee t = [(0 \rightarrow t) \rightarrow t] \wedge [(t \rightarrow 0) \rightarrow 0] = (1 \rightarrow t) \wedge (s \rightarrow t) = t \wedge 0 = 0.$$

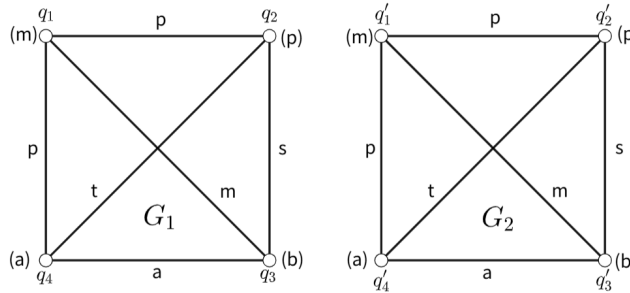
Therefore,

$$\bigvee_{q \in F_1} r_{\mu_1}(n^* f^2, q) \neq \bigvee_{q \in F_2} r_{\mu_2}(n^* f^2, q).$$

Consequently,  $A(Z_1(G))$  and  $A(Z_2(G))$  are not equivalent.

*Remark 3.34.* Let  $G$  be a cycle SH-graph with  $2k + 1$  vertices, and let  $Z_1(G)$  and  $Z_2(G)$  be symmetric zero forcing sets. Then the above example shows that  $A(Z_1(G))$  and  $A(Z_2(G))$  may not be equivalent.

**Example 3.35.** Consider the semihoop in Example 3.11, let  $G_1 = (\alpha_1, \beta_1)$  and  $G_2 = (\alpha_2, \beta_2)$  be two SH-graphs on  $G_1^* = (V_1, E_1)$  and  $G_2^* = (V_2, E_2)$ , respectively, as in Fig. 29, where  $V_1 = \{q_1, q_2, q_3, q_4\}$ ,  $E_1 = \{q_1q_2, q_2q_3, q_3q_4, q_4q_1, q_1q_3, q_2q_4\}$ ,  $\alpha_1(q_1) = m$ ,  $\alpha_1(q_2) = p$ ,  $\alpha_1(q_3) = b$ ,  $\alpha_1(q_4) = a$ ,  $\beta_1(q_1q_2) = p$ ,  $\beta_1(q_2q_3) = s$ ,  $\beta_1(q_3q_4) = a$ ,  $\beta_1(q_4q_1) = p$ ,  $\beta_1(q_1q_3) = m$ ,  $\beta_1(q_2q_4) = t$ ,  $V_2 = \{q'_1, q'_2, q'_3, q'_4\}$ ,  $E_2 = \{q'_1q'_2, q'_2q'_3, q'_3q'_4, q'_4q'_1, q'_1q'_3, q'_2q'_4\}$ ,  $\alpha_2(q'_1) = m$ ,  $\alpha_2(q'_2) = p$ ,  $\alpha_2(q'_3) = b$ ,  $\alpha_2(q'_4) = a$ ,  $\beta_2(q'_1q'_2) = p$ ,  $\beta_2(q'_2q'_3) = s$ ,  $\beta_2(q'_3q'_4) = a$ ,  $\beta_2(q'_4q'_1) = p$ ,  $\beta_2(q'_1q'_3) = m$ ,  $\beta_2(q'_2q'_4) = t$ . Additionally, let  $h$  be a bijection from  $G_1$  into  $G_2$ , where  $h(q_i) = q'_i$ . Then,  $G_1$  and  $G_2$  are isomorphic SH-graphs.



**FIGURE 29.** The SH-graphs  $G_1$  and  $G_2$ .

Let  $A(Z(G_1)) = (Q_1, X, \mu_1, F_1, \sigma_1)$  and  $A(Z(G_2)) = (Q_2, X, \mu_2, F_2, \sigma_2)$  be two SH-graph automata related to  $G_1$  and  $G_2$ , respectively, as in Fig. 30, where  $Q_1 = \{q_1, q_2, q_3, q_4\}$ ,  $Z(G_1) = \{q_2, q_3, q_4\}$ ,  $F_1 = \{q_1\}$ ,  $\mu_1(q_2, n, q_3) = \mu_1(q_3, n, q_2) = 1$ ,  $\mu_1(q_3, n, q_4) = \mu_1(q_4, n, q_3) = 1$ ,  $\mu_1(q_2, n, q_4) = \mu_1(q_4, n, q_2) = 1$ ,  $\mu_1(q_2, f, q_1) = p$ ,  $\mu_1(q_3, f, q_1) = m$ ,  $\mu_1(q_4, f, q_1) = p$ ,  $\sigma_1(q_2) = \sigma_1(q_3) = \sigma_1(q_4) = 1$  and  $\sigma_1(q_1) = 0$ ,  $Q_2 = \{q'_1, q'_2, q'_3, q'_4\}$ ,  $Z(G_2) = \{q'_2, q'_3, q'_4\}$ ,  $F_2 = \{q'_1\}$ ,  $\mu_2(q'_2, n, q'_3) = \mu_2(q'_3, n, q'_2) = 1$ ,  $\mu_2(q'_3, n, q'_4) = \mu_2(q'_4, n, q'_3) = 1$ ,  $\mu_2(q'_2, n,$

$q'_4) = \mu_2(q'_4, n, q'_2) = 1, \mu_2(q'_2, f, q'_1) = p, \mu_2(q'_3, f, q'_1) = m, \mu_2(q'_4, f, q'_1) = p,$   
 $\sigma_2(q'_2) = \sigma_2(q'_3) = \sigma_2(q'_4) = 1$  and  $\sigma_2(q'_1) = 0$ . Then,

$$\bigvee_{q \in F_1} r_{\mu_1}(n^*f, q) = r_{\mu_1}(n^*f, q_1) = \mu_1^*(q_2, n^*f, q_1) \vee \mu_1^*(q_3, n^*f, q_1) \vee \mu_1^*(q_4, n^*f, q_1)$$

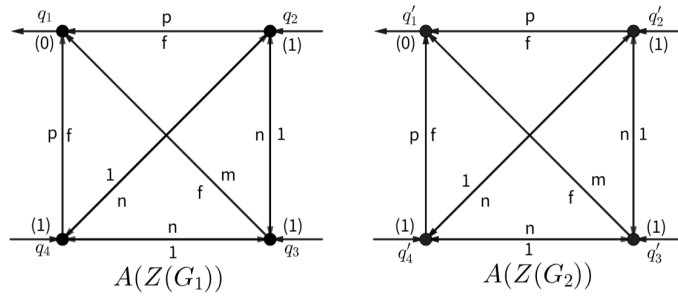
$$= p \vee m \vee p = [(p \rightarrow m) \rightarrow m] \wedge [(m \rightarrow p) \rightarrow p] \vee p = (m \wedge 1) \vee p = m \vee p =$$

$$[(m \rightarrow p) \rightarrow p] \wedge [(p \rightarrow m) \rightarrow m] = 1 \wedge m = m, \text{ and}$$

$$\bigvee_{q \in F_2} r_{\mu_2}(n^*f, q) = r_{\mu_2}(n^*f, q'_1) = \mu_2^*(q'_2, n^*f, q'_1) \vee \mu_2^*(q'_3, n^*f, q'_1) \vee \mu_2^*(q'_4, n^*f, q'_1)$$

$$= p \vee m \vee p = m.$$

Therefore,  $A(Z(G_1))$  and  $A(Z(G_2))$  are equivalent.



**FIGURE 30.** The related *SH*-graph automata  $A(Z(G_1))$  and  $A(Z(G_2))$ .

**Example 3.36.** Consider the semihoop in Example 3.11, let  $G_1 = (\alpha_1, \beta_1)$  and  $G_2 = (\alpha_2, \beta_2)$  be two *SH*-graphs on  $G_1^* = (V_1, E_1)$  and  $G_2^* = (V_2, E_2)$ , respectively, as in Fig. 31, where  $V_1 = \{q_1, q_2, q_3, q_4\}$ ,  $E_1 = \{q_1q_2, q_2q_3, q_3q_4, q_4q_1, q_1q_3, q_2q_4\}$ ,  $\alpha_1(q_1) = m, \alpha_1(q_2) = p, \alpha_1(q_3) = b, \alpha_1(q_4) = a, \beta_1(q_1q_2) = s, \beta_1(q_2q_3) = s, \beta_1(q_3q_4) = a, \beta_1(q_4q_1) = p, \beta_1(q_1q_3) = m, \beta_1(q_2q_4) = t, V_2 = \{q'_1, q'_2, q'_3, q'_4\}$ ,  $E_2 = \{q'_1q'_2, q'_2q'_3, q'_3q'_4, q'_4q'_1, q'_1q'_3, q'_2q'_4\}$ ,  $\alpha_2(q'_1) = m, \alpha_2(q'_2) = p, \alpha_2(q'_3) = b, \alpha_2(q'_4) = a, \beta_2(q'_1q'_2) = s, \beta_2(q'_2q'_3) = s, \beta_2(q'_3q'_4) = a, \beta_2(q'_4q'_1) = p, \beta_2(q'_1q'_3) = m, \beta_2(q'_2q'_4) = s$ . Additionally, let  $h$  be a bijection from  $G_1$  into  $G_2$ , where  $h(q_i) = q'_i$ . Then,  $G_1$  and  $G_2$  are isomorphic *SH*-graphs.

Let  $A(Z(G_1)) = (Q_1, X, \mu_1, F_1, \sigma_1)$  and  $A(Z(G_2)) = (Q_2, X, \mu_2, F_2, \sigma_2)$  be two *SH*-graph automata related to  $G_1$  and  $G_2$ , respectively, as in Fig. 32, where  $Q_1 = \{q_1, q_2, q_3, q_4\}$ ,  $Z(G_1) = \{q_1, q_3, q_4\}$ ,  $F_1 = \{q_2\}$ ,  $\mu_1(q_1, n, q_3) = \mu_1(q_3, n, q_1) = 1, \mu_1(q_3, n, q_4) = \mu_1(q_4, n, q_3) = 1, \mu_1(q_1, n, q_4) = \mu_1(q_4, n, q_1) = 1, \mu_1(q_1, f, q_2) = s, \mu_1(q_3, f, q_2) = s, \mu_1(q_4, f, q_2) = t, \sigma_1(q_1) = \sigma_1(q_3) = \sigma_1(q_4) = 1$  and  $\sigma_1(q_2) = 0, Q_2 = \{q'_1, q'_2, q'_3, q'_4\}$ ,  $Z(G_2) = \{q'_1, q'_3, q'_4\}$ ,  $F_2 = \{q'_2\}$ ,  $\mu_2(q'_1, n, q'_3) = \mu_2(q'_3, n, q'_1) = 1, \mu_2(q'_3, n, q'_4) = \mu_2(q'_4, n, q'_3) = 1, \mu_2(q'_1, n, q'_4) = \mu_2(q'_4, n, q'_1) = 1, \mu_2(q'_1, f, q'_2) = s, \mu_2(q'_3, f, q'_2) = s, \mu_2(q'_4, f, q'_2) = s, \sigma_2(q'_1) = \sigma_2(q'_3) = \sigma_2(q'_4) = 1$  and  $\sigma_2(q'_2) = 0$ . Then,

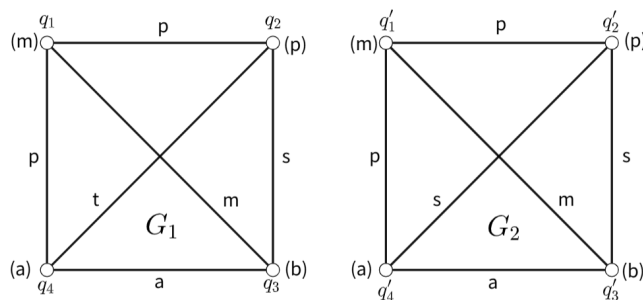


FIGURE 31. The SH-graphs  $G_1$  and  $G_2$ .

$$\bigvee_{q \in F_1} r_{\mu_1}(n^*f, q) = r_{\mu_1}(n^*f, q_2) = \mu_1^*(q_1, n^*f, q_2) \vee \mu_1^*(q_3, n^*f, q_2) \vee \mu_1^*(q_4, n^*f, q_2) = s \vee s \vee t = 1, \text{ and}$$

$$\bigvee_{q \in F_2} r_{\mu_2}(n^*f, q) = r_{\mu_2}(n^*f, q'_2) = \mu_2^*(q'_1, n^*f, q'_2) \vee \mu_2^*(q'_3, n^*f, q'_2) \vee \mu_2^*(q'_4, n^*f, q'_2) = s \vee s \vee s = s.$$

Therefore,  $A(Z(G_1))$  and  $A(Z(G_2))$  are not equivalent.

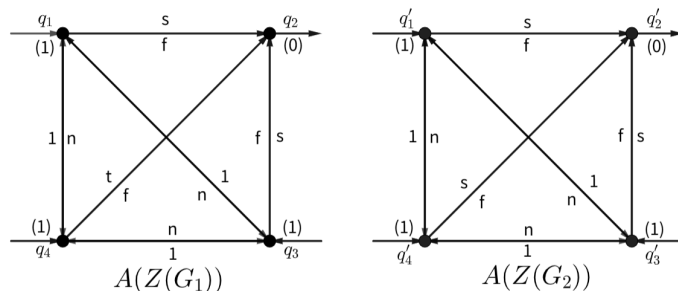
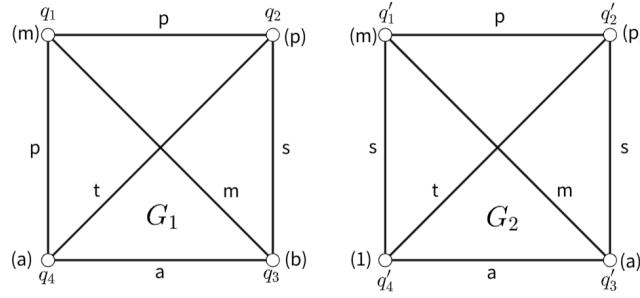


FIGURE 32. The related SH-graph automata  $A(Z(G_1))$  and  $A(Z(G_2))$ .

**Example 3.37.** Consider the semihoop in Example 3.11, let  $G_1 = (\alpha_1, \beta_1)$  and  $G_2 = (\alpha_2, \beta_2)$  be two SH-graphs on  $G_1^* = (V_1, E_1)$  and  $G_2^* = (V_2, E_2)$ , respectively, as in Fig. 33, where  $V_1 = \{q_1, q_2, q_3, q_4\}$ ,  $E_1 = \{q_1q_2, q_2q_3, q_3q_4, q_4q_1, q_1q_3, q_2q_4\}$ ,  $\alpha_1(q_1) = m$ ,  $\alpha_1(q_2) = p$ ,  $\alpha_1(q_3) = b$ ,  $\alpha_1(q_4) = a$ ,  $\beta_1(q_1q_2) = p$ ,  $\beta_1(q_2q_3) = s$ ,  $\beta_1(q_3q_4) = a$ ,  $\beta_1(q_4q_1) = p$ ,  $\beta_1(q_1q_3) = m$ ,  $\beta_1(q_2q_4) = t$ ,  $V_2 = \{q'_1, q'_2, q'_3, q'_4\}$ ,  $E_2 = \{q'_1q'_2, q'_2q'_3, q'_3q'_4, q'_4q'_1, q'_1q'_3, q'_2q'_4\}$ ,  $\alpha_2(q'_1) = m$ ,  $\alpha_2(q'_2) = p$ ,  $\alpha_2(q'_3) = a$ ,  $\alpha_2(q'_4) = 1$ ,  $\beta_2(q'_1q'_2) = p$ ,  $\beta_2(q'_2q'_3) = s$ ,  $\beta_2(q'_3q'_4) = a$ ,  $\beta_2(q'_4q'_1) = s$ ,  $\beta_2(q'_1q'_3) = m$ ,  $\beta_2(q'_2q'_4) = t$ . Additionally, let  $h$  be a bijection from  $G_1$  into



**FIGURE 33.** The SH-graphs  $G_1$  and  $G_2$ .

$G_2$ , where  $h(q_i) = q'_i$ . Then,  $G_1$  and  $G_2$  are not isomorphic SH-graphs. Let  $A(Z(G_1)) = (Q_1, X, \mu_1, F_1, \sigma_1)$  and  $A(Z(G_2)) = (Q_2, X, \mu_2, F_2, \sigma_2)$  be two SH-graph automata related to  $G_1$  and  $G_2$ , respectively, as in Fig. 34, where  $Q_1 = \{q_1, q_2, q_3, q_4\}$ ,  $Z(G_1) = \{q_1, q_2, q_4\}$ ,  $F_1 = \{q_3\}$ ,  $\mu_1(q_1, n, q_2) = \mu_1(q_2, n, q_1) = 1$ ,  $\mu_1(q_1, n, q_4) = \mu_1(q_4, n, q_1) = 1$ ,  $\mu_1(q_2, n, q_4) = \mu_1(q_4, n, q_2) = 1$ ,  $\mu_1(q_1, f, q_3) = m$ ,  $\mu_1(q_2, f, q_3) = s$ ,  $\mu_1(q_4, f, q_3) = a$ ,  $\sigma_1(q_1) = \sigma_1(q_2) = \sigma_1(q_4) = 1$  and  $\sigma_1(q_3) = 0$ ,  $Q_2 = \{q'_1, q'_2, q'_3, q'_4\}$ ,  $Z(G_2) = \{q'_1, q'_2, q'_4\}$ ,  $F_2 = \{q'_3\}$ ,  $\mu_2(q'_1, n, q'_2) = \mu_2(q'_2, n, q'_1) = 1$ ,  $\mu_2(q'_1, n, q'_4) = \mu_2(q'_4, n, q'_1) = 1$ ,  $\mu_2(q'_2, n, q'_4) = \mu_2(q'_4, n, q'_2) = 1$ ,  $\mu_2(q'_1, f, q'_3) = m$ ,  $\mu_2(q'_2, f, q'_3) = s$ ,  $\mu_2(q'_4, f, q'_3) = a$ ,  $\sigma_2(q'_1) = \sigma_2(q'_2) = \sigma_2(q'_4) = 1$  and  $\sigma_2(q'_3) = 0$ . Then,

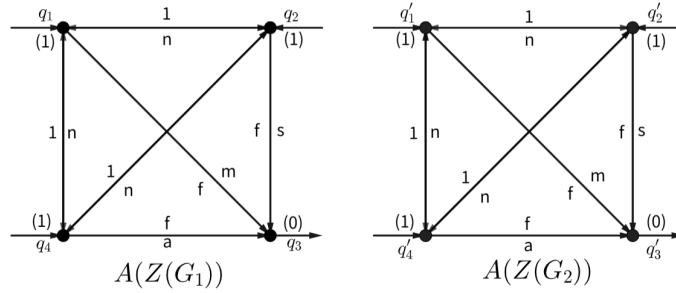
$$\bigvee_{q \in F_1} r_{\mu_1}(n^*f, q) = r_{\mu_1}(n^*f, q_3) = \mu_1^*(q_1, n^*f, q_3) \vee \mu_1^*(q_2, n^*f, q_3) \vee \mu_1^*(q_4, n^*f, q_3) = m \vee s \vee a = a, \text{ and}$$

$$\bigvee_{q \in F_2} r_{\mu_2}(n^*f, q) = r_{\mu_2}(n^*f, q'_3) = \mu_2^*(q'_1, n^*f, q'_3) \vee \mu_2^*(q'_2, n^*f, q'_3) \vee \mu_2^*(q'_4, n^*f, q'_3) = m \vee s \vee a = a.$$

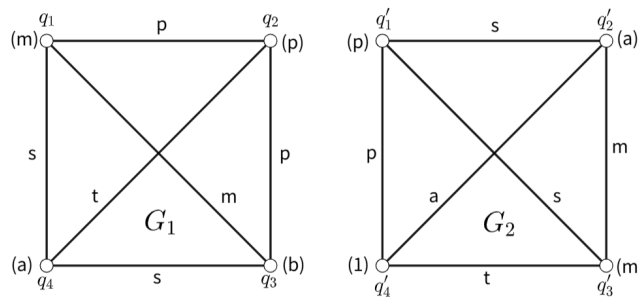
Therefore,  $A(Z(G_1))$  and  $A(Z(G_2))$  are equivalent.

**Example 3.38.** Consider the semihoop in Example 3.11, let  $G_1 = (\alpha_1, \beta_1)$  and  $G_2 = (\alpha_2, \beta_2)$  be two SH-graphs on  $G_1^* = (V_1, E_1)$  and  $G_2^* = (V_2, E_2)$ , respectively, as in Fig. 35, where  $V_1 = \{q_1, q_2, q_3, q_4\}$ ,  $E_1 = \{q_1q_2, q_2q_3, q_3q_4, q_4q_1, q_1q_3, q_2q_4\}$ ,  $\alpha_1(q_1) = m$ ,  $\alpha_1(q_2) = p$ ,  $\alpha_1(q_3) = b$ ,  $\alpha_1(q_4) = a$ ,  $\beta_1(q_1q_2) = p$ ,  $\beta_1(q_2q_3) = p$ ,  $\beta_1(q_3q_4) = s$ ,  $\beta_1(q_4q_1) = s$ ,  $\beta_1(q_1q_3) = m$ ,  $\beta_1(q_2q_4) = t$ ,  $V_2 = \{q'_1, q'_2, q'_3, q'_4\}$ ,  $E_2 = \{q'_1q'_2, q'_2q'_3, q'_3q'_4, q'_4q'_1, q'_1q'_3, q'_2q'_4\}$ ,  $\alpha_2(q'_1) = p$ ,  $\alpha_2(q'_2) = a$ ,  $\alpha_2(q'_3) = m$ ,  $\alpha_2(q'_4) = 1$ ,  $\beta_2(q'_1q'_2) = s$ ,  $\beta_2(q'_2q'_3) = m$ ,  $\beta_2(q'_3q'_4) = t$ ,  $\beta_2(q'_4q'_1) = p$ ,  $\beta_2(q'_1q'_3) = s$ ,  $\beta_2(q'_2q'_4) = a$ . Additionally, let  $h$  be a bijection from  $G_1$  into  $G_2$ , where  $h(q_i) = q'_i$ . Then,  $G_1$  and  $G_2$  are not isomorphic SH-graphs.

Let  $A(Z(G_1)) = (Q_1, X, \mu_1, F_1, \sigma_1)$  and  $A(Z(G_2)) = (Q_2, X, \mu_2, F_2, \sigma_2)$  be two SH-graph automata related to  $G_1$  and  $G_2$ , respectively, as in Fig. 36,



**FIGURE 34.** The related SH-graph automata  $A(Z(G_1))$  and  $A(Z(G_2))$ .



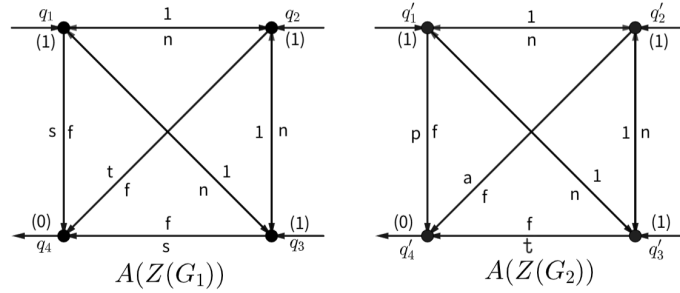
**FIGURE 35.** The SH-graphs  $G_1$  and  $G_2$ .

where  $Q_1 = \{q_1, q_2, q_3, q_4\}$ ,  $Z(G_1) = \{q_1, q_2, q_3\}$ ,  $F_1 = \{q_4\}$ ,  $\mu_1(q_1, n, q_2) = \mu_1(q_2, n, q_1) = 1$ ,  $\mu_1(q_1, n, q_3) = \mu_1(q_3, n, q_1) = 1$ ,  $\mu_1(q_2, n, q_3) = \mu_1(q_3, n, q_2) = 1$ ,  $\mu_1(q_1, f, q_4) = s$ ,  $\mu_1(q_2, f, q_4) = t$ ,  $\mu_1(q_3, f, q_4) = s$ ,  $\sigma_1(q_1) = \sigma_1(q_2) = \sigma_1(q_3) = 1$  and  $\sigma_1(q_4) = 0$ ,  $Q_2 = \{q'_1, q'_2, q'_3, q'_4\}$ ,  $Z(G_2) = \{q'_1, q'_2, q'_3\}$ ,  $F_2 = \{q'_4\}$ ,  $\mu_2(q'_1, n, q'_2) = \mu_2(q'_2, n, q'_1) = 1$ ,  $\mu_2(q'_1, n, q'_3) = \mu_2(q'_3, n, q'_1) = 1$ ,  $\mu_2(q'_2, n, q'_3) = \mu_2(q'_3, n, q'_2) = 1$ ,  $\mu_2(q'_1, f, q'_4) = p$ ,  $\mu_2(q'_2, f, q'_4) = a$ ,  $\mu_2(q'_3, f, q'_4) = t$ ,  $\sigma_2(q'_1) = \sigma_2(q'_2) = \sigma_2(q'_3) = 1$  and  $\sigma_2(q'_4) = 0$ . Then,

$$\bigvee_{q \in F_1} r_{\mu_1}(n^*f, q) = r_{\mu_1}(n^*f, q_4) = \mu_1^*(q_1, n^*f, q_4) \vee \mu_1^*(q_2, n^*f, q_4) \vee \mu_1^*(q_3, n^*f, q_4) = s \vee t \vee s = 1, \text{ and}$$

$$\bigvee_{q \in F_2} r_{\mu_2}(n^*f, q) = r_{\mu_2}(n^*f, q'_4) = \mu_2^*(q'_1, n^*f, q'_4) \vee \mu_2^*(q'_2, n^*f, q'_4) \vee \mu_2^*(q'_3, n^*f, q'_4) = p \vee a \vee t = a.$$

Therefore,  $A(Z(G_1))$  and  $A(Z(G_2))$  are not equivalent.



**FIGURE 36.** The related *SH*-graph automata  $A(Z(G_1))$  and  $A(Z(G_2))$ .

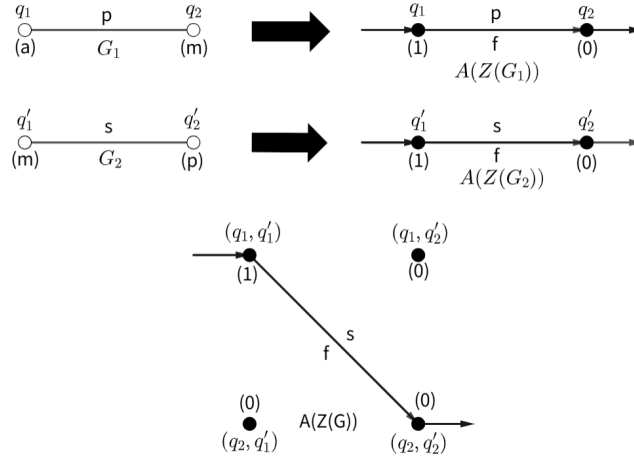
*Remark 3.39.* The above four examples express *SH*-graph automata  $A(Z(G_1))$  and  $A(Z(G_2))$  are equivalent can not be characterized by *SH*-graphs  $G_1$  and  $G_2$  being isomorphic.

**Definition 3.40.** Consider two *SH*-graphs  $G_1 = (\alpha_1, \beta_1)$  and  $G_2 = (\alpha_2, \beta_2)$  on  $G_1^* = (V_1, E_1)$  and  $G_2^* = (V_2, E_2)$ , respectively. Let the related *SH*-graph automata be  $A(Z(G_1)) = (Q_1, X, \mu_1, F_1, \sigma_1)$  and  $A(Z(G_2)) = (Q_2, X, \mu_2, F_2, \sigma_2)$ . Then,  $A(Z(G_1)) \times A(Z(G_2)) = A(Z(G)) = (Q, X, \mu, F, \sigma)$  is a direct product of two *SH*-graph automata, where  $Q = Q_1 \times Q_2$ ,  $F = F_1 \times F_2$ ,  $\mu((q_i, q'_k), \mu, (q_j, q'_t)) = \mu_1(q_i, u, q_j) \odot \mu_2(q'_k, u, q'_t)$ , and  $\sigma(q, q') = \begin{cases} 1 & \text{if } q \in Z(G_1), q' \in Z(G_2), \\ 0 & \text{if otherwise.} \end{cases}$

**Example 3.41.** Consider the semihoop in Example 3.11, let  $G_1 = (\alpha_1, \beta_1)$  and  $G_2 = (\alpha_2, \beta_2)$  be two *SH*-graphs on  $G_1^* = (V_1, E_1)$  and  $G_2^* = (V_2, E_2)$ , respectively, as in Fig. 37, where  $V_1 = \{q_1, q_2\}$ ,  $E_1 = \{q_1q_2\}$ ,  $\alpha_1(q_1) = a$ ,  $\alpha_1(q_2) = m$ ,  $\beta_1(q_1q_2) = p$ ,  $V_2 = \{q'_1, q'_2\}$ ,  $E_2 = \{q'_1q'_2\}$ ,  $\alpha_2(q'_1) = m$ ,  $\alpha_2(q'_2) = p$ ,  $\beta_2(q'_1q'_2) = s$ . Let  $A(Z(G_1)) = (Q_1, X, \mu_1, F_1, \sigma_1)$  and  $A(Z(G_2)) = (Q_2, X, \mu_2, F_2, \sigma_2)$  be *SH*-graph automata related to  $G_1$  and  $G_2$ , respectively, as in Fig. 37, where  $Q_1 = \{q_1, q_2\}$ ,  $Z(G_1) = \{q_1\}$ ,  $F_1 = \{q_2\}$ ,  $\mu_1(q_1, f, q_2) = p$ ,  $\sigma_1(q_1) = 1$  and  $\sigma_1(q_2) = 0$ ,  $Q_2 = \{q'_1, q'_2\}$ ,  $Z(G_2) = \{q'_1\}$ ,  $F_2 = \{q'_2\}$ ,  $\mu_2(q'_1, f, q'_2) = s$ ,  $\sigma_2(q'_1) = 1$  and  $\sigma_2(q'_2) = 0$ . Then,  $A = (Q, X, \mu, F, \sigma)$  is a direct product of two *SH*-graph automata, as in Fig. 37, where  $Q = Q_1 \times Q_2$ ,  $F = \{(q_2, q'_2)\}$ ,  $\mu((q_1, f, q'_1), f, (q_2, f, q'_2)) = p \odot s = s$ ,  $\sigma(q_1, q'_1) = 1$  and  $\sigma(q_1, q'_2) = \sigma(q_2, q'_1) = \sigma(q_2, q'_2) = 0$ . So,

$$\bigvee_{q \in F_1} r_{\mu_1}(n^*f, q) = r_{\mu_1}(n^*f, q_2) = \mu_1^*(q_1, n^*f, q_2) = p.$$

$$\bigvee_{q \in F_2} r_{\mu_2}(n^*f, q) = r_{\mu_2}(n^*f, q'_2) = \mu_2^*(q'_1, n^*f, q'_2) = s.$$



**FIGURE 37.** The SH-graph  $G_1$ , the SH-graph automata  $A(Z(G_1))$ , the SH-graph  $G_2$ , the SH-graph automata  $A(Z(G_2))$  and their direct product SH-graph automata  $A(Z(G))$ .

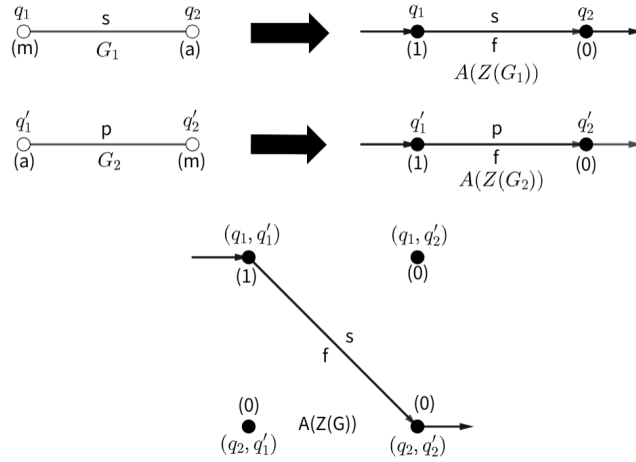
$$\bigvee_{q \in F} r_\mu(n^*f, (q, q')) = r_\mu(n^*f, (q_2, q'_2)) = \mu((q_1, f, q'_1), f, (q_2, f, q'_2)) = s.$$

Therefore,  $A(Z(G))$  and  $A(Z(G_1))$  are not equivalent,  $A(Z(G))$  and  $A(Z(G_2))$  are equivalent.

**Example 3.42.** Consider the semihoop in Example 3.11, let  $G_1 = (\alpha_1, \beta_1)$  and  $G_2 = (\alpha_2, \beta_2)$  be two SH-graphs on  $G_1^* = (V_1, E_1)$  and  $G_2^* = (V_2, E_2)$ , respectively, as in Fig. 38, where  $V_1 = \{q_1, q_2\}$ ,  $E_1 = \{q_1q_2\}$ ,  $\alpha_1(q_1) = m$ ,  $\alpha_1(q_2) = p$ ,  $\beta_1(q_1q_2) = s$ ,  $V_2 = \{q'_1, q'_2\}$ ,  $E_2 = \{q'_1q'_2\}$ ,  $\alpha_2(q'_1) = a$ ,  $\alpha_2(q'_2) = m$ ,  $\beta_2(q'_1q'_2) = p$ . Let  $A(Z(G_1)) = (Q_1, X, \mu_1, F_1, \sigma_1)$  and  $A(Z(G_2)) = (Q_2, X, \mu_2, F_2, \sigma_2)$  be SH-graph automata related to  $G_1$  and  $G_2$ , respectively, as in Fig. 38, where  $Q_1 = \{q_1, q_2\}$ ,  $Z(G_1) = \{q_1\}$ ,  $F_1 = \{q_2\}$ ,  $\mu_1(q_1, f, q_2) = s$ ,  $\sigma_1(q_1) = 1$  and  $\sigma_1(q_2) = 0$ ,  $Q_2 = \{q'_1, q'_2\}$ ,  $Z(G_2) = \{q'_1\}$ ,  $F_2 = \{q'_2\}$ ,  $\mu_2(q'_1, f, q'_2) = p$ ,  $\sigma_2(q'_1) = 1$  and  $\sigma_2(q'_2) = 0$ . Then,  $A = (Q, X, \mu, F, \sigma)$  is a direct product of two SH-graph automata, as in Fig. 38, where  $Q = Q_1 \times Q_2$ ,  $F = \{(q_2, q'_2)\}$ ,  $\mu((q_1, f, q'_1), f, (q_2, f, q'_2)) = s \odot p = s$ ,  $\sigma(q_1, q'_1) = 1$  and  $\sigma(q_1, q'_2) = \sigma(q_2, q'_1) = \sigma(q_2, q'_2) = 0$ . So,

$$\bigvee_{q \in F_1} r_{\mu_1}(n^*f, q) = r_{\mu_1}(n^*f, q_2) = \mu_1^*(q_1, n^*f, q_2) = s.$$

$$\bigvee_{q \in F_2} r_{\mu_2}(n^*f, q) = r_{\mu_2}(n^*f, q'_2) = \mu_2^*(q'_1, n^*f, q'_2) = p.$$



**FIGURE 38.** The *SH*-graph  $G_1$ , the *SH*-graph automaton  $A(Z(G_1))$ , the *SH*-graph  $G_2$ , the *SH*-graph automaton  $A(Z(G_2))$  and their direct product *SH*-graph automaton  $A(Z(G))$ .

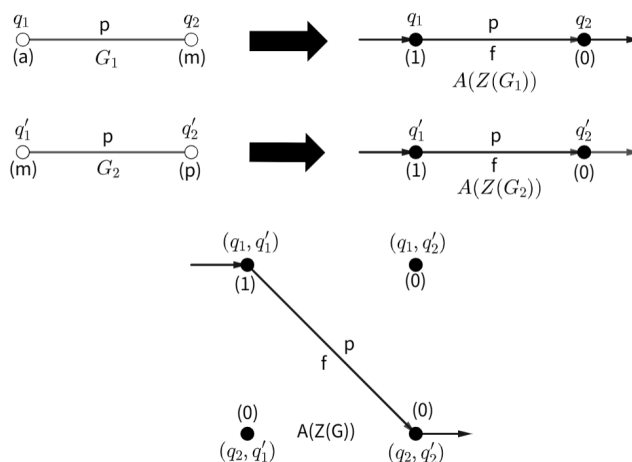
$$\bigvee_{q \in F} r_\mu(n^* f, (q, q')) = r_\mu(n^* f, (q_2, q'_2)) = \mu((q_1, f, q'_1), f, (q_2, f, q'_2)) = s.$$

Therefore,  $A(Z(G))$  and  $A(Z(G_1))$  are equivalent,  $A(Z(G))$  and  $A(Z(G_2))$  are not equivalent.

**Example 3.43.** Consider the semihoop in Example 3.11, let  $G_1 = (\alpha_1, \beta_1)$  and  $G_2 = (\alpha_2, \beta_2)$  be two *SH*-graphs on  $G_1^* = (V_1, E_1)$  and  $G_2^* = (V_2, E_2)$ , respectively, as in Fig. 39, where  $V_1 = \{q_1, q_2\}$ ,  $E_1 = \{q_1 q_2\}$ ,  $\alpha_1(q_1) = a$ ,  $\alpha_1(q_2) = m$ ,  $\beta_1(q_1 q_2) = p$ ,  $V_2 = \{q'_1, q'_2\}$ ,  $E_2 = \{q'_1 q'_2\}$ ,  $\alpha_2(q'_1) = m$ ,  $\alpha_2(q'_2) = p$ ,  $\beta_2(q'_1 q'_2) = p$ . Let  $A(Z(G_1)) = (Q_1, X, \mu_1, F_1, \sigma_1)$  and  $A(Z(G_2)) = (Q_2, X, \mu_2, F_2, \sigma_2)$  be *SH*-graph automata related to  $G_1$  and  $G_2$ , respectively, as in Fig. 39, where  $Q_1 = \{q_1, q_2\}$ ,  $Z(G_1) = \{q_1\}$ ,  $F_1 = \{q_2\}$ ,  $\mu_1(q_1, f, q_2) = p$ ,  $\sigma_1(q_1) = 1$  and  $\sigma_1(q_2) = 0$ ,  $Q_2 = \{q'_1, q'_2\}$ ,  $Z(G_2) = \{q'_1\}$ ,  $F_2 = \{q'_2\}$ ,  $\mu_2(q'_1, f, q'_2) = p$ ,  $\sigma_2(q'_1) = 1$  and  $\sigma_2(q'_2) = 0$ . Then,  $A = (Q, X, \mu, F, \sigma)$  is a direct product of two *SH*-graph automata, as in Fig. 39, where  $Q = Q_1 \times Q_2$ ,  $F = \{(q_2, q'_2)\}$ ,  $\mu((q_1, f, q'_1), f, (q_2, f, q'_2)) = p \odot p = p$ ,  $\sigma(q_1, q'_1) = 1$  and  $\sigma(q_1, q'_2) = \sigma(q_2, q'_1) = \sigma(q_2, q'_2) = 0$ . So,

$$\bigvee_{q \in F_1} r_{\mu_1}(n^* f, q) = r_{\mu_1}(n^* f, q_2) = \mu_1^*(q_1, n^* f, q_2) = p.$$

$$\bigvee_{q \in F_2} r_{\mu_2}(n^* f, q) = r_{\mu_2}(n^* f, q'_2) = \mu_2^*(q'_1, n^* f, q'_2) = p.$$

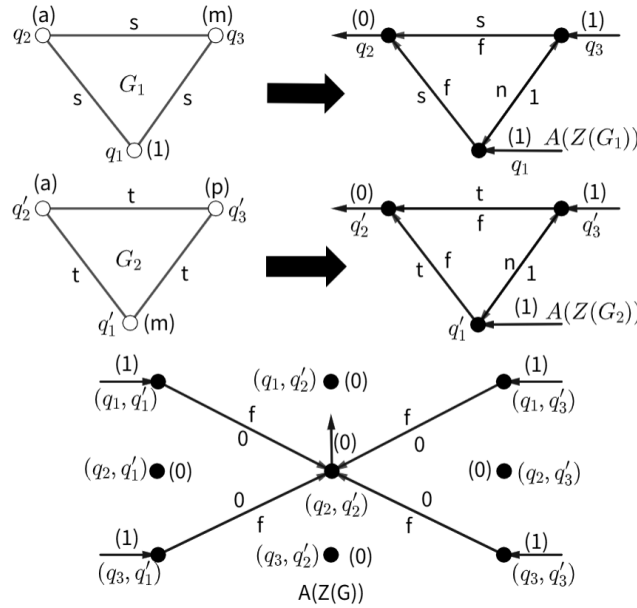


**FIGURE 39.** The SH-graph  $G_1$ , the SH-graph automaton  $A(Z(G_1))$ , the SH-graph  $G_2$ , the SH-graph automaton  $A(Z(G_2))$  and their direct product SH-graph automaton  $A(Z(G))$ .

$$\bigvee_{q \in F} r_\mu(n^* f, (q, q')) = r_\mu(n^* f, (q_2, q'_2)) = \mu((q_1, f, q'_1), f, (q_2, f, q'_2)) = p.$$

Therefore,  $A(Z(G))$  and  $A(Z(G_1))$  are equivalent,  $A(Z(G))$  and  $A(Z(G_2))$  are equivalent, too.

**Example 3.44.** Consider the semihoop in Example 3.11, let  $G_1 = (\alpha_1, \beta_1)$  and  $G_2 = (\alpha_2, \beta_2)$  be two SH-graphs on  $G_1^* = (V_1, E_1)$  and  $G_2^* = (V_2, E_2)$ , respectively, as in Fig. 40, where  $V_1 = \{q_1, q_2, q_3\}$ ,  $E_1 = \{q_1 q_2, q_2 q_3, q_3 q_1\}$ ,  $\alpha_1(q_1) = 1$ ,  $\alpha_1(q_2) = a$ ,  $\alpha_1(q_3) = m$ ,  $\beta_1(q_1 q_2) = s$ ,  $\beta_1(q_2 q_3) = s$ ,  $\beta_1(q_3 q_1) = s$ ,  $V_2 = \{q'_1, q'_2, q'_3\}$ ,  $E_2 = \{q'_1 q'_2, q'_2 q'_3, q'_3 q'_1\}$ ,  $\alpha_2(q'_1) = m$ ,  $\alpha_2(q'_2) = a$ ,  $\alpha_2(q'_3) = p$ ,  $\beta_2(q'_1 q'_2) = t$ ,  $\beta_2(q'_2 q'_3) = t$ ,  $\beta_2(q'_3 q'_1) = t$ . Let  $A(Z(G_1)) = (Q_1, X, \mu_1, F_1, \sigma_1)$  and  $A(Z(G_2)) = (Q_2, X, \mu_2, F_2, \sigma_2)$  be SH-graph automata related to  $G_1$  and  $G_2$ , respectively, as in Fig. 40, where  $Q_1 = \{q_1, q_2, q_3\}$ ,  $Z(G_1) = \{q_1, q_3\}$ ,  $F_1 = \{q_2\}$ ,  $\mu_1(q_1, n, q_3) = \mu_1(q_3, n, q_1) = 1$ ,  $\mu_1(q_1, f, q_2) = s$ ,  $\mu_1(q_3, f, q_2) = s$ ,  $\sigma_1(q_1) = \sigma_1(q_3) = 1$ , and  $\sigma_1(q_2) = 0$ ,  $Q_2 = \{q'_1, q'_2, q'_3\}$ ,  $Z(G_2) = \{q'_1, q'_3\}$ ,  $F_2 = \{q'_2\}$ ,  $\mu_2(q'_1, n, q'_3) = \mu_2(q'_3, n, q'_1) = 1$ ,  $\mu_2(q'_1, f, q'_2) = t$ ,  $\mu_2(q'_3, f, q'_2) = t$ ,  $\sigma_2(q'_1) = \sigma_2(q'_3) = 1$ , and  $\sigma_2(q'_2) = 0$ . Then,  $A = (Q, X, \mu, F, \sigma)$  is a direct product of two SH-graph automata, as in Fig. 40, where  $Q = Q_1 \times Q_2$ ,  $F = \{(q_2, q'_2)\}$ ,  $\mu((q_1, f, q'_1), f, (q_2, f, q'_2)) = s \odot t = 0$ ,  $\mu((q_1, f, q'_3), f, (q_2, f, q'_2)) = s \odot t = 0$ ,  $\mu((q_3, f, q'_1), f, (q_2, f, q'_2)) = s \odot t = 0$ ,  $\mu((q_3, f, q'_3), f, (q_2, f, q'_2)) = s \odot t = 0$ ,  $\sigma(q_1, q'_1) = \sigma(q_1, q'_3) = \sigma(q_3, q'_1) = \sigma(q_3, q'_3) = 1$  and  $\sigma(q_1, q'_2) = \sigma(q_2, q'_1) = \sigma(q_2, q'_2) = \sigma(q_2, q'_3) = \sigma(q_3, q'_1) = \sigma(q_3, q'_2) = 0$ . So,



**FIGURE 40.** The  $SH$ -graph  $G_1$ , the  $SH$ -graph automaton  $A(Z(G_1))$ , the  $SH$ -graph  $G_2$ , the  $SH$ -graph automaton  $A(Z(G_2))$  and their direct product  $SH$ -graph automaton  $A(Z(G))$ .

$$\bigvee_{q \in F_1} r_{\mu_1}(n^*f, q) = r_{\mu_1}(n^*f, q_2) = \mu_1^*(q_1, n^*f, q_2) \vee \mu_1^*(q_3, n^*f, q_2) = s \vee s = s.$$

$$\bigvee_{q \in F_2} r_{\mu_2}(n^*f, q) = r_{\mu_2}(n^*f, q'_2) = \mu_2^*(q'_1, n^*f, q'_2) \vee \mu_2^*(q'_3, n^*f, q'_2) = t \vee t = t.$$

$$\bigvee_{q \in F} r_{\mu}(n^*f, (q, q')) = r_{\mu}(n^*f, (q_2, q'_2)) = \mu((q_1, f, q'_1), f, (q_2, f, q'_2)) \vee \mu((q_1, f, q'_3), f, (q_2, f, q'_2)) \vee \mu((q_3, f, q'_1), f, (q_2, f, q'_2)) \vee \mu((q_3, f, q'_3), f, (q_2, f, q'_2)) = 0 \vee 0 \vee 0 \vee 0 = 0.$$

Therefore,  $A(Z(G))$  and  $A(Z(G_1))$  are not equivalent,  $A(Z(G))$  and  $A(Z(G_2))$  are not equivalent, too.

*Remark 3.45.* The above four examples express that there are various relationships with equivalence between  $A(Z(G))$  and  $A(Z(G_1))$ ,  $A(Z(G_2))$ , including that: (i)  $A(Z(G))$  is equivalent to  $A(Z(G_1))$  and also equivalent to  $A(Z(G_2))$ ; (ii)  $A(Z(G))$  is not equivalent to  $A(Z(G_1))$  and also not equivalent to  $A(Z(G_2))$ ; (iii)  $A(Z(G))$  is equivalent to  $A(Z(G_1))$ , but not equivalent to  $A(Z(G_2))$ ; (iv)  $A(Z(G))$  is equivalent to  $A(Z(G_2))$ , but not equivalent to  $A(Z(G_1))$ .

#### 4. Some Applications of SH-graphs and automata

Mordeson and other authors have recently mentioned many applications for fuzzy graphs in their books , but these graphs are not used in some cases, while the SH-graphs can be useful in other applications in addition to those applications. For instance, it can be used to categorize different drugs with the same properties in a drugstore and music styles that can be played by musical instruments. Moreover, it can be used to categorize books of the library to make a connection between them and determine the minimum books that have covered all subjects.

**Definition 4.1.** Let  $(S, \wedge, \odot, \rightarrow, 1)$  be a smeihoop and  $G = (\alpha, E\beta)$  be the SH-graph on  $G^* = (V, E)$ . Then  $C(G) = \{q_i | q_i \in V, \forall \alpha(q_i) = 1\}$  is a cover set of this SH-graph when it has a minimum number of vertices.

Assume that the musician has three musical instruments named  $b_1, b_2, b_3$  and they can play four musical styles and such that musical instrument  $b_1$  can play three musical styles  $a_1, a_2, a_3$ , musical instrument  $b_2$  can play two musical styles  $a_1, a_2$ , musical instrument  $b_3$  can play two musical styles  $a_1, a_4$ . Consider  $S = (P(X), \cap, \odot, \rightarrow, \emptyset, X)$ , where  $X = \{a_1, a_2, a_3, a_4\}$ ,  $A \odot B = A \cap B$ , and  $A \rightarrow$

$$B = \begin{cases} X & \text{if } A \subseteq B, \\ B & \text{if otherwise.} \end{cases} \text{ Therefore, this is modeled by SH-graph } G = (\alpha, \beta)$$

on  $G^* = (V, E)$ , as in Fig. 41, where  $V = \{b_1, b_2, b_3\}$ ,  $E = \{b_1b_2, b_2b_3, b_3b_1\}$ ,  $\alpha(b_1) = \{a_1, a_2, a_3\}$ ,  $\alpha(b_2) = \{a_1, a_2\}$ ,  $\alpha(b_3) = \{a_1, a_4\}$ ,  $\beta(b_1b_2) = \{a_1, a_2\}$ ,  $\beta(b_2b_3) = \{a_1\}$ ,  $\beta(b_3b_1) = \{a_1\}$ .

$$\alpha(b_1) \vee \alpha(b_2) = \{a_1, a_2, a_3\} \vee \{a_1, a_2\} = [(\{a_1, a_2, a_3\} \rightarrow \{a_1, a_2\}) \rightarrow \{a_1, a_2\}] \wedge [(\{a_1, a_2\} \rightarrow \{a_1, a_2, a_3\}) \rightarrow \{a_1, a_2, a_3\}] = (\{a_1, a_2\} \rightarrow \{a_1, a_2\}) \wedge (X \rightarrow \{a_1, a_2, a_3\})$$

$$= X \cap \{a_1, a_2, a_3\} = \{a_1, a_2, a_3\}.$$

$$\alpha(b_1) \vee \alpha(b_3) = \{a_1, a_2, a_3\} \vee \{a_1, a_4\} = [(\{a_1, a_2, a_3\} \rightarrow \{a_1, a_4\}) \rightarrow \{a_1, a_4\}] \wedge [(\{a_1, a_4\} \rightarrow \{a_1, a_2, a_3\}) \rightarrow \{a_1, a_2, a_3\}] = (\{a_1, a_4\} \rightarrow \{a_1, a_4\}) \wedge (\{a_1, a_2, a_3\} \rightarrow \{a_1, a_2, a_3\}) = X \cap X = X.$$

$$\alpha(b_2) \vee \alpha(b_3) = \{a_1, a_2\} \vee \{a_1, a_4\} = [(\{a_1, a_2\} \rightarrow \{a_1, a_4\}) \rightarrow \{a_1, a_4\}] \wedge [(\{a_1, a_4\} \rightarrow \{a_1, a_2\}) \rightarrow \{a_1, a_2\}] = (\{a_1, a_4\} \rightarrow \{a_1, a_4\}) \wedge (\{a_1, a_2\} \rightarrow \{a_1, a_2\}) = X \cap X = X.$$

Thus,  $\{b_1, b_3\}$  and  $\{b_2, b_3\}$  are cover sets of  $G$ .  $\{b_1, b_3\}$  is a cover set of  $G$  because  $b_1$  can play three musical styles  $a_1, a_2, a_3$  and  $b_3$  can play two musical styles  $a_1, a_4$ . Although  $b_2$  can play two musical styles  $a_1, a_2$  and  $b_3$  can play two musical styles  $a_1, a_4$ , when  $b_2$  and  $b_3$  can create a musical style  $a_3$  by playing simultaneously, then  $\{b_2, b_3\}$  is a cover set of  $G$ .

Mordeson and other authors also have recently mentioned many applications for fuzzy automata in their books , but these automata are not used in some sectors, while the SH-graphs automata can be used more sectors. For instance, it can be used to find which diseases have most of the similar symptoms among

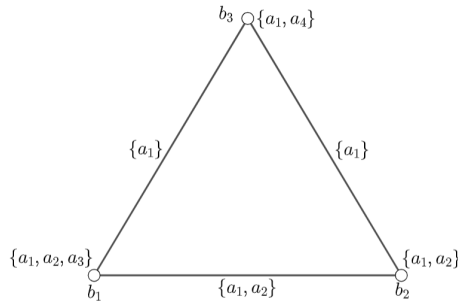


FIGURE 41. The  $SH$ -graph  $G$ .

many diseases, which food has most of the similar ingredients among much food and so on.

Let four food  $a_1, a_2, a_3, a_4$  and six ingredients  $b_1, b_2, b_3, b_4, b_5, b_6$ , and let  $\alpha(a_1) = \{b_1, b_2, b_4, b_5\}$ ,  $\alpha(a_2) = \{b_1, b_3, b_6\}$ ,  $\alpha(a_3) = \{b_1, b_4, b_5\}$ ,  $\alpha(a_4) = \{b_1, b_3\}$ . Let  $S = (P(X), \cap, \odot, \rightarrow, \emptyset, X)$ , where  $X = \{b_1, b_2, b_3, b_4, b_5\}$ ,  $A \odot B = A \cap B$ , and  $A \rightarrow B = \begin{cases} X & \text{if } A \subseteq B, \\ B & \text{if otherwise.} \end{cases}$

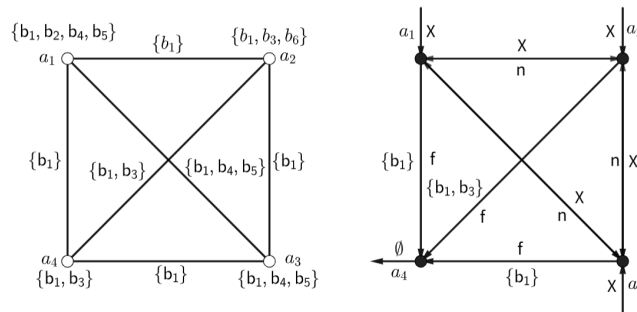
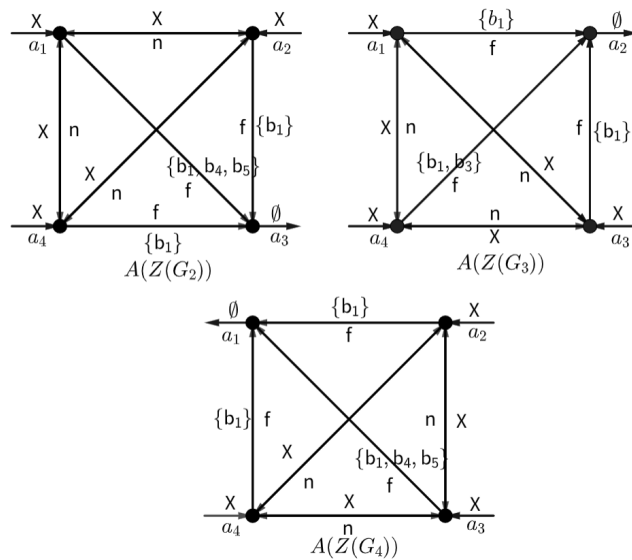


FIGURE 42. The  $SH$ -graph  $G$  and the related  $SH$ -graph automaton  $A(Z_1(G))$ .

Then  $G = (\alpha, \beta)$  on  $G^*$  is the  $SH$  graph, as in Fig. 42, where  $V = \{a_1, a_2, a_3, a_4\}$ ,  $E = \{a_1a_2, a_2a_3, a_3a_4, a_4a_1, a_1a_3, a_2a_4\}$ ,  $\beta(a_1a_2) = \{b_1\}$ ,  $\beta(a_2a_3) = \{b_1\}$ ,  $\beta(a_3a_4) = \{b_1\}$ ,  $\beta(a_4a_1) = \{b_1\}$ ,  $\beta(a_1a_3) = \{b_1, b_4, b_5\}$  and  $\beta(a_2a_4) = \{b_1, b_3\}$ . Besides,  $A(Z_1(G)) = (Q, X, \mu_1, F_1, \sigma_1)$ , as in Fig. 42,  $A(Z_2(G)) = (Q, X, \mu_2, F_2, \sigma_2)$ ,  $A(Z_3(G)) = (Q, X, \mu_3, F_3, \sigma_3)$ ,  $A(Z_4(G)) = (Q, X, \mu_4, F_4, \sigma_4)$ , as in Fig. 43, are related  $SH$ -graph automata, where  $Q = \{a_1, a_2, a_3, a_4\}$ ,  $Z_1(G) = \{a_1, a_2, a_3\}$ ,  $\mu_1(a_1, n, a_2) = \mu_1(a_2, n, a_1) = X$ ,  $\mu_1(a_1, n, a_3) = \mu_1(a_3, n, a_1) = X$ ,  $\mu_1(a_2, n, a_3) = \mu_1(a_3, n, a_2) = X$ ,  $\mu_1(a_1, f, a_4) = \{b_1\}$ ,  $\mu_1(a_2, f, a_4) = \{b_1, b_3\}$ ,

$\mu_1(a_3, f, a_4) = \{b_1\}$ ,  $\sigma_1(a_1) = \sigma_1(a_2) = \sigma_1(a_3) = X$  and  $\sigma_1(a_4) = \emptyset$ ,  $F_1 = \{a_4\}$ ,  $Z_2(G) = \{a_1, a_2, a_4\}$ ,  $\mu_2(a_1, n, a_2) = \mu_2(a_2, n, a_1) = X$ ,  $\mu_2(a_1, n, a_4) = \mu_2(a_4, n, a_1) = X$ ,  $\mu_2(a_2, n, a_4) = \mu_2(a_4, n, a_2) = X$ ,  $\mu_2(a_1, f, a_3) = \{b_1, b_4, b_5\}$ ,  $\mu_2(a_2, f, a_3) = \{b_1\}$ ,  $\mu_2(a_4, f, a_3) = \{b_1\}$ ,  $\sigma_2(a_1) = \sigma_2(a_2) = \sigma_2(a_4) = X$  and  $\sigma_2(a_3) = \emptyset$ ,  $F_2 = \{a_3\}$ ,  $Z_3(G) = \{a_1, a_3, a_4\}$ ,  $\mu_3(a_1, n, a_3) = \mu_3(a_3, n, a_1) = X$ ,  $\mu_3(a_1, n, a_4) = \mu_3(a_4, n, a_1) = X$ ,  $\mu_3(a_3, n, a_4) = \mu_3(a_4, n, a_3) = X$ ,  $\mu_3(a_1, f, a_2) = \{b_1\}$ ,  $\mu_3(a_3, f, a_2) = \{b_1\}$ ,  $\mu_3(a_4, f, a_2) = \{b_1, b_3\}$ ,  $\sigma_3(a_1) = \sigma_3(a_3) = \sigma_3(a_4) = X$  and  $\sigma_3(a_2) = \emptyset$ ,  $F_3 = \{a_2\}$ ,  $Z_4(G) = \{a_2, a_3, a_4\}$ ,  $\mu_4(a_2, n, a_3) = \mu_4(a_3, n, a_2) = X$ ,  $\mu_4(a_2, n, a_4) = \mu_4(a_4, n, a_2) = X$ ,  $\mu_4(a_3, n, a_4) = \mu_4(a_4, n, a_3) = X$ ,  $\mu_4(a_2, f, a_1) = \{b_1\}$ ,  $\mu_4(a_3, f, a_1) = \{b_1, b_4, b_5\}$ ,  $\mu_4(a_4, f, a_1) = \{b_1\}$ ,  $\sigma_4(a_2) = \sigma_4(a_3) = \sigma_4(a_4) = X$  and  $\sigma_4(a_1) = \emptyset$ ,  $F_4 = \{a_1\}$ . Hence, we can know that food  $a_1$  and  $a_3$  have most of the similar ingredients  $b_1$  and  $b_4$  and  $b_5$ , food  $a_2$  and  $a_4$  have most of the similar ingredients  $b_1$  and  $b_3$ .



**FIGURE 43.** The related SH-graph  $G$  automata  $A(Z_2(G))$ ,  $A(Z_3(G))$  and  $A(Z_4(G))$ .

### 5. Conclusions

In this article, because semihoops are the most basic residuated structure that contain all logical algebras based on Galois connections, we first introduced the concepts of SH-graphs and SH-graph automata based on semihoops. Then, we proposed the concepts of isomorphism between two SH-graphs and isomorphism between two SH-graph automata. After that, we obtained several

important conclusions about isomorphism. In addition, the concept of equivalence between two *SH*-graph automata was introduced. Then, some theorems and conclusions about equivalence were obtained. Later, the concept of the direct product of *SH*-graph automata was introduced. Then, the relationships between the direct product of *SH*-graph automata  $A(Z(G))$  and component *SH*-graph automata  $A(Z(G_1))$ ,  $A(Z(G_2))$  were obtained. Finally, we introduced some applications about *SH*-graphs and *SH*-graph automata.

Due to the limited and simple applications introduced in this article, we are willing to further study this topic in the future, attempting to apply it in urban planning and transportation, prevention of disease transmission, selection of the best drugs for specific diseases, and providing the highest therapeutic effects. This will enable us to gain a better understanding of these structures and measure their complexity.

## 6. Author Contributions

All authors contributed to this article .

## 7. Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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## 9. Ethical considerations

We declare that we have complied with the ethical standards for publishing articles in this journal.

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## 11. Conflict of interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

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