Journal of Mahani Mathematical Research
Print ISSN: 2251-7952 Online ISSN: 2645-4505

A NUMERICAL CARDINAL STRATEGY TO SOLVE MULTI-ORDER FRACTIONAL DIFFERENTIAL EQUATIONS WITH CAPUTO DERIVATIVES

S. Mirshojaei, E. Shivanian ✉, B. Ghayebi, and S.J. Hosseini Ghoncheh

Article type: Research Article
(Received: 22 March 2025, Received in revised form 20 April 2025)
(Accepted: 13 May 2026, Published Online: 13 May 2026)

Abstract. In this article, a numerical method to solve multi-order fractional differential equations with Caputo derivatives is suggested. Shifted Chebyshev cardinal functions are employed as basic functions. The corresponding fractional derivative operator matrix for these cardinal functions is computed. By approximating the unknown expression of the problem in terms of the shifted Chebyshev cardinal functions, applying their fractional derivative operator matrix, and utilizing the collocation method, equation is converted into solving a system of algebraic equations. By solving this system, the approximate solution of the problem is obtained. Finally, the accuracy and efficiency of the proposed method are examined by solving several numerical examples. The results show that the method presented in this article is an efficient and highly accurate method to solve such multi-order fractional differential equations.

Keywords: Multi-order Differential equations; Caputo derivative; Shifted Chebyshev cardinal functions; Fractional derivative operator matrix.
2020 MSC: 35R11, 34B10.

1. Introduction

Fractional calculus (calculus based on fractional order) is a subdivision of mathematics that aims to extend the concepts of differentiation and integration from traditional calculus to non-integer (fractional) orders. Unlike regular calculus, in which derivatives and integrals are restricted to integer orders, in fractional calculus, concepts such as the derivative of order 0.5 or the integral of order 1.7 can be defined. This theory enjoys extensive applications in physics, engineering, economics, and life sciences, particularly in modeling phenomena that display non-local, memory-retentive, or anomalous behavior. Such features have resulted in a proliferation of research on their applications in science and engineering. For instance, recent studies have documented the

✉ shivanian@sci.ikiu.ac.ir, ORCID: 0000-0001-9383-6779
https://doi.org/10.22103/jmmr.2026.25014.1780

Publisher: Shahid Bahonar University of Kerman
How to cite: S. Mirshojaei, E. Shivanian, B. Ghayebi, S.J. Hosseini Ghoncheh, *A numerical cardinal strategy to solve multi-order fractional differential equations with Caputo derivatives*, J. Mahani Math. Res. 2026; 15(2): 433-455.

applications of fractional derivatives in epidemic models, Maxwell nanofluids, Zika virus transmission, reversible enzymatic reactions, etc.

Fractional differential equations refer to a specific type of differential equations in which derivatives of non-integer (fractional) order appear. These equations are a generalization of ordinary and partial differential equations and allow for more accurate modeling of phenomena that exhibit memory and non-local behavior. Some of the applications of these equations include describing complex systems in physics, biology, engineering, control, and finance, for instance [1, 7, 13, 26]. Arriving at an analytical solution for fractional equations is often difficult and in some cases impossible. Accordingly, researchers employ numerical methods in order to solve and analyze these types of equations. Recently, considerable efforts have been made to numerically solve fractional differential equations. The Legendre wavelet method [12], the local Galerkin approach [19], the Chebyshev discrete polynomial strategy [21], the least squares methodology [23], and the radial basis function technique [22], the matrix method [14], predict–correct technique [27], Legendre wavelets and eigenfunction approach [25], Genocchi-based spectral technique [16], Generalized Bessel polynomial method [15], Gegenbauer operational matrix method [24], spectral approach [2], and Bernstein polynomials approach [3] are among the examples of these methods.

In this paper, we propose a local method based on Chebyshev cardinal functions to solve multi-order fractional differential equations. Hence, we focus on the following problem:

$$ {}_0^C D_t^{\mu} \vartheta(t) + \sum_{i=1}^{n} \eta_i(t)\, {}_0^C D_t^{\alpha_i} \vartheta(t) + \eta_{n+1}(t)\, \vartheta(t) = f(t), \quad t \in [0, t_f], \tag{1} $$

With initial conditions of

$$ \vartheta^{(i)}(0) = \xi_i, \qquad i = 0, 1, \ldots, q-1, \tag{2} $$

in which η_i are real functions for $i = 1, 2, \ldots, n+1$, with $q-1 < \mu \leq q$, $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_n < \mu$. Also, the values ξ_i $(i = 1, 2, \ldots, q-1)$ are known values. The function $f(t)$ is a given function. The expression ${}_0^C D_t^{\mu}$ describes the fractional Caputo derivative of order μ.

In recent years, numerous researchers have employed cardinal polynomials for various fractional problems. For instance, see references [4–6, 8, 10, 17, 18].

While earlier works [4, 8, 10, 17, 18] successfully employed general cardinal polynomials to tackle various fractional problems, recent approaches have increasingly shifted towards Chebyshev-cardinal-based methods. Compared to the previous techniques in [4, 8, 10, 17, 18], the new Chebyshev-cardinal approaches offer distinct computational advantages. First, they provide exceptionally high approximation accuracy for complex fractional behaviors. Second, unlike some general cardinal bases that require complex numerical root-finding, Chebyshev cardinal functions benefit from an explicit analytical formula for the roots of Chebyshev polynomials. This explicit formulation drastically simplifies the construction of the basis polynomials, reduces computational overhead, and makes them highly effective for solving multi-order fractional differential equations.

The two main reasons for such extensive applications lie in the ease of calculating their integrals and fractional derivatives and the high accuracy of their approximations. In recent years, Chebyshev cardinal functions have received growing attention as the most important family of these cardinal functions. The main reason for this, regardless of their high approximation accuracy, is the existence of an explicit formula to calculate the roots of Chebyshev polynomials that are used to construct these polynomials.

Some of the most important advantages of the established method are listed in the following:

(1) The basis functions used in the established approach posses many useful properties, such as orthogonality, interpolation, and spectral accuracy.
(2) The operational matrices used in the developed approach significantly reduce the computational cost.
(3) Only a small number of basis functions are required to achieve highly accurate results.
(4) The solution of the main fractional problem transforms into the solution of a linear system of algebraic equations, which can be easily solved.

The present article seeks to address the numerical solution of a multi-order fractional differential equation with Caputo derivative using shifted Chebyshev cardinal functions. Indeed, by approximating the unknown solution of the problem according to these basic functions and employing fractional derivative matrices of these polynomials, the problem under study is transformed into a system of algebraic equations that can be easily solved.

This work presents a spectral method that leverages a set of basis functions for the numerical solution of partial differential equations. The proposed approach shares a foundational philosophy with several established numerical schemes. Specifically, it can be interpreted as a global counterpart to the Galerkin method [19], as both project the solution onto a finite-dimensional function space; however, while traditional Galerkin methods often use local, piecewise polynomial basis functions (e.g., in Finite Elements), our method employs global, smooth basis functions spanning the entire domain, leading to exponential convergence for smooth solutions. Furthermore, the methodology is closely related to the Radial Basis Function (RBF) methods [22], as both utilize a set of non-polynomial basis functions for approximation. The primary distinction lies in the choice and distribution of these bases. Finally, the numerical implementation, which involves enforcing the differential equation at collocation points, is conceptually analogous to the Least-Squares Approximation [23] framework, where the objective is to minimize the residual of the governing equation in a discrete norm, thus ensuring an accurate fit to the underlying differential operator.

This article is structured as follows: in sections 2 and 3, the shifted Chebyshev cardinal functions are defined respectively, and then their fractional derivative operator matrices are calculated. The numerical method intended to solve the multi-order fractional differential equation using the Caputo fractional derivative is discussed in Section 4. The convergence analysis of the presented scheme is studied in Section 5.The accuracy and efficiency of the proposed method are examined through three numerical examples in Section 6. Finally, the article concludes with a summary in Section 7.

2. Chebyshev cardinal functions

In this section, the Chebyshev cardinal functions will be introduced. Afterward, the approximation of a function employing these basic functions will be presented.
First, note that using the following relation, we can determine the zeros of Chebyshev polynomials of degree $(m+1)$ $(m \in \mathbb{N} \bigcup \{0\})$:

$$\theta_k = -\cos\left(\frac{(2k-1)\pi}{2\,(m+1)}\right), \quad k = 1, 2, \ldots, m+1.$$

Chebyshev cardinal functions of degree m over the interval $[0, t_f]$ can be defined as [9]

$$\phi_i(t) = \prod_{k=1, k\neq i}^{m+1} \frac{t - t_k}{t_i - t_k}, \quad i = 1, 2, \ldots, m+1, \tag{3}$$

in which $t_k = \frac{t_f}{2}(1+\theta_k)$. The cardinal functions mentioned above can be expressed more precisely as follows:

$$\phi_i(t) = \frac{1}{\gamma_i} \sum_{k=0}^{m} b_{ik} t^{m-k}, \ \ i = 1, 2, \ldots, m+1, \tag{4}$$

Such that

$$\gamma_i = \prod_{\ell=1, \ell\neq i}^{m+1} (t_i - t_\ell), \quad b_{ik} = \begin{cases} 1, & k = 0, \\ \dfrac{-1}{k} \displaystyle\sum_{\ell=1}^{k} a_{i\ell} b_{i\,k-\ell}, & k = 1, 2, \ldots, m, \end{cases}$$

$$a_{i\ell} = \sum_{r=1, r\neq i}^{m+1} t_r^{\ell}, \ \ 1 \leq \ell \leq k.$$

Using the above-mentioned basic functions, any continuous function $h \in L_2([0,t_f])$ can be approximated as

$$h(t) \simeq \sum_{i=1}^{m+1} h_i \phi_i(t) \triangleq \mathbf{H}^T \Phi_m(t), \tag{5}$$

such that $\mathbf{H} = [h_1\, h_2\, \ldots\, h_{m+1}]^T$, and

$$\Phi_m(t) = [\phi_1(t)\, \phi_2(t)\, \ldots\, \phi_{m+1}(t)]^T, \tag{6}$$

with $h_i = h(t_i)$.

3. Operational matrices of fractional and ordinary derivative

In this section, the method for the calculation of the ordinary and Caputo fractional derivative operators for Chebyshev cardinal functions is explained.

Definition 3.1. ([20]) Let the function $f(t)$ be q-times differentiable on the interval $[0,t_f]$. The Caputo fractional derivative of order $q-1<\alpha\le q$ of the function $f(t)$ is defined as follows:

$$ {}_0^C D_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(q-\alpha)} \displaystyle\int_0^t (t-s)^{q-\alpha-1} \dfrac{d^q f(s)}{ds^q}\, ds, & q-1<\alpha<q, \\ \dfrac{d^q f(t)}{dt^q}, & \alpha = q. \end{cases} \tag{7}$$

Corollary 3.2. *For $q-1<\alpha<q$ and $k \in \mathbb{R}^+ \cup \{0\}$, we have*

$$ {}_0^C D_t^\alpha t^k = \begin{cases} 0, & k<q, \\ \dfrac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}\, t^{k-\alpha}, & k \ge q. \end{cases} \tag{8}$$

Theorem 3.3. *([11]) Suppose $q-1<\alpha\le q$ and $\Phi_m(t)$ is the vector given in (6). Thus, we get*

$$ {}_0^C D_t^\alpha \Phi_m(t) \simeq \mathbf{D}^{(\alpha)}\, \Phi_m(t), \tag{9}$$

such that $\mathbf{D}^{(\alpha)}$ is a square matrix of order $m+1$ defined as

$$\mathbf{D}^{(\alpha)} = \begin{pmatrix} d_{00}^{(\alpha)} & d_{01}^{(\alpha)} & \cdots & d_{0m}^{(\alpha)} \\ d_{10}^{(\alpha)} & d_{11}^{(\alpha)} & \cdots & d_{1m}^{(\alpha)} \\ \vdots & \vdots & \ddots & \vdots \\ d_{m0}^{(\alpha)} & d_{m1}^{(\alpha)} & \cdots & d_{mm}^{(\alpha)} \end{pmatrix},$$

with the coefficients

$$d_{ij}^{(\alpha)} = \frac{1}{\gamma_i} \sum_{k=0}^{m-1} \sum_{l=q}^{m-k} b_{ik} \frac{\Gamma(m-k+1)}{\Gamma(m-k-l+1)\Gamma(l-\alpha+1)} t_j^{l-\alpha}. \tag{10}$$

For example, for $\alpha = 0.95$, $t_f = 1$ and $m = 4$, result in

$$\mathbf{D}_t^{(0.95)} = \begin{pmatrix} -0.96057 & -0.26174 & 0.049831 & -0.059227 & 0.090912 \\ 1.3947 & -0.02523 & -0.29353 & 0.16856 & -0.32024 \\ -0.66388 & 0.40286 & 0.038101 & -0.44507 & 0.63339 \\ 0.33262 & -0.16372 & 0.26982 & 0.13404 & -1.4115 \\ -0.10285 & 0.047821 & -0.064217 & 0.2017 & 1.0074 \end{pmatrix}.$$

Theorem 3.4. *([11]) The ordinary derivative of order r of the vector $\Phi_m(t)$ expressed in (6) is calculated by the following relation:*

$$\frac{d^r \Phi_m(t)}{dt^r} = \mathbf{D}_t^{(r)} \Phi_m(t), \tag{11}$$

in which $\mathbf{D}_t^{(r)}$ is a square matrix of order $(m+1)$ with the coefficients described as

$$d_{ij}^{(r)} = \frac{1}{\gamma_i} \sum_{l=0}^{m-r} b_{il}(m-l)(m-l-1)\cdots(m-l-r+1)t_j^{m-l-r}, \qquad i,j = 1,2,\cdots,m+1.$$

To better understand, the procedure is concisely explained in the following step-by-step algorithm:

Algorithm of the operator matrix of fractional derivative

Input: The natural numbers m, n and q; the constant $t_f, \in \mathbb{R}^+$, $\mu \in (q-1, q)$ and $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \mu$.

Step 1: Compute $\theta_k = -\cos\left(\frac{(2k-1)\pi}{2(m+1)}\right)$ and $t_k = \frac{t_f}{2}(1+\theta_k)$ for $k = 1, 2, \ldots, m+1$.

Step 2: Calculate $d_{ij}^{(\alpha)}$ using the relation (10) for $i, j = 1, 2, \cdots, m+1$.

Output: The operation matrix is: ${}_0^C D_t^{\alpha} \Phi_m(t) \simeq \mathbf{D}^{(\alpha)} \Phi_m(t)$.

For example, for $r = 1, 2$, $t_f = 1$ and $m = 4$, we have

$$\mathbf{D}_t^{(1)} = \begin{pmatrix} -0.99596 & -0.21029 & 0.064984 & -0.049643 & 0.10515 \\ 1.4414 & -0.089806 & -0.27528 & 0.17013 & -0.34026 \\ -0.68052 & 0.42058 & 0 & -0.42058 & 0.68052 \\ 0.34026 & -0.17013 & 0.27528 & 0.089806 & -1.4414 \\ -0.10515 & 0.049643 & -0.064984 & 0.21029 & 0.99596 \end{pmatrix},$$

and

$$\mathbf{D}_t^{(2)} = \begin{pmatrix} 0.61666 & 0.26933 & -0.027331 & 0.0039876 & 0.18733 \\ -1.284 & -0.45666 & 0.18733 & -0.027331 & -0.58933 \\ 1.0693 & 0.21067 & -0.32 & 0.21067 & 1.0693 \\ -0.58933 & -0.027331 & 0.18733 & -0.45666 & -1.284 \\ 0.18733 & 0.0039876 & -0.027331 & 0.26933 & 0.61666 \end{pmatrix}.$$

4. The suggested strategy

In this section, a numerical method based on Chebyshev cardinal functions is presented to solve nonlinear multi-order fractional differential equations. First, the unknown function of the problem (1) using the considered basis functions is approximated as

$$\vartheta(t) \simeq \sum_{i=1}^{m+1} \upsilon_i \phi_i(t) \triangleq \Upsilon^T \Phi_m(t), \tag{12}$$

such that $\Upsilon^T = [\upsilon_1 \ \upsilon_2 \ \cdots \ \upsilon_{m+1}]$ is the vector of unknown coefficients. Utilizing Theorem 3.3 and the approximation above, we achieve

$$\begin{cases} {}_0^C D_t^{\mu} \vartheta(t) \simeq \Upsilon^T \mathbf{D}_t^{(\mu)} \Phi_m(t), \\ {}_0^C D_t^{\alpha_i} \vartheta(t) \simeq \Upsilon^T \mathbf{D}_t^{(\alpha_i)} \Phi_m(t), \quad i = 1, 2, \cdots, n. \end{cases} \tag{13}$$

Based on Theorem 3.4 and the approximation offered in (12), we conclude that

$$\vartheta^{(i)}(t) \simeq \Upsilon^T \mathbf{D}_t^{(i)} \Phi_m(t), i = 1, 2, \cdots, q-1. \tag{14}$$

By substituting (12) and (13) into equation (1), we get

$$Res(t) = \Upsilon^T \mathbf{D}_t^{(\mu)} \Phi_m(t) + \sum_{i=1}^{n} \eta_i(t) \Upsilon^T \mathbf{D}_t^{(\alpha_i)} \Phi_m(t) + \eta_{n+1}(t) \Upsilon^T \Phi_m(t) - f(t) \simeq 0. \tag{15}$$

Next, considering the initial conditions (2) and the relations (12) and (14), results in

$$\begin{cases} \mathbf{E}_0 \triangleq \Upsilon^T \Phi_m(0) - \xi_0 \simeq 0, \\ \mathbf{E}_1 \triangleq \Upsilon^T \mathbf{D}_t^{(1)} \Phi_m(0) - \xi_1 \simeq 0, \\ \vdots \\ \mathbf{E}_{q-1} \triangleq \Upsilon^T \mathbf{D}_t^{(q-1)} \Phi_m(0) - \xi_{q-1} \simeq 0. \end{cases} \tag{16}$$

Employing the relations obtained in (15) and (16), a linear system of order $(m+1)$ is achieved as

$$\begin{cases} Res(t_i) \simeq 0, & i = q, q+1, \cdots, m, \\ \mathbf{E}_i \simeq 0, & i = 0, 1, \cdots, q-1. \end{cases} \tag{17}$$

By substituting these values into relation, the values of the vector of unknown coefficients are obtained. By substituting these values into relation (12), we obtain the approximate solution to the problem under investigation. It should be noted that the above linear system was solved using the "Solve" command in MATLAB, we obtain the approximate solution to the problem under investigation. In this project, a laptop with the following specifications was used: Intel Core i7 processor, 16GB RAM, and 512GB SSD.

5. Analysis of convergence

This section is devoted to investigating the analysis of convergence for the method expressed in Section 4. Assume $F(t,\vartheta(t)):[0,t_f]\to\mathbb{R}$ is a continuous function and satisfies condition

$$\left\|F(t,\vartheta(t))-F\left(t,\widehat{\vartheta}(t)\right)\right\|_2\le\mathcal{F}\left\|\vartheta(t)-\widehat{\vartheta}(t)\right\|_2,\tag{18}$$

for every $t\in[0,t_f]$ and $\vartheta,\widehat{\vartheta}\in\mathbb{R}$ and the positive constant $\mathcal{F}$.

Theorem 5.1. *Assume $\vartheta(t)\in C^{\varsigma+1}[0,t_f]$ and $\vartheta^*(t)$ are the best approximation of $\vartheta(t)$ in the space $\mathbb{T}$, where*

$$\mathbb{T}=span\{\phi_1(t),\ \phi_1(t),\ \cdots,\phi_{m+1}(t)\}.$$

Then, we have the error bound

$$\|\vartheta(t)-\vartheta^*(t)\|_2\le\frac{N\,M_1}{(\varsigma+1)!},\tag{19}$$

where $N=\max_{r=0,1,\dots,\varsigma+1}\left|\vartheta^{(r)}(t)\right|$ and $M_1=\sqrt{\dfrac{t_f^{2\varsigma+3}}{(2\varsigma+3)}}$.

Proof. Since $\vartheta(t)\in C^{\varsigma+1}[0,t_f]$, then

$$\exists N>0:\forall t\in[0,t_f],\ \left|\vartheta^{(r)}(t)\right|\le N,\quad r=0,1,\dots,\varsigma+1.\tag{20}$$

Using the Taylor expansion, we expand $\vartheta(t)$ as follows:

$$\vartheta(t)=\sum_{r=0}^{\varsigma}\frac{1}{r!}\left(t\frac{d}{dt}\right)^r\vartheta(t)\,|_{(t)=0}+\frac{1}{(\varsigma+1)!}\left(t\frac{d}{dt}\right)^{\varsigma+1}\vartheta(t)\,\Big|_{(t)=(\widehat{t})},\tag{21}$$

such that $\widehat{t}\in[0,t_f]$. Let

$$\widehat{\vartheta}(t)=\sum_{r=0}^{\varsigma}\frac{1}{r!}\left(t\frac{d}{dt}\right)^r\vartheta(t)\,|_{(t)=(0)}.\tag{22}$$

Thus, there is a function $\vartheta^*(t)\in\mathbb{T}$ that satisfies the relation

$$\begin{aligned}\|\vartheta(t)-\vartheta^*(t)\|_2&\le\left\|\vartheta(t)-\widehat{\vartheta}(t)\right\|_2\\&=\left\|\frac{1}{(\varsigma+1)!}\left(t\frac{d}{dt}\right)^{\varsigma+1}\vartheta(\widehat{t})\right\|_2\\&=\left[\int_0^{t_f}\left(\frac{1}{(\varsigma+1)!}\left(t\frac{d}{dt}\right)^{\varsigma+1}\vartheta\left(\widehat{t}\right)\right)^2dt\right]^{\frac{1}{2}}.\end{aligned}\tag{23}$$

Moreover, we have

$$\left(t\frac{d}{dt}\right)^{\varsigma+1}\vartheta(\widehat{t})\le Nt^{\varsigma+1}.\tag{24}$$

So, from relations (23) and (24), we obtain

$$\begin{aligned}
&\left[\int_0^{t_f}\left(\frac{1}{(\varsigma+1)!}\left(t\frac{d}{dt}\right)^{\varsigma+1}\vartheta(\hat{t})\right)^2 dt\right]^{\frac{1}{2}}\\
&\leq\left[\int_0^{t_f}\left(\frac{N}{(\varsigma+1)!}t^{\varsigma+1}\right)^2 dt\right]^{\frac{1}{2}}\\
&=\frac{N}{(\varsigma+1)!}\underbrace{\sqrt{\frac{t_f^{2\varsigma+3}}{(2\varsigma+3)}}}_{M_1}\\
&=\frac{N\,M_1}{(\varsigma+1)!}.
\end{aligned}\tag{25}$$

Finally, using equations (25) and (23), we obtain

$$\|\vartheta(t)-\vartheta^*(t)\|_2\leq\frac{N\,M_1}{(\varsigma+1)!}.$$

□

Corollary 5.2. *Due to the above Theorem, we have*

$$\left\|{}_0^C D_t^{\mu}\vartheta(t)-{}_0^C D_t^{\mu}\vartheta^*(t)\right\|_2\leq N\,M_2,\tag{26}$$

in which

$$M_2=\sqrt{\frac{t_f^{2\varsigma-2\mu+1}}{\Gamma(\varsigma-\mu+2)(2\varsigma-2\mu+1)}}.$$

Proof. The proof of this Corollary is similar to the above Theorem. Therefore, we leave it. □

In the continuation, we consider the multi-order fractional problem given in (1) in the following form:

$$\underbrace{{}_0^C D_t^{\mu}\vartheta(t)+\sum_{i=1}^{n}\eta_i(t)\,{}_0^C D_t^{\alpha_i}\vartheta(t)+\eta_{n+1}(t)\,\vartheta(t)-f(t)}_{F(x,t,\vartheta(t))}\simeq 0.\tag{27}$$

Theorem 5.3. *Let $\vartheta(t)$ and $\widehat{\vartheta}(t)$ be the analytical and approximation solutions of Caputo fractional problem (27) , respectively. So, the following relation is correct*

$$\|ER(t)\|_2\leq\frac{NM_3}{(\varsigma+1)!},\tag{28}$$

where $M_3 = (M_2(\varsigma+1)! + M_1)$ *and* $ER(t)$ *is the solution error, which is defined as*

$$ER(t) = \vartheta(t) - \widehat{\vartheta}(t).$$

Proof. By substituting $\widehat{\vartheta}(t)$ into the Caputo fractional problem (27), we obtain

$$ {}_{0}^{C}D_t^{\mu}\widehat{\vartheta}(t) + F\left(x, t, \widehat{\vartheta}(t)\right) + \mathrm{ER}(t) \simeq 0. \tag{29}$$

Based on the problems expressed in (27) and (28), we get

$$\|\mathrm{ER}(t)\|_2 \leq \left\| {}_{0}^{C}D_t^{\mu}\vartheta(t) - {}_{0}^{C}D_t^{\mu}\widehat{\vartheta}(t)\right\|_2 + \left\|F\left(t, \vartheta(t)\right) - F\left(t, \widehat{\vartheta}(t)\right)\right\|_2. \tag{30}$$

Moreover, using Assumption 5 and Theorem 5.1, we have

$$\left\|F\left(t, \vartheta(t)\right) - F\left(t, \widehat{\vartheta}(t)\right)\right\|_2 \leq \mathcal{F}\left\|\vartheta(t) - \widehat{\vartheta}(t)\right\|_2 \leq \frac{\mathcal{F}NM_1}{(\varsigma+1)!}, \tag{31}$$

in which $\mathcal{F}$ is a constant.
From relations (30) and (31) along with Corollary 5.2, we achieve

$$\|\mathrm{ER}(t)\|_2 \leq NM_2 + \frac{N\,M_1}{(\varsigma+1)!} = \frac{N}{(\varsigma+1)!}\underbrace{(M_2(\varsigma+1)! + M_1)}_{M_3}.$$

□

Remark 5.4. It is important to note that the error analysis and convergence results established in this work provide a theoretical upper bound on the approximation error. While these bounds guarantee the convergence and reliability of the proposed Chebyshev-cardinal method, they do not constitute a sharp spectral estimate, which would provide a tighter, more exact characterization of the error decay rate in practice.

Remark 5.5. While the preceding theoretical analysis provides a rigorous upper bound on the approximation error, it is notably conservative. In the subsequent numerical examples, we observe fast, exponential convergence. This behavior occurs because Chebyshev cardinal functions inherently possess spectral approximation properties. When the exact solution of the fractional differential equation is sufficiently smooth, the numerical approximation leverages this smoothness, leading to an exponential decay in the error that is much faster than the algebraic decay suggested by the general theoretical upper bounds.

6. **Numerical examples**

In this section, we examine several numerical examples to determine the accuracy and efficiency of the proposed method. To evaluate the accuracy of the results, we employ the following formula:

$$e_m(t) = \left|\vartheta(t) - \widetilde{\vartheta}_m(t)\right|, \tag{32}$$

in which ϑ and $\widetilde{\vartheta}_m$ are the exact and approximate solutions, respectively. In addition, the numerical convergence order (CO) of the suggested strategy is calculated via the following relation:

$$CO = \left| \frac{\ln\left(\dfrac{ER(m_2)}{ER(m_1)}\right)}{\ln\left(\dfrac{m_2}{m_1}\right)} \right|,$$

with m_1 and m_2 as the number of basis functions employed in the first and second performances, respectively.

Example 6.1. *In this example, we consider the following multi-order fractional differential equation*

$$\begin{cases} {}_0^C D_t^{\mu} \vartheta(t) + \exp(t)\, {}_0^C D_t^{\alpha} \vartheta(t) + \exp(-t)\, \vartheta(t) = f(t), \\ t \in [0, 6],\ 1 < \mu \le 2,\ 0 < \alpha \le 1, \\ \vartheta(0) = \vartheta'(0) = 0, \end{cases}$$

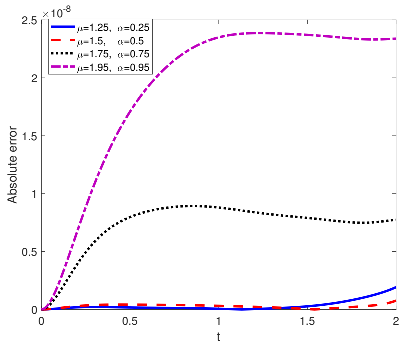
in which

$$f(t) = \frac{8!}{\Gamma(9-\mu)} t^{8-\mu} + \frac{8!}{\Gamma(9-\alpha)} t^{8-\alpha} \exp(t) + \exp(-t) t^8.$$

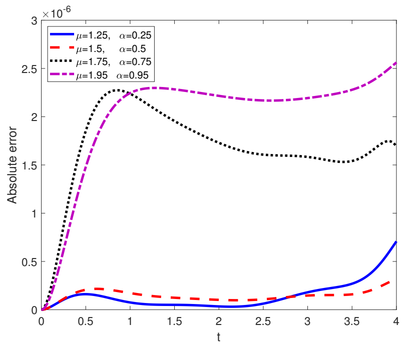
The exact solution to this fractional problem is $u(t) = t^8$. This multi-order fractional differential equation was solved using the proposed method in the previous section. The absolute error for $m = 10$ and various values of μ, α, and t_f are provided in Figure 1. Moreover, the best choice is for $m = 10$. The CPU time(Sec) of the method are reported in Table 1. According to Table 1, we find that as the number of bases and time interval increase, the running time becomes longer. The results clearly indicate the efficiency and high accuracy of the proposed method.

Table 1. Report of CPU time for four different intervals of $[0, t_f]$ with $m = 10$ in Example 6.1.

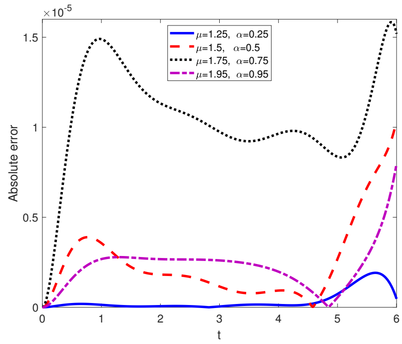
m	$[0, t_f]$	*CPU time*
5	$t \in [0, 2]$	$1.51s$
	$t \in [0, 4]$	$2.31s$
	$t \in [0, 6]$	$3.64s$
10	$t \in [0, 2]$	$5.36s$
	$t \in [0, 4]$	$7.67s$
	$t \in [0, 6]$	$9.56s$
20	$t \in [0, 2]$	$15.01s$
	$t \in [0, 4]$	$30.54s$
	$t \in [0, 6]$	$78.92s$



(A) $t \in [0, 2]$



(B) $t \in [0, 4]$



(C) $t \in [0, 6]$

FIGURE 1. Graphs of the absolute error function for $m = 10$ over different intervals and for several fractional order values ($\mu = 1.25, 1.5, 1.75, 1.95,\ \alpha = 0.25, 0.5, 0.75, 0.95$) in Example 6.1.

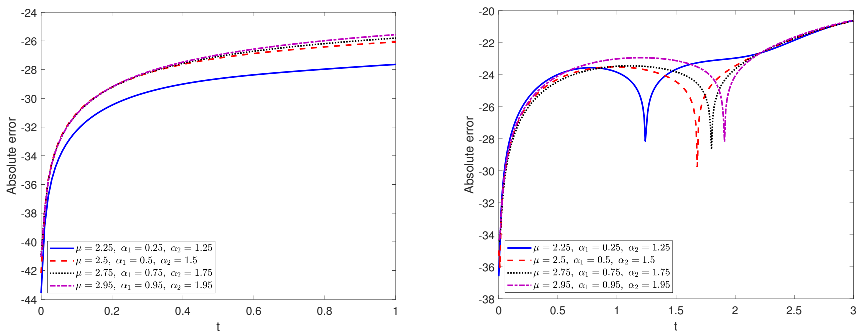
Example 6.2. *In this example, we consider the following multi-order fractional differential equation:*

$$\begin{cases} {}^C_0D^{\mu}_t \vartheta(t) + \sin(t)\, {}^C_0D^{\alpha_1}_t \vartheta(t) + \cos(t)\, {}^C_0D^{\alpha_2}_t \vartheta(t) + t^4\vartheta(t) = f(t), \quad t \in [0, 20], \\ \vartheta(0) = \vartheta'(0) = \vartheta''(0) = 0, \end{cases}$$

in which $2 < \mu \leq 3,\ 0 < \alpha_1 \leq 1,\ 1 < \alpha_2 \leq 2$ *and*

$$f(t) = \frac{6!}{\Gamma(7-\mu)} t^{6-\mu} + \frac{6!}{\Gamma(7-\alpha_1)} t^{6-\alpha_1}\sin(t) + \frac{6!}{\Gamma(7-\alpha_2)} t^{6-\alpha_2}\cos(t) + t^{10}.$$

The exact solution to this fractional problem is $u(t) = t^6$. *This multi-order fractional differential equation was solved using the proposed method in the* *previous section. The absolute error for* $m = 8$ *and various values of* μ, α_i $(i = 1, 2)$, *and* t_f *are given in Figure 2. Also, the absolute error and approximate solution for* $m = 8$, $\mu = 2.25$, $\alpha_1 = 0.25$, *and* $\alpha_2 = 1.25$ *over the intervals* $[0, 10]$ *and* $[0, 20]$ *are shown in Figures 3 and 4. Moreover, the best choice is for* $m = 8$. *The results clearly demonstrate the efficiency and high accuracy of the proposed method.*



(A) $t \in [0, 1]$ (B) $t \in [0, 3]$

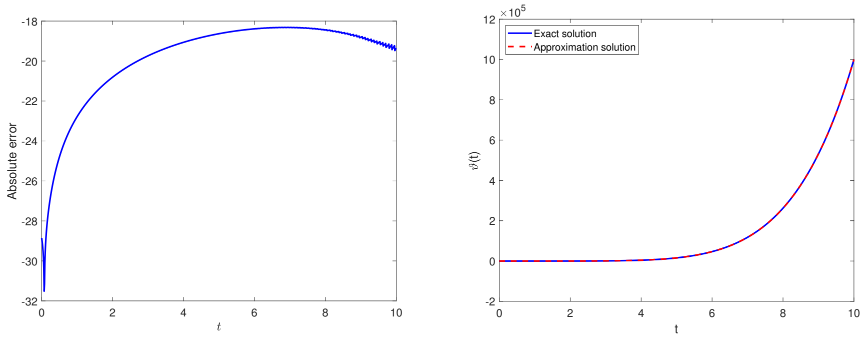
FIGURE 2. Absolute error function (log-scale format) Graphs for $m = 8$ over different intervals and for several fractional order values ($\mu = 2.25, 2.5, 2.75, 2.95$, $\alpha_1 = 0.25, 0.5, 0.75, 0.95$, $\alpha_2 = 1.25, 1.5, 1.75, 1.95$) in Example 6.2.

Example 6.3. *In this example, we consider the following multi-order fractional differential equation:*

$$\begin{cases} {}^{C}_{0}D^{\mu}_{t}\vartheta(t) + \exp(-t)\,{}^{C}_{0}D^{\alpha_1}_{t}\vartheta(t) + \cos(t)\,{}^{C}_{0}D^{\alpha_2}_{t}\vartheta(t) + \exp(t)\,{}^{C}_{0}D^{\alpha_3}_{t}\vartheta(t) \\ +\sin(t)\vartheta(t) = f(t), \\ \vartheta(0) = \vartheta'(0) = \vartheta''(0) = \vartheta'''(0) = 0, \end{cases}$$

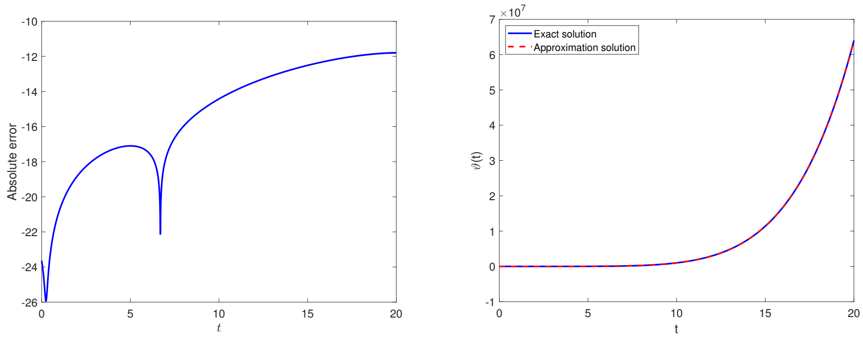
such that $t \in [0, 10]$, $3 < \mu \leq 4$, $0 < \alpha_1 \leq 1$, $1 < \alpha_2 \leq 2$, $2 < \alpha_3 \leq 3$ *and*

$$\begin{aligned} f(t) &= \frac{5!}{\Gamma(6-\mu)}t^{5-\mu} + \frac{7!}{\Gamma(8-\mu)}t^{7-\mu} \\ &+ \left(\frac{5!}{\Gamma(6-\alpha_1)}t^{5-\alpha_1} + \frac{7!}{\Gamma(8-\alpha_1)}t^{7-\alpha_1}\right)\exp(-t) \\ &+ \left(\frac{5!}{\Gamma(6-\alpha_2)}t^{5-\alpha_2} + \frac{7!}{\Gamma(8-\alpha_2)}t^{7-\alpha_2}\right)\cos(t) \\ &+ \left(\frac{5!}{\Gamma(6-\alpha_3)}t^{5-\alpha_3} + \frac{7!}{\Gamma(8-\alpha_3)}t^{7-\alpha_3}\right)\exp(t) + \sin(t)\left(t^5 + t^7\right). \end{aligned}$$



(A) Absolute error (B) Exact and approximate solutions

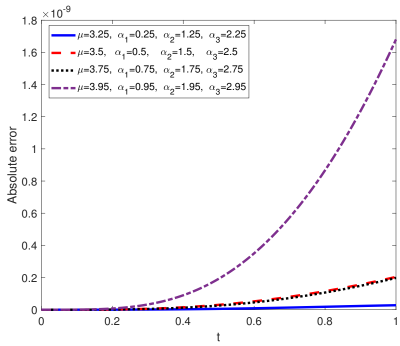
FIGURE 3. Absolute error function (log-scale format) Graphs for $m = 8$, $\mu = 2.25$, $\alpha_1 = 0.25$ and $\alpha_2 = 1.25$ and over interval $[0, 10]$ in Example 6.2.



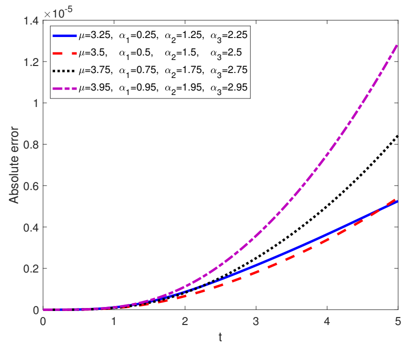
(A) Absolute error (B) Approximate solution

FIGURE 4. Absolute error function (log-scale format) Graphs for $m = 8$, $\mu = 2.25$, $\alpha_1 = 0.25$ and $\alpha_2 = 1.25$ and over interval $[0, 20]$ in Example 6.2.

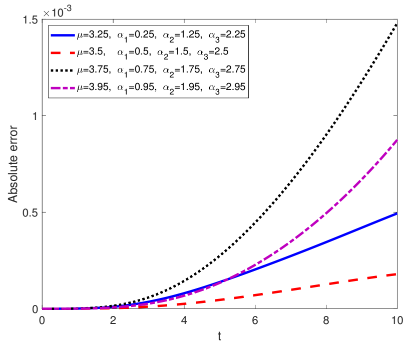
The exact solution to this fractional problem is $\vartheta(t) = t^5 + t^7$. This multi-order fractional differential equation was solved using the proposed method in the previous section. The absolute error for $m = 9$ and various values of μ, α and t_f are presented in Figure 5. Moreover, the best choice is for $m = 9$. The results clearly indicate the efficiency and high accuracy of the proposed method.



(A) $t \in [0, 1]$



(B) $t \in [0, 5]$



(C) $t \in [0, 10]$

FIGURE 5. Absolute error function Graphs for $m = 9$ over different intervals and for several fractional order values ($\mu = 3.25, 3.5, 3.75, 3.95$, $\alpha_1 = 0.25, 0.5, 0.75, 0.95$, $\alpha_2 = 1.25, 1.5, 1.75, 1.95$, $\alpha_3 = 2.25, 2.5, 2.75, 2.95$) in Example 6.3.

Example 6.4. *In this example, we consider the following fractional differential equation*

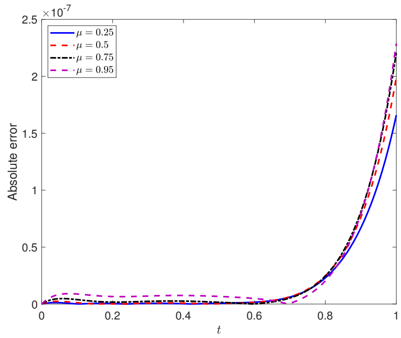
$$\begin{cases} {}_0^C D_t^{\mu} \vartheta(t) + t^4 \, \vartheta(t) = f(t), \quad t \in [0, 5], \; 0 < \mu \leq 1, \\ \vartheta(0) = 1, \end{cases}$$

in which

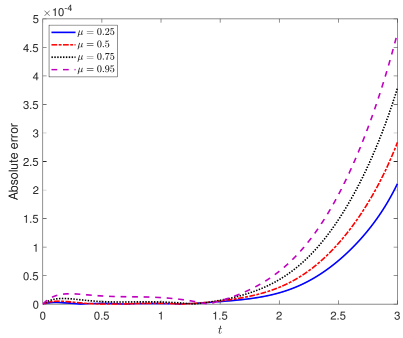
$$f(t) = -t^{1-\mu} \sum_{r=0}^{15} \frac{(-1)^r t^r}{\Gamma(r - \mu + 2)} + t^4 \exp(-t).$$

The exact solution to this fractional problem is $\vartheta(t) = \exp(-t)$. *This multi-order fractional differential equation was solved using the proposed method in the previous section. The absolute error for* $m = 8$ *and various values of* μ

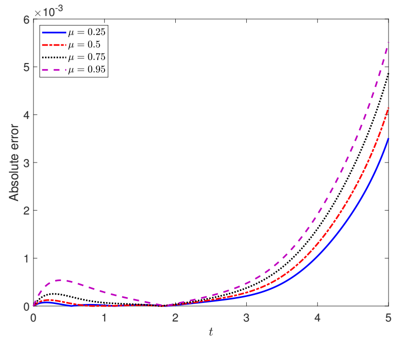
and t_f *are provided in Figure 6. Moreover, approximation solution and exact solution are shown in Figure 7. Moreover, the best choice is for* $m = 10$*. The ER and CO of the method for some values of* t *and* m *are summarized in Table 2. The results clearly indicate the efficiency and high accuracy of the proposed method.*



(A) $t \in [0, 1]$



(B) $t \in [0, 3]$

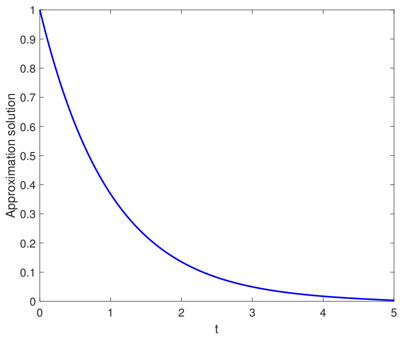


(C) $t \in [0, 5]$

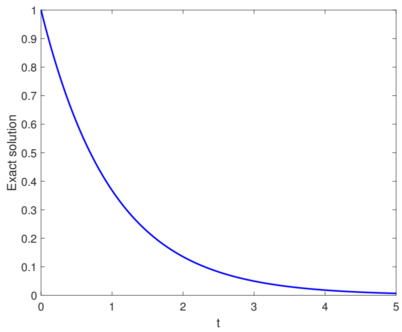
FIGURE 6. Graphs of the absolute error function for $m = 10$ over different intervals and for several fractional order values ($\mu = 0.25, 0.5, 0.75, 0.95$) in Example 6.4.

Example 6.5. *In this example, we consider the following multi-order fractional differential equation*

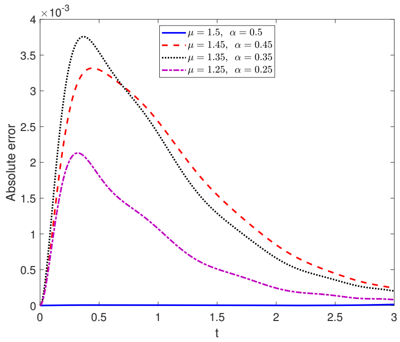
$$\begin{cases} {}_0^C D_t^{\mu} \vartheta(t) + {}_0^C D_t^{\alpha} \vartheta(t) + \exp(-t)\,\vartheta(t) = f(t), \quad t \in [0, 3],\ 1 < \mu \leq 2,\ 0 < \alpha \leq 1, \\ \vartheta(0) = \vartheta'(0) = 0, \end{cases}$$



(A) $t \in [0, 5]$



(B) $t \in [0, 5]$



(C) $t \in [0, 3]$

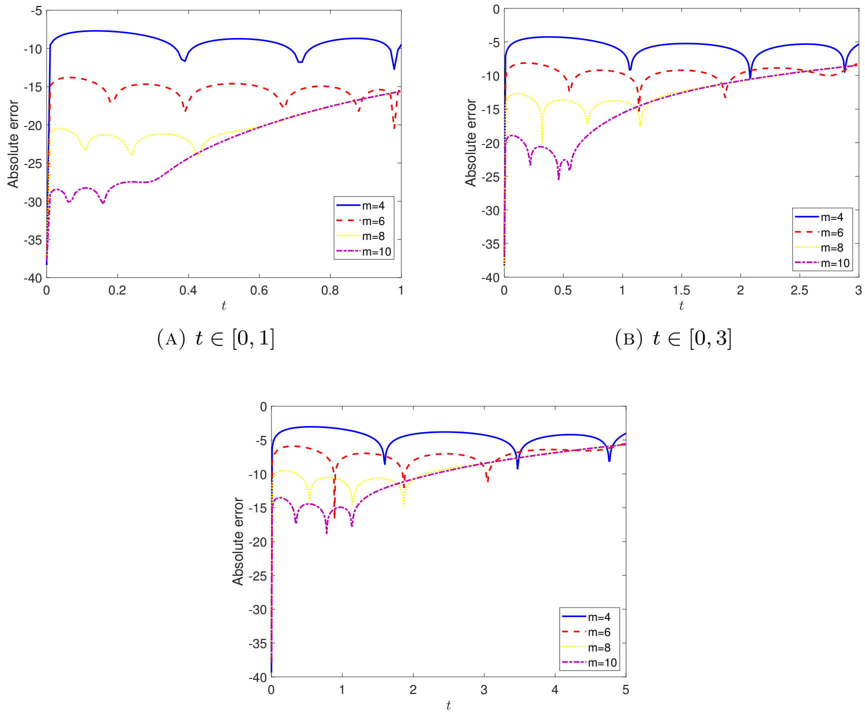
FIGURE 7. Graphs of the approximation solution and exact solution and error functions for $m = 10$ in Example 6.4.

TABLE 2. Values of absolute error (ER) and CO for $\mu = 0.25$ in Example 6.4.

m	$t=1$		$t=2$		$t=3$		$t=4$		$t=5$	
	ER	CO	ER	CO	ER	CO	ER	CO	ER	CO
4	4.7121E-1	-	5.0133E-2	-	2.197E-1	-	8.4166E-3	-	1.1279E-1	-
6	2.1193E-2	7.6496	7.9827E-3	4.5316	8.7144E-3	7.9595	1.1601E-3	4.8875	1.3900E-2	5.1635
8	7.4318E-6	27.6540	2.3109E-3	4.3091	1.0074E-3	7.5000	4.7155E-4	3.1293	1.9760E-3	6.7811
10	5.2170E-7	11.9050	3.6648E-6	28.8900	7.4823E-4	1.3329	4.9520E-5	10.1000	1.9672E-4	10.3390

in which

$$f(t) = \frac{\Gamma(9+\mu+\alpha)}{\Gamma(9+\alpha)} t^{8+\alpha} + \frac{\Gamma(9+\mu+\alpha)}{\Gamma(9+\mu)} t^{8+\mu} + \exp(-t) t^{8+\mu+\alpha}.$$



(A) $t \in [0, 1]$

(B) $t \in [0, 3]$

(C) $t \in [0, 5]$

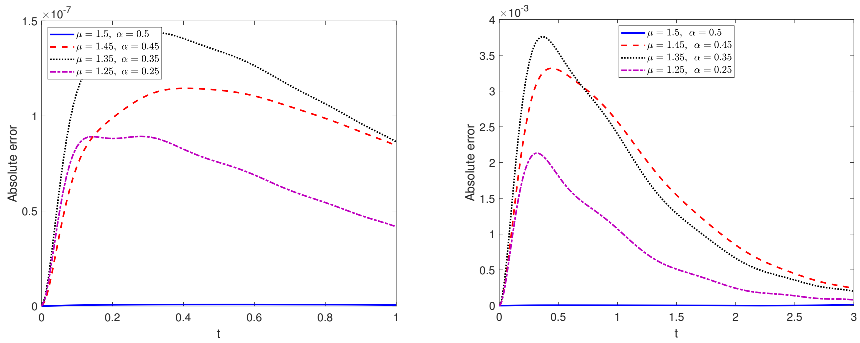
FIGURE 8. Graphs of the absolute error function for $m = 4, 6, 8, 10$ and $\mu = 0.25$ over different intervals in Example 6.4.

The exact solution to this fractional problem is $\vartheta(t) = t^{8+\alpha+\mu}$. *This multi-order fractional differential equation was solved using the proposed method in the previous section. The absolute error for* $m = 12$ *and various values of* μ, α, *and* t_f *are provided in Figure 9. Moreover, the best choice is for* $m = 12$. *The results clearly indicate the efficiency and high accuracy of the proposed method.*

Example 6.6. *In this example, we consider the following fractional differential equation*

$$\begin{cases} {}_0^C D_t^{\mu} \vartheta(t) + \sin(t)\, \vartheta(t) = f(t), \quad t \in [0, 1],\ 0 < \mu \le 1, \\ \vartheta(0) = 0, \end{cases}$$

The exact solution to this fractional problem is $\vartheta(t) = \ln(t+1)$. *This fractional differential equation was solved using the proposed method in the previous section. The absolute error for* $m = 10$ *and various values of* μ, *with* $t_f = 1$



(A) $t \in [0, 1]$ (B) $t \in [0, 3]$

FIGURE 9. Graphs of the absolute error function for $m = 12$ over different intervals and for several fractional order values ($\mu = 1.5, 1.45, 1.35, 1.25,\ \alpha = 0.5, 0.45, 0.35, 0.25$) in Example 6.5.

are provided in Figure 10. The results clearly indicate the efficiency and high accuracy of the proposed method.

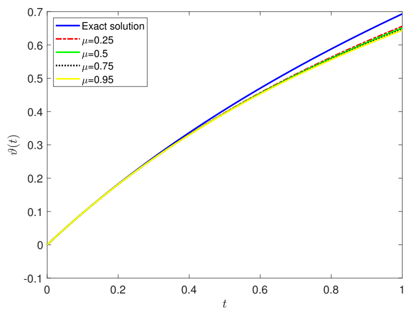


FIGURE 10. Graph of the approximation solution and exact solution for $m = 10$ over interval $[0, 1]$ in Example 6.6.

7. Conclusion

In this paper, a cardinal numerical method was presented to solve multi-order fractional differential equations with Caputo derivatives. Shifted Chebyshev cardinal functions were utilized as basic functions. The corresponding Caputo fractional derivative operator matrix for these cardinal functions was computed. By approximating the unknown expression of the problem in terms of the shifted Chebyshev cardinal functions, applying their fractional derivative operator matrix, and using the collocation method, solving the equation at hand was reduced to solving a system of algebraic equations. Finally, the accuracy and efficiency of the proposed method were examined by solving several numerical examples. The results indicate that the method presented in this paper is an efficient and highly accurate approach to solving such multi-order fractional differential equations. As future research direction, the basis functions utilized in this paper can be utilized to solve fractional versions of various applied problems, such as the Rosenau equation, Rosenau-Hyman equation, Ginzburg-Landau equation, etc. Also, it is possible to modify the numerical strategy provided in this study for the similar equations with other fractional derivatives, such as ψ-Caputo derivative, Caputo-Hadamard derivative, and tempered Caputo derivative.

8. Author Contributions

All authors contributed equally to this paper.

9. Data Availability Statement

No data was used for the research described in the article.

10. Aknowledgement

We would like to thank the reviewers for their thoughtful comments and efforts towards improving our manuscript.

11. Ethical considerations

The authors avoided data fabrication and falsification.

12. Funding

This research did not receive any specific grant from the public, commercial, or not-for-profit funding agencies.

13. Conflict of interest

The authors declare no conflict of interest.

References

[1] Abdelkawy, M. A., Izadi, M., and Adel, W. (2024). Robust and accurate numerical framework for multi-dimensional fractional-order telegraph equations using Jacobi/Jacobi-Romanovski spectral technique. Boundary Value Problems, 2024(1), 131. https://doi.org/10.1186/s13661-024-01831-5

[2] Abdelkawy, M., Izadi, M., and Adel, W. (2024). Robust and accurate numerical framework for multi-dimensional fractional-order telegraph equations using Jacobi/Jacobi-Romanovski spectral technique. Boundary Value Problems, 2024(11), 131. https://doi.org/10.1186/s13661-024-01831-5

[3] Algazaa, S. A. T., and Saeidian, J. (2023). Solving nonlinear multi-order fractional differential equations using Bernstein polynomials. IEEE Access, 11, 128032–128043. https://doi.org/10.1109/ACCESS.2023.3330000

[4] Avazzadeh, Z., and Heydari, M. (2017). Chebyshev cardinal functions for solving age-structured population models. International Journal of Applied and Computational Mathematics, 3, 2139–2149. https://doi.org/10.1007/s40819-016-0287-3

[5] Chegini, T. G., Dinmohammadi, A., Jafarabadi, A., and Shivanian, E. (2025). Analysis of a time-dependent source function for the heat equation with nonlocal boundary conditions through a local meshless procedure. Computational and Applied Mathematics, 44(6), 282.

[6] Chegini, T. G., Fairooz, M. Z., Hosseinzadeh, N., and Shivanian, E. (2025). A robust RBF-FD technique combined with polynomial enhancements for valuing European options in jump-diffusion frameworks. International Journal of Dynamics and Control, 13(6), 212.

[7] Ganie, A. H., Mofarreh, F., and Khan, A. (2023). A fractional analysis of Zakharov–Kuznetsov equations with the Liouville–Caputo operator. Axioms, 12(6), 609. https://doi.org/10.3390/axioms12060609

[8] Heydari, M. H., Razzaghi, M., and Rouzegar, J. (2022). Chebyshev cardinal polynomials for delay distributed-order fractional fourth-order sub-diffusion equation. Chaos, Solitons & Fractals, 162, 112495. https://doi.org/10.1016/j.chaos.2022.112495

[9] Heydari, M. H. (2023). A cardinal approach for two-dimensional modified anomalous space–time fractional sub-diffusion equation. Results in Physics, 49, 106545. https://doi.org/10.1016/j.rinp.2023.106545

[10] Hosseininia, M., Heydari, M. H., and Razzaghi, M. (2023). A hybrid spectral approach based on 2D cardinal and classical second-kind Chebyshev polynomials for time fractional 3D Sobolev equation. Mathematical Methods in the Applied Sciences, 46(18), 18768–18788. https://doi.org/10.1002/mma.9453

[11] Hosseininia, M., Heydari, M. H., Baleanu, D., and Bayram, M. (2025). A hybrid method based on classical/piecewise Chebyshev cardinal functions for multi-dimensional fractional Rayleigh–Stokes equations. Results in Applied Mathematics, 25, 100541.

[12] İdiz, F., Tanoğlu, G., and Aghazadeh, N. (2024). A numerical method based on Legendre wavelet and quasilinearization technique for fractional Lane–Emden type equations. Numerical Algorithms, 95(1), 181–206. https://doi.org/10.1007/s11075-023-01618-8

[13] Izadi, M., Yüzbaşı, Ş., and Adel, W. (2022). Accurate and efficient matrix techniques for solving the fractional Lotka–Volterra population model. Physica A: Statistical Mechanics and its Applications, 600, 127558. https://doi.org/10.1016/j.physa.2022.127558

[14] Izadi, M., and Srivastava, H. M. (2021). A novel matrix technique for multi-order pantograph differential equations of fractional order. Proceedings of the Royal Society A, 477(2253), 20210321. https://doi.org/10.1098/rspa.2021.0321

[15] Izadi, M., and Cattani, C. (2020). Generalized Bessel polynomial for multi-order fractional differential equations. Symmetry, 12(8), 1260. https://doi.org/10.3390/sym12081260

[16] Izadi, M., Ansari, Kh. J., and Srivastava, H. M. (2025). A highly accurate and efficient Genocchi-based spectral technique applied to singular fractional order boundary value problems. Fractional Differential Calculus, 48(1), 905–925.

[17] Jebreen, H. B., and Dassios, I. (2024). The collocation method based on the new Chebyshev cardinal functions for solving fractional delay differential equations. Mathematics, 12(21), 3388. https://doi.org/10.3390/math12213388

[18] Lakestani, M., and Dehghan, M. (2010). The use of Chebyshev cardinal functions for the solution of a partial differential equation with an unknown time-dependent coefficient subject to an extra measurement. Journal of Computational and Applied Mathematics, 235(3), 669–678. https://doi.org/10.1016/j.cam.2010.05.020

[19] Masti, I., and Sayevand, Kh. (2024). On collocation-Galerkin method and fractional B-spline functions for a class of stochastic fractional integro-differential equations. Mathematics and Computers in Simulation, 216, 263–287.

[20] Podlubny, I. (1998). Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, methods of their solution and some of their applications. Academic Press, San Diego. https://doi.org/10.1016/S0076-5392(98)80002-2

[21] Partohaghighi, M., Mortezaee, M., and Akgül, A. (2024). Numerical estimation of the fractional Klein–Gordon equation with discrete Chebyshev polynomials. Alexandria Engineering Journal, 90, 44–53. https://doi.org/10.1016/j.aej.2024.01.020

[22] Saberi Zafarghandi, F., Mohammadi, M., Babolian, E., and Javadi, Sh. (2019). Radial basis functions method for solving the fractional diffusion equations. Applied Mathematics and Computation, 342, 224–246. https://doi.org/10.1016/j.amc.2018.09.045

[23] Solhi, E., Mirzaee, F., and Naserifar, Sh. (2024). Enhanced moving least squares method for solving the stochastic fractional Volterra integro-differential equations of Hammerstein type. Numerical Algorithms, 95(4), 1921–1951. https://doi.org/10.1007/s11075-023-01633-9

[24] Soufivand, F., Soltanian, F., and Mamehrashi, K. (2023). A numerical approach for solving two-dimensional variable-order fractional optimal control problems using the Gegenbauer operational matrix. IMA Journal of Mathematical Control and Information, 40(1), 1–19. https://doi.org/10.1093/imamci/dnac034

[25] Shivani, R., Gupta, S., and Dileep Kumar, S. (2024). Numerical solution of the multi-order fractional differential equation using Legendre wavelets and eigenfunction approach. Partial Differential Equations in Applied Mathematics, 10, 100739. https://doi.org/10.1016/j.padiff.2024.100739

[26] Waleed, A., Hari Mohan, H., Izadi, M., Elsonbaty, A., and El-Mesady, A. (2024). Dynamics and numerical analysis of a fractional-order toxoplasmosis model incorporating human and cat populations. Boundary Value Problems, 2024(1), 152. https://doi.org/10.1186/s13661-024-01852-0

[27] Zanariah Abdul, Z., Nur Amirah, Z., Adem, K., and Zarina Bibi, I. (2022). Numerical solution of fractional differential equations with Caputo derivative using a fractional predict–correct technique. Advances in Continuous and Discrete Models, 2022(1), 26. https://doi.org/10.1186/s13662-022-03634-7

Sara Mirshojaei
Orcid number: 0009-0002-8542-4023
Department of Mathematics
Qazvin Branch, Islamic Azad University
Qazvin, Iran
Email address: Mir.412.sm@gmail.com

Elyas Shivanian
Orcid number: 0000-0001-9383-6779
Department of Applied Mathematics
Imam Khomeini International University
Qazvin, 34148-96818, Iran
Email address: shivanian@sci.ikiu.ac.ir

Bakhtyar Ghayebi
Orcid number: 0009-0001-7560-0332
Department of Mathematics
Qazvin Branch, Islamic Azad University
Qazvin, Iran
Email address: b.ghayebi@qiau.ac.ir

Seyyed Jalal Hosseini Ghoncheh
Orcid number: 0000-0002-3320-0661
Department of Mathematics
Qazvin Branch, Islamic Azad University
Qazvin, Iran
Email address: sjhghoncheh@gmail.com