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RIGHT DC-NOETHERIAN S-POSETS

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Abstract. In the present paper, the right dc-Noetherianity for S-posets is defined and studied. Some fundamental properties of right dc-Noetherian S-posets are presented, and the relation of right dc-Noetherian S-posets with sub S-posets, factor S-posets, products, and coproducts are studied. Finally, the relations between right po-Noetherian S-posets, right dc-Noetherian S-posets and right Noetherian ordered pomonoids are investigated.

Keywords: S-poset, down closed sub S-poset, right dc-Noetherian.
2020 MSC: 06F05, 18G05, 20M30, 20M50.

1. Introduction

The concept of Noetherianity, which is a finiteness condition, is essential in ring and module theory. Noetherian rings were introduced by Noether in the early 1900s and have been extensively studied since then. Hotzel initiated the study of right Noetherian semigroups in [10], and Kozhukhov further studied them in [19]. These semigroups have been extensively studied in [4, 9, 20]. Noetherian acts were introduced by Normak in [21] and have been studied in some papers [6, 7, 13–15, 18].

In [11], the concept of weakly right po-Noetherian ordered semigroups was introduced and has been studied in [24, 26, 27]. Also, right Noetherian ordered semigroups were introduced and studied in [25], and in [16], right po-Noetherian S-posets were defined and studied.

The present paper is devoted to introducing and studying the right dc-Noetherian S-posets. An S-poset is called *right dc-Noetherian* if each down closed sub S-poset of it has a finite generating set. In [26], it is concluded that Noetherianity with respect to down closed right ideals (poideals) of S have a natural behavior with respect to regular injectivity and that is why the author studies right Noetherianity with respect to down closed sub S-posets instead of general sub S-posets. It is proved that an S-poset is right dc-Noetherian if and only if the ascending chain condition holds on its down closed sub S-posets, equivalently, if the maximal condition holds on its down closed sub S-posets. We present an alternative way of defining the right dc-Noetherian property

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using down closed cyclic sub S-posets. Also, we study the behavior of right dc-Noetherianity with some constructions and clarify the relations between right po-Noetherian and right dc-Noetherian S-posets. Finally, the relation between right po-Noetherian S-posets and right Noetherian ordered pomonoids is investigated.

To begin, we provide a brief overview of the definition and key algebraic properties of the category of ordered semigroups and (right) S-posets that will be used later. For further details, refer to [2, 5, 17].

Recall that a (*right*) *S-act* A is a set equipped with a map $\lambda : A \times S \to A$, called its *action* and S is a semigroup (or a monoid with identity 1), such that, denoting $\lambda(a, s)$ by as, we have $a(st) = (as)t$, (and $a1 = a$) for all $a \in A$, and $s, t \in S$.

A *posemigroup* is a semigroup S with a partial order $\leq$ with the property that $s \leq t, s' \leq t'$ imply $ss' \leq tt'$, that is, the binary operation and the partial order are compatible. A *pomonoid* is a posemigroup with an identity element.

A posemigroup S is called *weak finitely generated* if there exists a finite set $X \subseteq S$ such that for each element $s \in S$, $s \leq x_1 x_2 \ldots x_n$ where $x_1, x_2, \ldots, x_n \in X, n \in \mathbb{N}$.

A *right poideal* of a posemigroup S is a (possibly empty) subset I of S if it is both a semigroup right ideal ($IS \subseteq I$) and a poset ideal (that is, a down closed subset of S: $a \leq b, b \in I$ imply $a \in I$).

Let S be a posemigroup and $X \subseteq S$. Then

$$< X >= \downarrow (XS^1) = \{s \in S : \exists x \in X, \exists t \in S^1, s \leq xt\}$$

is the smallest right poideal of S containing X, and it is referred to as the right poideal of S generated by X. $< X >$ is called finitely generated if X is finite, and it is called principal if X is a singleton.

A poset A which is also an S-act over a posemigroup S, with the order-preserving action $\lambda : A \times S \to A$, where $A \times S$ considered as a poset with componentwise order, is called a (*right*) *S-poset*. A sub poset B of an S-poset A is called a sub S-poset of A if for each $b \in B, s \in S$, $bs \in B$.

A possibly empty sub S-poset A of an S-poset B is said to be *down closed* in B if for each $a \in A$ and $b \in B$ with $b \leq a$ we have $b \in A$.

Let S be a posemigroup and B be a down closed sub S-poset of an S-poset A. Then $X \subseteq A$ is called a *generating set* for B if $B =< X >= \downarrow (XS^1) = \{a \in A : \exists x \in X, \exists s \in S^1, a \leq xs\}$. B is called *finitely generated* if it has a finite generating set, and it is called *cyclic* if it has a singleton generating set.

A map between S-posets that is action and order preserving is called an *S-poset map.*

A (right) S-act congruence α on an S-poset B, that is, an equivalence relation on B which is compatible with the S-action, for which the factor act B/α has the structure of an S-poset and the canonical map $B \to B/\alpha$ is an S-poset morphism is called a (*right*) *S-poset congruence.* Let $\alpha \subseteq B \times B$. One can define a relation $\leq_\alpha$ on B as the following:

$$b \leq_\alpha b' \Leftrightarrow b \leq b_1\alpha b'_1 \leq ... \leq b_k\alpha b'_k \leq b',$$

for some natural number k and $b_1, b'_1, \ldots, b_k, b'_k \in B$. Then an S-act congruence α on B is an S-poset congruence if and only if $b \leq_\alpha b' \leq_\alpha b$, for every $b, b' \in B$ implies $b\alpha b'$. A semigroup congruence α on a posemigroup S is called an *order congruence* if it is an S-poset congruence on S where S is considered as an S-poset.

Recall from [28] that the S-poset congruence $\alpha(X)$ on an S-poset B generated by $X \subseteq B \times B$, is characterized as follows: $b\alpha(X)b'$ if and only if $b = b'$ or there exist $t_1, t_2, ..., t_k, w_1, w_2, ..., w_l \in S^1$ such that

$$b \leq p_1t_1, q_1t_1 \leq p_2t_2, q_2t_2 \leq p_3t_3, ..., q_kt_k \leq b';$$

$$b' \leq u_1w_1, v_1w_1 \leq u_2w_2, v_2w_2 \leq u_3w_3, ..., v_lw_l \leq b,$$

where $(p_i, q_i), (u_j, v_j) \in X \cup X^{-1}$ for $1 \leq i \leq k; 1 \leq j \leq l$.

Each sub S-poset $B \subseteq A$ defines the *Rees congruence* ρ_B on A, by setting $a\rho_B b$ if $a, b \in B$ or $a = b$. The resulting factor S-poset is denoted by A/B and it is called *the Rees factor S-poset of A by the sub S-poset B*.

Recall from [25] that a posemigroup (or pomonoid) S is called *right Noetherian* if each right order congruence on S has a finite generating set.

Recall from [22] that Green's relations $\mathcal{L}, \mathcal{R}$ and $\mathcal{J}$ on a posemigroup are given as follows: For two elements $s, t \in S$, $s\mathcal{L}t$ if they generate the same principal left poideal, that is $\downarrow (S^1s) = \downarrow (S^1t)$, and $s\mathcal{R}t$ if they generate the same principal right poideal, that is $\downarrow (sS^1) = \downarrow (tS^1)$. In a similar way, we define Green's relations on an S-poset A. For example, $a\mathcal{R}b$ if they generate the same down closed cyclic sub S-poset, that is $\downarrow (aS^1) = \downarrow (bS^1)$. Then the $\mathcal{R}$-class $R_a, a \in A$, is the class of all elements $b \in A$ such that $\downarrow (aS^1) = \downarrow (bS^1)$.

Green's relation $\mathcal{R}$ defines a preorder $\leq_\mathcal{R}$ on an S-poset A, given by

$$a \leq_\mathcal{R} b \Leftrightarrow \downarrow (aS^1) \subseteq \downarrow (bS^1).$$

The preorder $\leq_\mathcal{R}$ induces a partial order on the set of $\mathcal{R}$-classes of A given by

$$R_a \leq R_b \text{ if and only if } a \leq_\mathcal{R} b.$$

2. **Right dc-Noetherian S-posets**

In this section, the concept of weak right Noetherianity for S-posets is introduced and some fundamental properties are presented. Equivalent characterizations of right dc-Noetherian S-posets in terms of down closed cyclic sub S-posets and the poset of $\mathcal{R}$-classes are provided. Also, the behavior of right dc-Noetherian S-posets with respect to sub S-posets, factor S-posets, products and coproducts is studied.

Note that the definition of dc-Noetherian S-posets given here can be viewed as a special case of the general notion of po-Noetherianity, where the finiteness condition is restricted to down closed sub S-posets. We start with the following theorem, which is proved similarly to Proposition 2.1 of [14].

Theorem 2.1. *Suppose A is an S-poset over a posemigroup S. Then the following are equivalent:*

(i) *each down closed sub S-poset of A is finitely generated.*
(ii) *the ascending chain condition holds for down closed sub S-posets of A; each infinite ascending chain $B_1 \subseteq B_2 \subseteq \dots$ of down closed sub S-posets of A eventually terminates.*
(iii) *each non-empty set of down closed sub S-posets of A has a maximal element.*

Definition 2.2. An S-poset A is called *right dc-Noetherian* if one of the equivalent conditions in Theorem 2.1 holds.

Here, we establish a characterization of right dc-Noetherian S-posets based on their down closed cyclic sub S-posets.

Theorem 2.3. *The following are equivalent for an S-poset A:*

(i) *A is right dc-Noetherian.*
(ii) *A does not have any set of pairwise incomparable down closed cyclic right sub S-posets, which is infinite, and the ascending chain condition holds on down closed cyclic right sub S-posets.*
(iii) *the poset of $\mathcal{R}$-classes of A contains no infinite strictly ascending chain or infinite antichain (a subset of some poset consisting of pairwise incomparable elements).*

Proof. (i) $\Rightarrow$ (ii) It is clear by Theorem 2.1 that A fulfills the ascending chain condition on down closed cyclic right sub S-posets.

We are going to prove that A does not contain an infinite set of pairwise incomparable down closed cyclic right sub S-posets. Let's assume the opposite for the sake of contradiction.

Consider the set $\{\downarrow (a_i S^1) : i \in \mathbb{N}\}$ of pairwise incomparable down closed cyclic right sub S-posets of A, which is infinite. Let B_n, $n \in \mathbb{N}$, be the down closed cyclic right sub S-posets of A generated by $a_1, \dots, a_n$ (i.e. $B_n = \downarrow (\{a_1, \dots, a_n\} S^1)$). Now, if there exists some $m \leq n, B_m = B_n$, then there exists some $i \leq m, a_n \in \downarrow (a_i S^1)$. Then $i = m = n$, as $\downarrow (a_i S^1)$ and $\downarrow (a_n S^1)$ are incomparable. Therefore, we get an infinite strictly ascending chain of down closed cyclic right sub S-posets of A,

$$B_1 \subset B_2 \subset \dots .$$

This contradicts the assumption.

(ii) $\Rightarrow$ (i) Let A be not right dc-Noetherian and satisfy the ascending chain condition on its down closed cyclic right sub S-posets. We find an infinite set of pairwise incomparable down closed cyclic right sub S-posets of A.

Since A is not right dc-Noetherian, there exists an infinite strictly ascending chain of down closed right sub S-posets of A,

$$B_1 \subset B_2 \subset \dots$$

Choose $a_1 \in B_1$ and for $k \geq 2, a_k \in B_k - B_{k-1}$. Now, since $\downarrow (a_j S^1) \subseteq B_j, a_k \in B_k - B_j, j < k$, then the down closed cyclic right sub S-poset $\downarrow (a_k S^1)$ is not contained in any down closed cyclic right sub S-poset $\downarrow (a_j S^1)$.

Let $\downarrow (a_{r_1} S^1)$ be a maximal element of the set $\{\downarrow (a_i S^1) : i \in \mathbb{N}\}$. Therefore, any $\downarrow (a_j S^1), j \neq r_1$ does not contain $\downarrow (a_{r_1} S^1)$. If this were not the case, then there exists an infinite strictly ascending chain of down closed cyclic right sub S-posets of A, which contradicts the hypothesis.

Denote a maximal element of the infinite set $\{\downarrow (a_i S^1) : i \geq r_1 + 1\}$ by $\downarrow (a_{r_2} S^1)$. It follows that for any $j > r_1, j \neq r_2$, $\downarrow (a_j S^1)$ does not contain $\downarrow (a_{r_2} S^1)$. This means that, for any j that belongs to the set of natural numbers, except for $j = r_2$, $\downarrow (a_j S^1)$ does not contain $\downarrow (a_{r_2} S^1)$. This is because any $\downarrow (a_j S^1), j < r_1$, does not contain $\downarrow (a_{r_2} S^1)$.

Continuing in this way infinitely, we obtain an infinite set $\{\downarrow (a_{r_i} S^1) : i \in \mathbb{N}\}$ of pairwise incomparable cyclic right sub S-posets of A.

(iii) $\Leftrightarrow$ (i) Since the S-poset map $\{R_a : a \in A\} \to \{\downarrow (aS^1) : a \in A\}$ from the poset of $\mathcal{R}$-classes of A into the poset of down closed cyclic right sub S-posets of A, given by $R_a = \{b \in S : \downarrow (aS^1) = \downarrow (bS^1)\} \mapsto \downarrow (aS^1)$, is an order isomorphism, the result follows. □

Corollary 2.4. *Let A be a right S-poset with a finite number of $\mathcal{R}$-classes. Then A is right dc-Noetherian. Particularly, all finite right S-posets and all right S-posets with no proper down closed sub S-poset are right dc-Noetherian.*

Proposition 2.5. *Every down closed sub S-poset of a right dc-Noetherian S-poset is right dc-Noetherian.*

Proof. Suppose B is a down closed sub S-poset of a right dc-Noetherian S-poset A, and let B' be a down closed sub S-poset of B. Since B is down closed in A, we get that B' is a down closed sub S-poset of A. Hence, B' is finitely generated since A is right dc-Noetherian. □

Proposition 2.6. *Every factor S-poset of a right dc-Noetherian S-poset by an S-poset congruence is right dc-Noetherian.*

Proof. Assume that $\bar{B}$ is a down closed sub S-poset of A/θ. Then a set $B = \{a \in A : [a]_\theta \in \bar{B}\}$ is a down closed sub S-poset of A and so it is generated by some finite set $X \subseteq A$. We claim that $\bar{B}$ is generated by the finite set $Y = \{[x]_\theta : x \in X\}$. To see this, let $[a]_\theta \in \bar{B}$. Then, $a \in B$ and so there exist $x \in X$ and $s \in S^1$ such that $a \leq xs$. Hence, $[a]_\theta \leq [x]_\theta s \in \downarrow (YS^1)$, as required. □

Corollary 2.7. *Let S be a pomonoid. Then S_S is right dc-Noetherian if and only if each cyclic S-poset is right dc-Noetherian.*

The following Proposition states some conditions under which the converse of Proposition 2.6 holds.

Proposition 2.8. *Suppose A is an S-poset over a posemigroup S and $\leq_\theta \subseteq \mathcal{R}$ is an S-poset congruence on A. Then A is right dc-Noetherian if and only if A/θ is right dc-Noetherian.*

Proof. It is enough to show that the poset of cyclic down closed right sub S-posets of A and the poset of cyclic down closed right sub S-posets of A/θ given by $\downarrow(aS^1) \mapsto \downarrow([a]_\theta S^1)$ are isomorphic. It is enough to show that $\downarrow(a_1S^1) \subseteq \downarrow(a_2S^1)$ if and only if $\downarrow([a_1]_\theta S^1) \subseteq \downarrow([a_2]_\theta S^1)$. We only prove the converse. Let $\downarrow([a_1]_\theta S^1) \subseteq \downarrow([a_2]_\theta S^1)$ then $[a_1]_\theta \leq [a_2]_\theta s, s \in S^1$, and so $a_1 \leq_\theta a_2 s$. Hence $\downarrow(a_1S^1) = \downarrow(a_2sS^1) \subseteq \downarrow(a_2S^1)$, as required. □

The following theorem shows that the coproduct of a finite number of right dc-Noetherian S-posets is also right dc-Noetherian.

Theorem 2.9. *The coproduct $A_1 \coprod A_2$ of right S-posets A_1 and A_2 is right dc-Noetherian if and only if A_1 and A_2 are right dc-Noetherian.*

Proof. Necessity is clear by Proposition 2.5. For sufficiency, let B be a down closed sub S-poset of $A_1 \coprod A_2$. Then $B \cap A_1$ is a down closed sub S-poset of A_1 and $B \cap A_2$ is a down closed sub S-poset of A_2. Hence there exist finite subsets $X_1 \subseteq B \cap A_1$ and $X_2 \subseteq B \cap A_2$ such that $B \cap A_1 = \downarrow(X_1S^1)$ and $B \cap A_2 = \downarrow(X_2S^1)$, since A_1 and A_2 are right dc-Noetherian. It is easily seen that B is generated by a finite set $X_1 \cup X_2$. □

Theorem 2.10. *Let B be a down closed sub S-poset of an S-poset A. Then A is right dc-Noetherian if and only if B and A/B both are right dc-Noetherian.*

Proof. It is enough to prove sufficiency. Let $B_1 \subseteq B_2 \subseteq \cdots$ be an infinite ascending chain of down closed sub S-posets of A. Put for each $B_i, i \in \mathbb{N}$, $B'_i = B_i \cap B$ and $\bar{B}_i = \{[b_i] : b_i \in B\}$ where for each element $a \in A, [a]$ denotes an image of element a by the canonical homomorphism $A \to A/B$. It is obvious that B'_i and $\bar{B}_i$ are down closed sub S-posets of B and A/B, respectively. So, we get the infinite ascending chain $B'_1 \subseteq B'_2 \subseteq \cdots$ of down closed sub S-posets of B and the infinite ascending chain $\bar{B}_1 \subseteq \bar{B}_2 \subseteq \cdots$ of down closed sub S-posets of A/B, respectively. Then there exist elements $n', \bar{n} \in \mathbb{N}$ such that $B'_{m'} = B'_{n'}$ and $\bar{B}_{\bar{m}} = \bar{B}_{\bar{n}}$ for each $m', \bar{m} \in \mathbb{N}, m' \geq n', \bar{m} \geq \bar{n}$, since B and A/B are right dc-Noetherian. Let $n = \max\{n', \bar{n}\}$. Hence, for each $m \in \mathbb{N}, m \geq n$, $B'_m = B'_n$ and $\bar{B}_m = \bar{B}_n$. We claim that for each $m \in \mathbb{N}, m \geq n$, $B_m = B_n$. Let $b \in B_m$. Then since $\bar{b} \in \bar{B}_m$ there exists $b' \in B_n$ such that $\bar{b} = \bar{b'}$. Two cases may occur:

Case 1. If $b \in B$ then $b \in B'_m = B'_n$ and hence $b \in B_n$.

Case 2. If $b \notin B$ then $\bar{b} = \bar{b'}$, and hence $b = b' \in B_n$. □

Now, the behavior of right dc-Noetherianity with respect to Rees short exact sequences are considered.

Recall from [12] that for S-posets A, B, and C, S-poset maps $f : A \to B$, and $g : B \to C$, the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is called *exact* at B if $\mathcal{I}_f = \mathcal{K}_g$ and it is called a *Rees short exact sequence* if f is a regular monomorphism ($a \leq$

$a' \Leftrightarrow f(a) \leq f(a')$), g is surjective, and $\mathcal{I}_f = \mathcal{K}_g$, where $\mathcal{I}_f = \{(b, b') \in B \times B : b \leq_{\theta_f} b'\}, \theta_f = (f(A) \times f(A)) \cup \Delta_B$ and $\mathcal{K}_g = \{(a, a') \in A \times A : g(a) \leq g(a')\}$.

Theorem 2.11. *In a Rees short exact sequence* $A \xrightarrow{f} B \xrightarrow{g} C$ *of* S*-posets,* B_S *is right dc-Noetherian if and only if both* A_S *and* C_S *are right dc-Noetherian.*

Proof. We only prove the sufficiency. The necessity follows from Proposition 2.5 and Proposition 2.6. Suppose

$$B_1 \subseteq B_2 \subseteq \dots$$

is an ascending chain of down closed sub S-posets of B. Then, putting $D_i = B_i \cap f(A), C_i = \downarrow g(B_i), i \in \mathbb{N}$,

$$D_1 \subseteq D_2 \subseteq \dots$$

is an ascending chain of down closed sub S-posets of $f(A)$ and

$$C_1 \subseteq C_2 \subseteq \dots$$

is an ascending chain of down closed sub S-posets of C. Hence there exist $n_1, n_2 \in \mathbb{N}$ such that for each $m \geq n_1, D_m = D_{n_1}$ and for each $m \geq n_2, C_m = C_{n_2}$. Put $n = \max\{n_1, n_2\}$. It is clear that for each $m \geq n, D_m = D_n$ and $C_m = C_n$. We claim that for each $m \geq n$, $B_m = B_n$. For, let $b_m \in B_m$. Two cases may occur:

Case 1. $b_m \in f(A)$. Then $b_m \in B_m \cap f(A) = B_n \cap f(A)$ and hence $b_m \in B_n$.

Case 2. $b_m \notin f(A)$. Then $g(b_m) \in g(B_m) \subseteq \downarrow g(B_m) = \downarrow g(B_n)$, and so $g(b_m) \leq g(b_n)$ for some $b_n \in B_n$. Hence $(b_m, b_n) \in \mathcal{K}_g = \mathcal{I}_f$. Hence $b_m = b_n \in B_n$, by the hypothesis. □

Proposition 2.12. *Let* S *be a posemigroup and* $\{A_i : i \in I\}$ *be a family of* S*-posets each with a minimal element* $\theta_i \in A_i$*. If* $A = \prod_{i \in I} A_i$ $(A = \coprod_{i \in I} A_i)$ *is right dc-Noetherian, then* I *is finite and each* A_i *is right dc-Noetherian.*

Proof. Suppose $A = \prod_{i \in I} A_i$ is right dc-Noetherian and suppose by the contrary, that I' is an infinite subset of elements of I which are distinct. Put $A'_n = \prod_{i \in I} C_i$ such that for $i \in I', i < n, C_i = A_i$ and for $i \in I', i \geq n$, or $i \notin I', C_i = \{\theta_i\}$. Then we have a strictly ascending chain of down closed sub S-posets of A

$$A'_1 \subset A'_2 \subset \dots,$$

which contradicts with right dc-Noetherianity of A. Hence in case I is infinite, A can not be right dc-Noetherian.

Now, let all but a finite number of A_i's be trivial and let, without loss of generality, for each $i > n$, $A_i = \{\theta_i\}$ be a trivial S-poset. Put $A'_{n+1} = \prod_{i > n} A_i$, which is a one-element S-poset, and so $A = A_1 \prod \cdots \prod A_n \prod A_{n+1}$. Hence, I could be considered as a finite set. The rest of the proof is clear. □

Corollary 2.13. *Let* S *be a posemigroup. Then* $A = \prod_{i=1}^n A_i$ $(A = \coprod_{i=1}^n A_i)$ *is right dc-Noetherian if and only if each* A_i *is right dc-Noetherian.*

3. Right Noetherianity and its relation with right dc-Noetherianity for S-posets

Recall from [16] that an S-poset A is called *right po-Noetherian* if each S-poset congruence on A is finitely generated. In this section, we show that each right po-Noetherian S-poset is right dc-Noetherian and by giving a counterexample it is shown that the converse does not hold in general. Similar to the theory of rings and modules, it is shown that a pomonoid S is right Noetherian if and only if every finitely generated S-poset is right po-Noetherian.

Proposition 3.1. *Suppose A is an S-poset. Then A is right dc-Noetherian whenever it is right po-Noetherian.*

Proof. Suppose B is a down closed sub S-poset of A and ρ_B is the Rees right congruence on A. Then for some finite set $X \subseteq B \times B$, ρ_B is generated by X, since A is right po-Noetherian. We prove that the set

$$Y = \{x \in B : (x, y) \in X \text{ for some } y \in B\},$$

which is finite, generates B. Let a be an element of B, and choose an element b of B such that $a \neq b$. Then $a\rho_B b$, and so there exists a chain

$$a \leq c_1s_1, d_1s_1 \leq c_2s_2, d_2s_2 \leq c_3s_3, ..., d_ns_n \leq b;$$

$$b \leq p_1t_1, q_1t_1 \leq p_2t_2, q_2t_2 \leq p_3t_3, ..., q_mt_m \leq a,$$

where $(c_i, d_i) \in, (p_j, q_j) \in X \cup X^{-1}$ for $i = 1, 2, ..., n; j = 1, 2, ..., m$ and $s_1, s_2, ..., s_n, t_1, t_2, ..., t_m \in S$. Particularly, for some $x \in Y$ and $s \in S^1$, $a \leq xs$. □

By the following example, similar to Example 3.1 in [9], it is shown that weak right Noetherianity does not imply right Noetherianity.

Example 3.2. *Suppose G is a non-weak finitely generated abelian pogroup and $x, 0$ are elements disjoint from elements of G. Let $S_0 = G, S_1 = \{xg : g \in G\}, S_2 = \{x^2g : g \in G\}, S_3 = \{x^3\}, S_4 = \{0\}$ (here x^2, x^3 are merely symbols). Let $S = \bigcup\{S_i : 0 \leq i \leq 4\}$. Define a multiplication and an order on S that extends those of G as follows:*
$g(xh) = x(gh) = (xh)g$,
$g(x^2h) = x^2(gh) = (x^2h)g$,
$gx^3 = x^3 = x^3g$,
$(xg)(xh) = x^2(gh) = (xh)(xg)$, $(xg)(x^2h) = x^3 = (x^2h)(xg)$,
$(xg)x^3 = 0 = x^3(xg)$,
$(x^2g)(x^2h) = 0 = (x^2h)(x^2g)$,
$(x^2g)x^3 = 0 = x^3(x^2g)$,
$x^3x^3 = 0 = 0s = s0$ *for all* $s \in S$,
and $x^ig_1 \leq x^ig_2, i = 1, 2,$ *if and only if* $g_1 \leq g_2$.
The only right poideals of S are $< 0 >, < x^3 >, < x^2 >, < x >$ and S, and hence S is a right dc-Noetherian pomonoid. We claim that $\mathbf{r}(x^2) = \{(u, v) :$

$x^2u = x^2v\}$ *is a non-finitely generated right S-poset congruence on S. On the contrary, suppose $r(x^2)$ is generated by the set*

$$H = \{(\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)\}.$$

Consider $g \in G$ with $g \neq 1_G$. Then, since $(xg, x) \in \mathbf{r}(x^2)$ and $xg \neq x$, there exists an H-sequence

$$xg \leq c_1s_1, d_1s_1 \leq c_2s_2, \ldots, d_ns_n \leq x,$$

$$x \leq p_1t_1, q_1t_1 \leq p_2t_2, \ldots, q_mt_m \leq xg$$

with $(c_i, d_i), (p_j, q_j) \in H \cup H^{-1}$ for $i = 1, 2, ..., n; j = 1, 2, ..., m$ and $s_1, s_2, ..., s_n, t_1, t_2, ..., t_m \in S$. It is concluded, by the construction of S, that each pair (c_i, d_i) is of the form (xh_i, xk_i) where $h_i, k_i \in G$ and each $s_i \in G$. One thus obtains that $g \leq h_1k_1^{-1} \ldots h_nk_n^{-1}$. This means that G is weak finitely generated, which is a contradiction. Therefore, $\mathbf{r}(x^2)$ is not finitely generated.

Proposition 3.3. *A right Noetherian S-poset A over a finite posemigroup S is finite.*

Proof. Let $A = \bigcup_{i\in\mathbb{N}} a_iS^1$. For each $n \in \mathbb{N}$, let B_n be a cyclic sub S-posets of A generated by $a_1, \ldots, a_n$ (i.e. $B_n =\downarrow (\{a_1, \ldots, a_n\}S^1)$). Now we can consider the following ascending chain of down closed sub S-posets of A,

$$B_1 \subset B_2 \subset \ldots.$$

For some $m \leq n, B_m = B_n$ as A is right dc-Noetherian. Since S is finite, $A = \bigcup_{i=1}^{n} a_iS^1$ is finite. □

Recalling from [8] that in the theory of rings and modules, a ring R is right Noetherian if and only if every finitely generated right R-module in Noetherian (that is, it satisfies the ascending chain condition on its submodules), we prove the analogue of this result for S-posets.

Recall from [23] that an S-poset A which is isomorphic to a quotient S-poset of a finitely generated free S-poset by a finitely generated S-poset congruence is called *finitely presented.*

Theorem 3.4. *Each finitely generated S-poset over a right Noetherian pomonoid S is finitely presented and right po-Noetherian.*

Proof. Let A be a finitely generated S-poset over a right Noetherian pomonoid S. Then by [1], $A \cong F(X)/\theta$, where $F(X)$ is a free S-poset on a finite set $X \subseteq A$ and θ is a right S-poset congruence on $F(X)$. Then by Proposition 2.7 (iii) of [16] and right Noetherianity of S, A is finitely presented. Hence A is isomorphic to a quotient S-poset of a finitely generated free S-poset by a finitely generated right S-poset congruence. Therefore, an S-poset A is right po-Noetherian by Proposition 2.7 (iii) and Lemma 2.3 (ii) of [16]. □

Corollary 3.5. *A pomonoid S is right Noetherian if and only if each finitely generated S-poset is right po-Noetherian.*

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