Shahid Bahonar University of Kerman
Journal of Mahani Mathematical Research
Print ISSN: 2251-7952 Online ISSN: 2645-4505

SCHATTEN CLASS TOEPLITZ OPERATORS ON BERGMAN SPACES WITH ALMOST STANDARD WEIGHTS

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Article type: Research Article

(Received: 26 January 2026, Received in revised form 08 May 2026)

(Accepted: 22 May 2026, Published Online: 22 May 2026)

Abstract. We study Schatten class Toeplitz operators on weighted Bergman spaces induced by almost standard radial weights on the unit disk. We obtain a complete characterization of such operators generated by positive Borel measures. The characterization is given in terms of the Berezin transform, integrability of localized averages with respect to the Möbius invariant measure, and discrete summability over pseudohyperbolic lattices.For Toeplitz operators generated by complex Borel measures, we establish sufficient conditions for Schatten class membership in terms of discrete lattice averages of the total variation, together with corresponding norm estimates. As an application, we derive Schatten class bounds for differences of such operators.

Keywords: Toeplitz operators, Schatten classes, weighted Bergman spaces, almost standard weights, complex measures.
2020 MSC: 47B35, 47B10; 30H20.

1. Introduction

Let $\omega : [0,1) \to \mathbb{R}$ be a positive, integrable function, extended radially to the unit disk $\mathbb{D}$ by $\omega(z) = \omega(|z|)$. The weighted Bergman space A^2_ω consists of analytic functions f on $\mathbb{D}$ satisfying

$$\|f\|^2_{A^2_\omega} = \int_{\mathbb{D}} |f(z)|^2 \omega(z)\, dA(z) < \infty,$$

where dA denotes the normalized area measure. Under the assumptions considered in this paper, A^2_ω is a closed subspace of $L^2(\mathbb{D}, \omega\, dA)$ and hence a Hilbert space; see, for example, [6, 18].

Toeplitz operators on Bergman spaces form one of the central classes of operators in function-theoretic operator theory. They provide a link between analytic function spaces, measure-theoretic embedding problems, and operator ideals. Given a finite positive Borel measure μ on $\mathbb{D}$, the associated Toeplitz operator is formally given by

$$T_\mu f(z) = \int_{\mathbb{D}} f(\xi) K^\omega_z(\xi)\, d\mu(\xi), \qquad f \in A^2_\omega,$$

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https://doi.org/10.22103/jmmr.2026.26719.1929
Publisher: Shahid Bahonar University of Kerman
How to cite: K. Esmaeili, *Schatten class Toeplitz operators on Bergman spaces with almost standard weights*, J. Mahani Math. Res. 2026; 15(2): 511-533.

where K_z^ω denotes the reproducing kernel of A^2_ω.

For a finite complex Borel measure μ on $\mathbb{D}$, one may instead define T_μ through the corresponding sesquilinear form. More precisely, whenever

$$(f,g) \mapsto \int_{\mathbb{D}} f(z)\overline{g(z)}\, d\mu(z)$$

is bounded on $A^2_\omega \times A^2_\omega$, it induces a unique bounded Toeplitz operator T_μ on A^2_ω via

$$\langle T_\mu f, g\rangle_{A^2_\omega} = \int_{\mathbb{D}} f(z)\overline{g(z)}\, d\mu(z).$$

In the positive-measure case, boundedness and compactness of Toeplitz operators are closely related to Carleson and vanishing Carleson measure conditions. More precisely, in many Bergman-type settings, T_μ is bounded, respectively compact, if and only if μ is an A^2_ω–Carleson, respectively vanishing Carleson, measure; see [3, 4, 11, 18].

A finer problem is to determine when T_μ belongs to a Schatten class. This question measures not only compactness, but also the rate at which the singular values of T_μ tend to zero. Schatten class criteria therefore give a more quantitative form of compactness and are useful in trace ideal theory, spectral estimates, and the study of compact perturbations of Toeplitz-type operators.

For standard weights $\omega_\alpha(z) = (1-|z|^2)^\alpha$, Schatten class properties of positive Toeplitz operators were characterized by Luecking [11] and Zhu [17] in terms of localized averages, Berezin transforms, and lattice summability conditions. Related results were obtained for exponential-type weights by Lin and Rochberg [10], and for radial weights satisfying doubling-type assumptions by Peláez and Rättyä [13]. Further related and recent work on Schatten class problems for Toeplitz and related operators has been carried out in several weighted Bergman-type settings, including large weighted Bergman spaces, exponential weighted Bergman spaces, Bergman–Besov Hilbert spaces, Bergman spaces induced by regular weights, and Fock-type spaces; see, for example, [2,5,9,14–16].

The aim of the present paper is to extend this circle of ideas to weighted Bergman spaces induced by almost standard radial weights. A fundamental obstacle in passing from standard weights to more general radial weights is the lack of explicit formulas for the reproducing kernels. Many classical proofs depend on sharp kernel asymptotics, and these arguments do not transfer directly to the almost standard setting. To overcome this difficulty, we use pseudohyperbolic geometry, localized kernel estimates, Carleson measure techniques, and lattice decompositions adapted to almost standard weights.

Recall that an almost standard weight is a radial weight ω satisfying $\omega(0) = 1$, such that ω is non-increasing and $\omega(r)/(1-r^2)^\delta$ is non-decreasing for some $\delta > 0$. This class contains the standard decreasing weights and their logarithmic perturbations, while still allowing enough control over local averages and reproducing kernels. Moreover, for such weights one has

$$H^p \subset A^p_\omega \subset A^p_\delta,$$

which situates these spaces between Hardy spaces and the standard weighted Bergman scale.

Our first main result gives a complete characterization of Schatten p–class Toeplitz operators induced by positive Borel measures on A^2_ω, for $1 \leq p < \infty$. The characterization is expressed in several equivalent forms: in terms of the Berezin transform, integrability of localized averages with respect to the Möbius invariant measure, and discrete summability over pseudohyperbolic lattices. Thus the classical Schatten class criteria for standard weighted Bergman spaces are recovered as a special case and extended to the almost standard setting.

We also consider Toeplitz operators induced by complex Borel measures. In this case, cancellations caused by the phase of the measure prevent a direct extension of the positive-measure characterization in terms of total variation. Nevertheless, we obtain sufficient Schatten class criteria formulated through discrete lattice averages of the total variation, together with corresponding norm estimates. As an application, we derive Schatten class bounds for differences of Toeplitz operators induced by complex measures. These estimates are useful for studying perturbations of Toeplitz operators and for understanding how cancellation phenomena affect trace ideal membership.

2. **Preliminaries**

Let $D_r(z)$ denote the Euclidean disk of radius r centered at z, and write $D_r = D_r(0)$. The pseudohyperbolic distance on $\mathbb{D}$ is defined by

$$\rho(\xi, z) = |\varphi_\xi(z)| = \left|\frac{\xi - z}{1 - \bar{\xi} z}\right|, \qquad \xi, z \in \mathbb{D},$$

where φ_ξ is the Möbius automorphism of $\mathbb{D}$ interchanging 0 and ξ. For $\xi \in \mathbb{D}$ and $0 < r < 1$, the pseudohyperbolic disk of center ξ and radius r is

$$E(\xi, r) = \{z \in \mathbb{D} : \rho(\xi, z) < r\}.$$

It is well known that $E(\xi, r)$ is a Euclidean disk with center $\frac{(1-r^2)\xi}{1-r^2|\xi|^2}$ and radius $\frac{(1-|\xi|^2)r}{1-r^2|\xi|^2}$ (see, e.g., [18, Chapter 2]).

For $\xi, z \in \mathbb{D}$ and $0 < r < 1$, the following identities will be used repeatedly:

$$1 - |\varphi_\xi(z)|^2 = \frac{(1 - |\xi|^2)(1 - |z|^2)}{|1 - \bar{\xi} z|^2}; \tag{1}$$

$$1 - \bar{\xi}\varphi_\xi(z) = \frac{(1 - |\xi|^2)}{1 - \bar{\xi} z}. \tag{2}$$

Moreover, for each $0 < r < 1$ there exists a constant $C = C(r) > 1$ such that

$$C^{-1}(1 - |\xi|^2) \leq 1 - |z|^2 \leq C(1 - |\xi|^2), \qquad z \in E(\xi, r).$$

A sequence $\{z_n\} \subset \mathbb{D}$ is called an r–lattice if there exists $r' \in (0, r)$ such that

(i) $\rho(z_n, z_m) \geq r'$ for all $n \neq m$,

(ii) $\{z_n\}$ is maximal among r'–separated subsets of $\mathbb{D}$.

For such a sequence, the disks $\{E(z_n, r')\}$ cover $\mathbb{D}$, and for each $\rho \in (r', 1)$ the family $\{E(z_n, \rho)\}$ has finite overlap. In particular, every point $z \in \mathbb{D}$ belongs to at most $N(\rho)$ of the sets $E(z_n, \rho)$, where $N(\rho)$ depends only on ρ (cf. [18, Lemma 4.7]).

Throughout the paper, $\delta > 0$ denotes the structural constant associated with the almost standard weight ω, and we use the abbreviations

$$[\omega](z) = (1 - |z|^2)^2 \omega(z), \qquad L^q(\mu) = L^q(\mathbb{D}, \mu).$$

We use $A \lesssim B$ to mean $A \le CB$ for some constant $C > 0$, and $A \asymp B$ if both $A \lesssim B$ and $B \lesssim A$ hold.

The following properties of almost standard weights will be used frequently (see, e.g., [7, 8, 12]):

Lemma 2.1. *Let ω be an almost standard weight and let $r \in (0, 1)$, $z \in \mathbb{D}$, and $\beta > \delta$. Then*

(i) $\displaystyle\int_{\mathbb{D}} \frac{\omega(\xi)}{|1 - \overline{z}\xi|^{\beta+2}}\, dA(\xi) \asymp \frac{\omega(z)}{(1 - |z|^2)^{\beta}}$;

(ii) $\omega(\xi) \asymp \omega(z)$ *for all* $\xi \in E(z, r)$;

(iii) $\|K_z^\omega\|_{A^2_\omega}^2 \asymp \frac{1}{[\omega](z)}$ *for* $z \in \mathbb{D}$.

Lemma 2.2. *Let ω be an almost standard weight and let $0 < q < \infty$. Then for each $r \in (0, 1)$ there exists $C = C(r, \omega) > 0$ such that*

$$|f(z)|^q \le \frac{C}{[\omega](z)} \int_{E(z,r)} |f(\xi)|^q \omega(\xi)\, dA(\xi), \qquad z \in \mathbb{D},\ f \in \mathcal{H}(\mathbb{D}).$$

Proof. Suppose $r \in (0, 1)$, $z \in \mathbb{D}$, and $f \in \mathcal{H}(\mathbb{D})$. Using identities (1) and (2), and a change of variables, we obtain

$$\begin{aligned} |f(z)|^q &\le \frac{1}{\pi r^2} \int\limits_{D_r} |f \circ \varphi_z(\xi)|^q \, dA(\xi) \\ &\le \frac{C_0(r)}{(1 - |z|^2)^2} \int_{E(z,r)} |f(\xi)|^q \, dA(\xi). \end{aligned}$$

Since ω is almost standard, Lemma 2.1(ii) yields a constant $C_1 = C_1(r, \omega) \ge 1$ such that $\omega(z) \le C_1\, \omega(\xi)$ for all $\xi \in E(z, r)$. Therefore

$$|f(z)|^q \le \frac{C(r, \omega)}{[\omega](z)} \int_{E(z,r)} |f(\xi)|^q \omega(\xi)\, dA(\xi).$$

□

A positive Borel measure μ on $\mathbb{D}$ is said to be a Carleson measure for A^q_ω if the inclusion $A^q_\omega \hookrightarrow L^q(\mu)$ is bounded.

Fix $r \in (0, 1)$. We write

$$\widehat{\mu}_{\omega,r}(z) = \frac{\mu(E(z, r))}{[\omega](z)}, \quad z \in \mathbb{D}.$$

Theorem 2.3. *Suppose that ω is an almost standard weight, $r \in (0,1)$, and $\{z_n\}$ is an r-lattice. Let μ be a positive Borel measure on $\mathbb{D}$, and let $1 < q < \infty$. Then μ is a Carleson measure for A^q_ω if and only if*

$$\sup_{n\in\mathbb{N}} \widehat{\mu}_{\omega,r}(z_n) < \infty.$$

In this case,

$$\sup_{\|f\|_{A^q_\omega}\le 1} \|f\|^q_{L^q(\mu)} \asymp \sup_{z\in\mathbb{D}} \widehat{\mu}_{\omega,r}(z) \asymp \sup_{n\in\mathbb{N}} \widehat{\mu}_{\omega,r}(z_n).$$

Proof. Necessity. Assume that μ is a Carleson measure for A^q_ω. For each $z \in \mathbb{D}$ define

$$f_z(\xi) = \frac{(1-|z|^2)^\delta}{\omega(z)^{1/q}(1-\bar{z}\xi)^{\delta+2/q}},$$

where $\delta > 0$ is the structural constant associated with the almost standardness of ω. Then

$$|f_z(\xi)|^q = \frac{(1-|z|^2)^{q\delta}}{\omega(z)\,|1-\bar{z}\xi|^{q\delta+2}}.$$

Applying Lemma 2.1(i) with $\beta = q\delta > \delta$, we obtain

$$\|f_z\|^q_{A^q_\omega} = \int_{\mathbb{D}} |f_z(\xi)|^q \omega(\xi)\,dA(\xi) \asymp 1,$$

uniformly in $z \in \mathbb{D}$.

For $\xi \in E(z,r)$ we have $|1-\bar{z}\xi| \asymp 1-|z|^2$ and $\omega(\xi) \asymp \omega(z)$ by Lemma 2.1(ii). Hence

$$|f_z(\xi)|^q \asymp \frac{1}{(1-|z|^2)^2\omega(z)} = \frac{1}{[\omega](z)}.$$

Therefore,

$$\widehat{\mu}_{\omega,r}(z) = \frac{\mu(E(z,r))}{[\omega](z)} \asymp \int_{E(z,r)} |f_z(\xi)|^q\,d\mu(\xi) \le \|f_z\|^q_{L^q(\mu)} \le \|I_\mu\|^q \|f_z\|^q_{A^q_\omega} \lesssim \|I_\mu\|^q,$$

where $I_\mu : A^q_\omega \to L^q(\mu)$ is the embedding operator. Taking the supremum over $z \in \mathbb{D}$ yields

$$\sup_{z\in\mathbb{D}} \widehat{\mu}_{\omega,r}(z) \lesssim \|I_\mu\|^q < \infty. \tag{3}$$

Sufficiency. Conversely, assume that

$$M := \sup_{n\in\mathbb{N}} \widehat{\mu}_{\omega,r}(z_n) < \infty.$$

There exists $r_0 \in (0,r)$ such that $\mathbb{D} = \bigcup_n E(z_n,r_0)$ and the family $\{E(z_n,r)\}$ has bounded overlap. Choose $r_1 > 0$ so that, whenever $z \in E(z_n,r_0)$, one has

$$E(z,r_1) \subset E(z_n,r).$$

By Lemma 2.2, for $z \in E(z_n,r_0)$,

$$|f(z)|^q \le \frac{C}{[\omega](z)} \int_{E(z,r_1)} |f(\xi)|^q \omega(\xi)\,dA(\xi) \lesssim \frac{1}{[\omega](z_n)} \int_{E(z_n,r)} |f(\xi)|^q \omega(\xi)\,dA(\xi),$$

where we use $[\omega](z) \asymp [\omega](z_n)$ on $E(z_n, r_0)$. Integrating over $E(z_n, r_0)$ with respect to $d\mu$ and using $E(z_n, r_0) \subset E(z_n, r)$, we get

$$\int_{E(z_n,r_0)} |f(z)|^q \, d\mu(z) \lesssim \frac{\mu(E(z_n, r_0))}{[\omega](z_n)} \int_{E(z_n,r)} |f(\xi)|^q \omega(\xi) \, dA(\xi)$$
$$\leq M \int_{E(z_n,r)} |f(\xi)|^q \omega(\xi) \, dA(\xi).$$

Summing over n and using bounded overlap, we obtain

$$\|f\|^q_{L^q(\mu)} \leq \sum_n \int_{E(z_n,r_0)} |f|^q \, d\mu \lesssim M \sum_n \int_{E(z_n,r)} |f|^q \omega \, dA \lesssim M \|f\|^q_{A^q_\omega}.$$

Hence μ is a Carleson measure for A^q_ω, and

$$\|I_\mu\|^q \lesssim \sup_{n\in\mathbb{N}} \widehat{\mu}_{\omega,r}(z_n). \tag{4}$$

Combining (3) and (4) yields the asserted equivalences. □

Remark 2.4. The standard weights provide a useful special case. Let

$$\omega_\alpha(z) = (1 - |z|^2)^\alpha, \qquad \alpha \geq 0.$$

Then

$$[\omega_\alpha](z) = (1 - |z|^2)^{\alpha+2}.$$

Thus Theorem 2.3 recovers the classical Carleson measure characterization for the standard weighted Bergman spaces $A^q_\alpha = A^q_{\omega_\alpha}$: a positive Borel measure μ is a Carleson measure for A^q_α if and only if

$$\sup_{n\in\mathbb{N}} \frac{\mu(E(z_n, r))}{(1 - |z_n|^2)^{\alpha+2}} < \infty.$$

This is the usual pseudohyperbolic lattice formulation of the Carleson measure condition for standard weighted Bergman spaces; see, e.g., [18, Chapter 7].

Let T be a compact operator on a separable Hilbert space H. Then there exist orthonormal sequences $\{e_n\}$ and $\{f_n\}$ in H and a sequence of nonnegative numbers $\{\lambda_n\}$ decreasing to 0, such that

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle f_n, \qquad x \in H.$$

The numbers $\lambda_n = \lambda_n(T)$ are called the singular values of T. For $0 < p < \infty$, the Schatten p–class $S_p(H)$ consists of all compact operators T on H such that $\{\lambda_n(T)\} \in \ell^p$. The singular values admit the characterization

$$\lambda_n(T) = \inf\{\|T - K\| : \operatorname{rank} K < n\}, \qquad n \geq 1.$$

In particular, finite rank operators belong to $S_p(H)$ for every $p > 0$, and membership in $S_p(H)$ measures the rate at which the singular values of T tend to zero. If $1 \le p < \infty$, $S_p(H)$ is a Banach space with norm

$$\|T\|_p = \left(\sum_{n=1}^{\infty} \lambda_n(T)^p\right)^{1/p}.$$

For $0 < p < 1$, $S_p(H)$ is a quasi-Banach space satisfying

$$\|T + S\|_p^p \le \|T\|_p^p + \|S\|_p^p.$$

We refer to [18] for further properties of Schatten class operators.

3. Schatten class Toeplitz operators on weighted Bergman spaces

In this section we characterize Schatten class membership of Toeplitz operators T_μ on A^2_ω for almost standard weights ω and $1 \le p < \infty$. The criteria are given in terms of the Berezin transform and localized averages of the inducing measure. We also record consequences for complex measures and differences of Toeplitz operators.

We denote by

$$k_z^\omega(\xi) = \frac{K_z^\omega(\xi)}{\|K_z^\omega\|_{A^2_\omega}}, \qquad z, \xi \in \mathbb{D},$$

the normalized reproducing kernel. The Berezin transform of μ (with respect to ω) is defined by

$$\widetilde{\mu}_\omega(z) = \langle T_\mu k_z^\omega, k_z^\omega \rangle_{A^2_\omega} = \int_{\mathbb{D}} |k_z^\omega(\xi)|^2 \, d\mu(\xi), \qquad z \in \mathbb{D}.$$

Finally, we denote by

$$d\lambda(z) = \frac{dA(z)}{(1 - |z|^2)^2}$$

the Möbius invariant area measure on $\mathbb{D}$.

We begin with auxiliary lemmas needed for the main characterization.

Lemma 3.1. *There exists $r_0 \in (0, 1)$ such that for each $z_0 \in \mathbb{D}$ and each $z \in E(z_0, r_0)$,*

$$\|K_z^\omega\|^2_{A^2_\omega} \asymp |k_{z_0}^\omega(z)|^2,$$

with constants independent of z_0.

Proof. Fix $z_0 \in \mathbb{D}$ and consider the closed subspace

$$M_{z_0} := \{f \in A^2_\omega : \; f(z_0) = 0\}.$$

Let $P_{M_{z_0}}$ denote the orthogonal projection from A^2_ω onto M_{z_0}, and let K^ω_{z,z_0} be the reproducing kernel of M_{z_0}.

Since M_{z_0} has codimension one, we have

$$P_{M_{z_0}} f = f - \frac{f(z_0)}{\|K^\omega_{z_0}\|^2_{A^2_\omega}} K^\omega_{z_0}, \qquad f \in A^2_\omega.$$

Applying this identity to K^ω_z yields

$$K^\omega_{z,z_0} = K^\omega_z - \frac{K^\omega_z(z_0)}{\|K^\omega_{z_0}\|^2_{A^2_\omega}} K^\omega_{z_0}.$$

Evaluating at z gives

$$\|K^\omega_z\|^2_{A^2_\omega} = K^\omega_{z,z_0}(z) + \frac{|K^\omega_z(z_0)|^2}{\|K^\omega_{z_0}\|^2_{A^2_\omega}} = K^\omega_{z,z_0}(z) + |k^\omega_{z_0}(z)|^2.$$

Since $K^\omega_{z,z_0}(z) \geq 0$, this immediately yields the upper estimate

$$|k^\omega_{z_0}(z)|^2 \leq \|K^\omega_z\|^2_{A^2_\omega}.$$

To obtain the lower bound, it suffices to show that there exist $r_0, \gamma \in (0,1)$ such that

$$\|K^\omega_{z,z_0}\|^2_{A^2_\omega} \leq \gamma \|K^\omega_z\|^2_{A^2_\omega}, \qquad z \in E(z_0, r_0). \tag{5}$$

Let $S_{z_0} : M_{z_0} \to A^2_\omega$ be defined by

$$S_{z_0} f(z) = \frac{f(z)}{z - z_0}.$$

Fix $r_1 \in (0,1)$ to be chosen later. By arguing as in [10, Lemma 3.6] and using Lemma 2.1(iii), there exists $C_1 = C_1(\omega) > 0$ such that

$$\int_{E(z_0,r_1)} |S_{z_0} f(z)|^2 \omega(z)\, dA(z) \leq C_1 \|S_{z_0} f\|^2_{A^2_\omega} \int_{E(z_0,r_1)} d\lambda(z).$$

Since λ is Möbius invariant, $\lambda(E(z_0, r_1)) = \lambda(E(0, r_1)) < \infty$. Choosing r_1 sufficiently small so that $C := C_1 \lambda(E(0, r_1)) < 1$, we obtain

$$(1 - C)\|S_{z_0} f\|^2_{A^2_\omega} \leq \int_{\mathbb{D}\setminus E(z_0,r_1)} |S_{z_0} f(z)|^2 \omega(z)\, dA(z).$$

For $z \in \mathbb{D} \setminus E(z_0, r_1)$, pseudohyperbolic geometry yields $|z - z_0| \gtrsim 1 - |z_0|^2$, and hence

$$\|S_{z_0} f\|^2_{A^2_\omega} \lesssim \frac{\|f\|^2_{A^2_\omega}}{(1 - |z_0|^2)^2}, \qquad f \in M_{z_0},$$

with a constant independent of z_0.

Let $U^z_{z_0}$ denote the point evaluation on M_{z_0} at z. For $f \in M_{z_0}$, we have

$$f(z) = (z - z_0)(S_{z_0} f)(z).$$

Thus

$$U^z_{z_0} = (z - z_0) L_z S_{z_0},$$

where L_z denotes the point evaluation on A^2_ω. Therefore

$$\|U^z_{z_0}\| \leq |z - z_0|\, \|L_z\|\, \|S_{z_0}\|.$$

Since $\|S_{z_0}\| \lesssim (1-|z_0|^2)^{-1}$ and $z \in E(z_0, r_0)$ implies

$$|z - z_0| \lesssim r_0(1-|z_0|^2),$$

we may choose $r_0 > 0$ sufficiently small so that

$$\|U^z_{z_0}\|^2 \le \gamma\,\|L_z\|^2 = \gamma\,\|K^\omega_z\|^2_{A^2_\omega}, \qquad z \in E(z_0, r_0),$$

for some $\gamma \in (0,1)$.

Since K^ω_{z,z_0} is the reproducing kernel of M_{z_0},

$$\|K^\omega_{z,z_0}\|^2_{A^2_\omega} = \|U^z_{z_0}\|^2,$$

and hence (5) holds.

Finally, since

$$K^\omega_{z,z_0}(z) \le \|K^\omega_{z,z_0}\|^2_{A^2_\omega},$$

combining this with (5) and the identity $\|K^\omega_z\|^2_{A^2_\omega} = K^\omega_{z,z_0}(z) + |k^\omega_{z_0}(z)|^2$ yields

$$(1-\gamma)\|K^\omega_z\|^2_{A^2_\omega} \le |k^\omega_{z_0}(z)|^2, \qquad z \in E(z_0, r_0).$$

□

We shall use the following local averaging estimate for positive measures.

Lemma 3.2. *Suppose that* $0 < r < 1$ *and that* μ *is a positive Borel measure on* $\mathbb{D}$*. Then there exists a constant* $C = C(r,\omega) > 0$ *such that*

$$\mu(E(z,r)) \le \frac{C}{[\omega](z)} \int_{E(z,r)} \mu(E(\xi,r))\,\omega(\xi)\,dA(\xi), \qquad z \in \mathbb{D}.$$

Proof. Fix $0 < r < 1$ and put $R = \tanh^{-1} r$. Let d denote the hyperbolic metric on $\mathbb{D}$, so that

$$d(a,b) = \tanh^{-1}\rho(a,b).$$

Suppose that $\zeta \in E(z,r)$, equivalently $d(z,\zeta) < R$. Let η be the midpoint of the hyperbolic geodesic segment joining z and ζ. Then

$$d(z,\eta) = d(\zeta,\eta) = \frac{1}{2}d(z,\zeta) < \frac{R}{2}.$$

Set

$$s = \tanh(R/2).$$

Then $0 < s < r$. If $w \in E(\eta, s)$, then $d(w,\eta) < R/2$. Hence, by the triangle inequality,

$$d(w,z) < R, \qquad d(w,\zeta) < R.$$

Equivalently,

$$E(\eta,s) \subset E(z,r) \cap E(\zeta,r).$$

Now fix $z \in \mathbb{D}$. By Tonelli's theorem,

$$\int_{E(z,r)} \mu(E(\xi,r))\,\omega(\xi)\,dA(\xi) = \int_{E(z,r)}\left(\int_{\mathbb{D}} \chi_{E(\xi,r)}(\zeta)\,d\mu(\zeta)\right)\omega(\xi)\,dA(\xi)$$
$$= \int_{\mathbb{D}}\left(\int_{E(z,r)} \chi_{E(\zeta,r)}(\xi)\,\omega(\xi)\,dA(\xi)\right)d\mu(\zeta).$$

If $\zeta \in E(z,r)$, then the preceding inclusion gives a point $\eta = \eta(z,\zeta) \in E(z,r)$ such that

$$E(\eta,s) \subset E(z,r) \cap E(\zeta,r).$$

Consequently,

$$\int_{E(z,r)} \chi_{E(\zeta,r)}(\xi)\,\omega(\xi)\,dA(\xi) \geq \int_{E(\eta,s)} \omega(\xi)\,dA(\xi).$$

Applying Lemma 2.1(ii), we have $\omega(\xi) \asymp \omega(\eta)$ for $\xi \in E(\eta,s)$. Using the standard area estimate $A(E(\eta,s)) \asymp (1-|\eta|^2)^2$ (with constants depending only on r), we obtain

$$\int_{E(\eta,s)} \omega(\xi)\,dA(\xi) \asymp \omega(\eta)\,(1-|\eta|^2)^2 = [\omega](\eta).$$

Since $\eta \in E(z,r)$, Lemma 2.1(ii), together with the standard estimate $1-|\eta|^2 \asymp 1-|z|^2$ on pseudohyperbolic disks, gives

$$[\omega](\eta) \asymp [\omega](z),$$

with constants depending only on r and ω.

Therefore, there exists $c = c(r,\omega) > 0$ such that for all $\zeta \in E(z,r)$,

$$\int_{E(z,r)} \chi_{E(\zeta,r)}(\xi)\,\omega(\xi)\,dA(\xi) \geq c\,[\omega](z).$$

Returning to the preceding identity and restricting to $\zeta \in E(z,r)$ (since the integrand is nonnegative), we obtain

$$\int_{E(z,r)} \mu(E(\xi,r))\,\omega(\xi)\,dA(\xi) \geq c\,[\omega](z)\,\mu(E(z,r)).$$

Rearranging gives the desired estimate.

□

Example 3.3. *In the unweighted case* $\omega \equiv 1$*, we have*

$$[\omega](z) = (1-|z|^2)^2.$$

Thus Lemma 3.2 becomes

$$\mu(E(z,r)) \leq \frac{C}{(1-|z|^2)^2}\int_{E(z,r)} \mu(E(\xi,r))\,dA(\xi), \qquad z \in \mathbb{D}.$$

This is the unit-disk analogue of Zhu's averaging estimate for positive measures on the unit ball with respect to Lebesgue volume measure; see, e.g., [17, Lemma 6].

We shall also use the trace formula for trace class operators.

Lemma 3.4. *If $T \in S_1(A^2_\omega)$, then*

$$\mathrm{tr}(T) = \int_{\mathbb{D}} \langle T K^\omega_z, K^\omega_z \rangle_{A^2_\omega} \, \omega(z) \, dA(z).$$

Proof. The proof is standard and follows by expanding $\mathrm{tr}(T)$ with respect to an orthonormal basis and using absolute convergence. See [18, Chapter 1]. □

We next note that the lattice summability condition is independent of the choice of lattice radius.

Lemma 3.5. *Let $0 < r, s < 1$ and $0 < p < \infty$. Assume that ω is an almost standard weight and let μ be a finite positive Borel measure on $\mathbb{D}$. If $\{z_n\}$ is an s–lattice in $\mathbb{D}$ such that*

$$\sum_{n=1}^{\infty} \left(\widehat{\mu}_{\omega,s}(z_n)\right)^p < \infty,$$

then for every r–lattice $\{\xi_m\}$ in $\mathbb{D}$ one has

$$\sum_{m=1}^{\infty} \left(\widehat{\mu}_{\omega,r}(\xi_m)\right)^p < \infty.$$

The implicit constants depend only on r, s, p, and ω.

Proof. By the covering and finite overlap properties of pseudohyperbolic lattices, there exists an integer $M = M(r, s)$ such that for each m,

$$E(\xi_m, r) \subset \bigcup_{j=1}^{M} E(z_{n(m,j)}, s)$$

for suitable indices $n(m, 1), \ldots, n(m, M)$. Hence

$$\mu(E(\xi_m, r)) \le \sum_{j=1}^{M} \mu(E(z_{n(m,j)}, s)).$$

Dividing by $[\omega](\xi_m)$ and using local comparability of $[\omega]$ for points lying in a fixed pseudohyperbolic neighborhood, we obtain

$$\widehat{\mu}_{\omega,r}(\xi_m) \lesssim \sum_{j=1}^{M} \widehat{\mu}_{\omega,s}(z_{n(m,j)}).$$

Using the elementary inequality

$$\left(\sum_{j=1}^{M} b_j\right)^p \lesssim \sum_{j=1}^{M} b_j^p, \qquad b_j \ge 0,$$

with constants depending only on M and p, we get

$$(\widehat{\mu}_{\omega,r}(\xi_m))^p \lesssim \sum_{j=1}^{M} \left(\widehat{\mu}_{\omega,s}(z_{n(m,j)})\right)^p .$$

Summing over m gives

$$\sum_m (\widehat{\mu}_{\omega,r}(\xi_m))^p \lesssim \sum_m \sum_{j=1}^{M} \left(\widehat{\mu}_{\omega,s}(z_{n(m,j)})\right)^p .$$

Since $\{z_n\}$ is an s–lattice and $\{\xi_m\}$ is an r–lattice, there exists $L = L(r, s)$ such that each disk $E(z_n, s)$ intersects at most L disks $E(\xi_m, r)$. Consequently, each index n appears among the indices $n(m, j)$ for at most L pairs (m, j), and therefore

$$\sum_m (\widehat{\mu}_{\omega,r}(\xi_m))^p \lesssim \sum_n (\widehat{\mu}_{\omega,s}(z_n))^p < \infty.$$

□

The following lemma provides a discrete estimate for localized averages of μ.

Lemma 3.6. *Let ω be an almost standard weight and let μ be a positive Borel measure on $\mathbb{D}$. Fix $r \in (0, 1)$, let $\{z_n\}$ be an r–lattice, and let $1 \le p < \infty$. If $\widehat{\mu}_{\omega,r} \in L^p(\mathbb{D}, d\lambda)$, then*

$$\sum_{n=1}^{\infty} (\widehat{\mu}_{\omega,r}(z_n))^p \lesssim \int_{\mathbb{D}} (\widehat{\mu}_{\omega,r}(z))^p \, d\lambda(z).$$

Proof. By Lemma 3.2, for some positive constant C,

$$\mu(E(z_n, r)) \le \frac{C}{[\omega](z_n)} \int_{E(z_n,r)} \mu(E(\xi, r))\, \omega(\xi)\, dA(\xi).$$

Using $[\omega](\xi) \asymp [\omega](z_n)$ on $E(z_n, r)$ (Lemma 2.1), we obtain

$$\widehat{\mu}_{\omega,r}(z_n) = \frac{\mu(E(z_n, r))}{[\omega](z_n)} \lesssim \int_{E(z_n,r)} \widehat{\mu}_{\omega,r}(\xi)\, d\lambda(\xi).$$

Since $p \ge 1$ and $\lambda(E(z_n, r)) = \lambda(E(0, r)) < \infty$, Hölder's inequality yields

$$(\widehat{\mu}_{\omega,r}(z_n))^p \lesssim \int_{E(z_n,r)} (\widehat{\mu}_{\omega,r}(\xi))^p \, d\lambda(\xi).$$

Summing over n and using the bounded overlap of $\{E(z_n, r)\}$ gives the result. □

Theorem 3.7. *Let ω be an almost standard weight, μ a finite positive Borel measure on $\mathbb{D}$, and let $1 \le p < \infty$. Fix $r \in (0, 1)$. For an r–lattice $\{z_n\}$ in $\mathbb{D}$, the following conditions are equivalent:*

(i) $T_\mu \in S_p(A^2_\omega)$;

(ii) $\widetilde{\mu}_\omega \in L^p(\mathbb{D}, d\lambda)$;

(iii) $\widehat{\mu}_{\omega,r} \in L^p(\mathbb{D}, d\lambda)$;

(iv) $\sum_{n=1}^{\infty} (\widehat{\mu}_{\omega,r}(z_n))^p < \infty$.

Moreover,

$$\|T_\mu\|_p \asymp \|\widetilde{\mu}_\omega\|_{L^p} \asymp \|\widehat{\mu}_{\omega,r}\|_{L^p} \asymp \left(\sum_{n=1}^{\infty} (\widehat{\mu}_{\omega,r}(z_n))^p\right)^{1/p}.$$

Proof. Since T_μ is positive on A^2_ω, its singular values coincide with its eigenvalues, and in particular

$$\|T_\mu\|_p^p = \sum_k \lambda_k(T_\mu)^p = \operatorname{tr}(T_\mu^p).$$

We now proceed to prove the equivalences.

(i)⇒(ii). Assume $T_\mu \in S_p(A^2_\omega)$. Then $T_\mu^p \in S_1(A^2_\omega)$. By [1, page 1017], for any positive operator S on a Hilbert space and $p \geq 1$,

$$\langle S^p x, x\rangle \geq \langle Sx, x\rangle^p, \qquad \|x\| = 1.$$

Applying this to $S = T_\mu$ and $x = k_z^\omega$, we obtain

$$\langle T_\mu^p k_z^\omega, k_z^\omega\rangle \geq \widetilde{\mu}_\omega(z)^p, \qquad z \in \mathbb{D}.$$

Lemma 3.4 and Lemma 2.1(iii) yield

$$\|T_\mu\|_p^p = \operatorname{tr}(T_\mu^p) \gtrsim \int_{\mathbb{D}} \widetilde{\mu}_\omega(z)^p \, d\lambda(z),$$

and hence $\widetilde{\mu}_\omega \in L^p(\mathbb{D}, d\lambda)$ with

$$\int_{\mathbb{D}} \widetilde{\mu}_\omega(z)^p \, d\lambda(z) \lesssim \|T_\mu\|_p^p.$$

(ii)⇒(iii). Let $r_0 \in (0,1)$ be as in Lemma 3.1. Then

$$\widehat{\mu}_{\omega,r_0}(z) \asymp \int_{E(z,r_0)} |k_z^\omega(\xi)|^2 \, d\mu(\xi) \lesssim \widetilde{\mu}_\omega(z),$$

so $\widehat{\mu}_{\omega,r_0} \in L^p(\mathbb{D}, d\lambda)$. Fix $r \in (0,1)$ and choose an r_0–lattice $\{z_n\}$. There exist $\rho = \rho(r, r_0) \in (r_0, 1)$ and an integer $M = M(r, r_0)$ such that, for every $z \in \mathbb{D}$,

$$E(z,r) \subset \bigcup_{n \in I(z)} E(z_n, r_0), \qquad I(z) := \{n : \ z \in E(z_n, \rho)\}.$$

For $n \in I(z)$ we have $z \in E(z_n, \rho)$, hence $[\omega](z) \asymp [\omega](z_n)$ by Lemma 2.1(ii). Therefore

$$\widehat{\mu}_{\omega,r}(z) = \frac{\mu(E(z,r))}{[\omega](z)} \lesssim \sum_{n \in I(z)} \widehat{\mu}_{\omega,r_0}(z_n).$$

Since $p \geq 1$ and $|I(z)| \leq M$, it follows that

$$(\widehat{\mu}_{\omega,r}(z))^p \lesssim \sum_{n\in I(z)} (\widehat{\mu}_{\omega,r_0}(z_n))^p .$$

Integrating with respect to $d\lambda$ and changing the order of summation, we have

$$\begin{aligned}\int_{\mathbb{D}} (\widehat{\mu}_{\omega,r}(z))^p \, d\lambda(z) &\lesssim \sum_{n=1}^{\infty} (\widehat{\mu}_{\omega,r_0}(z_n))^p \int_{\{z:\, n\in I(z)\}} d\lambda(z)\\ &\leq \lambda(E(0,\rho)) \sum_{n=1}^{\infty} (\widehat{\mu}_{\omega,r_0}(z_n))^p .\end{aligned}$$

By Lemma 3.6 (applied with $r = r_0$), the last sum is finite, and therefore $\widehat{\mu}_{\omega,r} \in L^p(\mathbb{D}, d\lambda)$.

(iii)⇒(iv). This follows from Lemma 3.6.

(iv)⇒(i). Let $\{z_n\}$ be an r–lattice in $\mathbb{D}$. Write

$$a_n := \widehat{\mu}_{\omega,r}(z_n), \qquad \text{and} \qquad N^* := \{n : a_n > 0\}.$$

Indices outside N^* do not contribute. We treat the cases $p = 1$ and $p > 1$ separately.

Step 1: Boundedness of T_μ. Since $(a_n) \in \ell^p$, the sequence (a_n) is bounded. Applying Theorem 2.3 with $q = 2$, we obtain that μ is an A^2_ω–Carleson measure. Hence T_μ is bounded, i.e., $T_\mu \in S_\infty(A^2_\omega)$.

Step 2: The case $p = 1$. Let $\{e_m\}$ be an orthonormal basis of A^2_ω. Using Lemma 2.1(iii), we obtain

$$\begin{aligned}\sum_m \langle T_\mu e_m, e_m\rangle &= \int_{\mathbb{D}} \left(\sum_m |e_m(z)|^2 \right) d\mu(z)\\ &= \int_{\mathbb{D}} \|K^\omega_z\|^2_{A^2_\omega} \, d\mu(z)\\ &\asymp \int_{\mathbb{D}} \frac{1}{[\omega](z)} \, d\mu(z).\end{aligned}$$

On each $E(z_n, r)$, local comparability yields $[\omega](z) \asymp [\omega](z_n)$, and therefore

$$\int_{\mathbb{D}} \frac{1}{[\omega](z)} \, d\mu(z) \asymp \sum_{n=1}^{\infty} \frac{\mu(E(z_n, r))}{[\omega](z_n)} = \sum_{n=1}^{\infty} \widehat{\mu}_{\omega,r}(z_n) = \sum_{n\in N^*} a_n < \infty.$$

Hence $T_\mu \in S_1(A^2_\omega)$.
Step 3: The case $1 < p < \infty$. For $\zeta \in \mathbb{C}$, define a (finite) complex measure

$$d\nu_\zeta := \sum_{n\in N^*} a_n^{p\zeta-1} \, \chi_{E(z_n,r)} \, d\mu.$$

The sum is well-defined since each point belongs to at most N of the sets $E(z_n, r)$. For each $n \in N^*$, $a_n^{p\zeta-1}$ is an entire function of ζ. Fix ζ with $0 \le \Re\zeta \le 1$ and let

$$N_0 := \{n \in N^* : a_n \ge 1\}.$$

For $n \notin N_0$, $a_n^{p\Re\zeta-1} \le a_n^{-1}$, and

$$\begin{aligned}\sum_{n\notin N_0} a_n^{p\Re\zeta-1}\,\mu(E(z_n,r)) &\le \sum_{n\notin N_0} a_n^{-1}\,\mu(E(z_n,r)) \\ &= \sum_{n\notin N_0} [\omega](z_n) \\ &\lesssim \sum_{n\notin N_0} \int_{E(z_n,r)} \omega(z)\,dA(z) \\ &\lesssim \int_{\mathbb{D}} \omega(z)\,dA(z) < \infty.\end{aligned}$$

On the other hand, since N_0 is finite, there exists $C_\mu > 0$ such that $a_n^{p\Re\zeta-1} \le C_\mu$ for all $n \in N_0$, and hence

$$\sum_{n\in N_0} a_n^{p\Re\zeta-1}\,\mu(E(z_n,r)) \le C_\mu\,\mu(\mathbb{D}) < \infty.$$

Combining these estimates, we conclude that

$$|\nu_\zeta|(\mathbb{D}) \le \sum_{n\in N^*} a_n^{p\Re\zeta-1}\,\mu(E(z_n,r)) < \infty.$$

Thus for every ζ with $0 \le \Re\zeta \le 1$, ν_ζ is a finite complex Borel measure, depending holomorphically on ζ.

Let $\zeta = it$. Then $|a_n^{p\zeta-1}| = a_n^{-1}$. For each k,

$$|\nu_\zeta|(E(z_k,r)) \le \sum_{n\in I(k)} a_n^{-1}\mu(E(z_n,r)) = \sum_{n\in I(k)} [\omega](z_n),$$

where $I(k) = \{n : E(z_n,r) \cap E(z_k,r) \ne \emptyset\}$. Lattice geometry and almost standardness give

$$[\omega](z_n) \le C[\omega](z_k), \quad |I(k)| \le N,$$

hence

$$\widehat{|\nu_\zeta|}_{\omega,r}(z_k) \le CN.$$

By Theorem 2.3, $|\nu_\zeta|$ is an A^2_ω–Carleson measure, and hence the operator T_{ν_ζ} defined by

$$\langle T_{\nu_\zeta} f, g\rangle = \int_{\mathbb{D}} f(\xi)\overline{g(\xi)}\,d\nu_\zeta(\xi)$$

is bounded on A^2_ω, with

$$M_0 := \sup_{\Re\zeta=0} \|T_{\nu_\zeta}\| < \infty.$$

Let $\zeta = 1 + it$. Then $|a_n^{p\zeta-1}| = a_n^{p-1}$. For each k,

$$|\nu_\zeta|(E(z_k, r)) \le \sum_{n \in I(k)} a_n^{p-1}\, \mu(E(z_n, r)) = \sum_{n \in I(k)} a_n^p [\omega](z_n).$$

Using the decomposition of $\operatorname{tr}(T_{|\nu_\zeta|})$ over the lattice and changing the order of summation, one obtains

$$\begin{aligned}
\operatorname{tr}(T_{|\nu_\zeta|}) &= \int_{\mathbb{D}} \|K_z^\omega\|_{A^2_\omega}^2 \, d|\nu_\zeta|(z) \\
&= \sum_{k \in N^*} \int_{E(z_k, r)} \|K_z^\omega\|_{A^2_\omega}^2 \, d|\nu_\zeta|(z) \\
&\lesssim \sum_{k \in N^*} \|K_{z_k}^\omega\|_{A^2_\omega}^2 \sum_{n \in I(k)} a_n^p [\omega](z_n) \\
&= \sum_{n \in N^*} a_n^p [\omega](z_n) \sum_{k \colon n \in I(k)} \|K_{z_k}^\omega\|_{A^2_\omega}^2 .
\end{aligned}$$

Since $|I(k)| \le N$ and $E(z_k, r) \cap E(z_n, r) \neq \emptyset$ implies $\|K_{z_k}^\omega\|_{A^2_\omega}^2 \asymp \|K_{z_n}^\omega\|_{A^2_\omega}^2$, we conclude

$$\sum_{k \colon n \in I(k)} \|K_{z_k}^\omega\|_{A^2_\omega}^2 \lesssim \|K_{z_n}^\omega\|_{A^2_\omega}^2 \asymp \frac{1}{[\omega](z_n)}.$$

Hence $T_{|\nu_\zeta|}$ is a positive trace class and

$$\|T_{|\nu_\zeta|}\|_1 = \operatorname{tr}(T_{|\nu_\zeta|}) \lesssim \sum_{n \in N^*} a_n^p < \infty.$$

Since $|T_{\nu_\zeta}| \le T_{|\nu_\zeta|}$, by standard properties of singular values,

$$\|T_{\nu_\zeta}\|_1 = \operatorname{tr}(|T_{\nu_\zeta}|) \le \operatorname{tr}(T_{|\nu_\zeta|}) \lesssim \sum_{n \in N^*} a_n^p .$$

Consequently,

$$M_1 := \sup_{\Re\zeta = 1} \|T_{\nu_\zeta}\|_1 \lesssim \sum_n a_n^p .$$

For fixed $f, g \in A^2_\omega$, the function $\zeta \mapsto \langle T_{\nu_\zeta} f, g \rangle_{A^2_\omega}$ is analytic on the open strip and continuous on its closure. By complex interpolation theorem [18, Theorem 2.6],

$$T_{\nu_{1/p}} \in S_p(A^2_\omega)$$

and

$$\|T_{\nu_{1/p}}\|_p^p \lesssim M_0^{p-1} M_1 \lesssim \sum_n a_n^p .$$

Note that $a_n^{p(1/p)-1} = a_n^0 = 1$, hence

$$d\nu_{1/p} = \sum_{n \in N^*} \chi_{E(z_n, r)} \, d\mu .$$

Since $\{E(z_n, r)\}$ covers $\mathbb{D}$, we have $\sum_n \chi_{E(z_n,r)} \geq 1$ on $\mathbb{D}$, and since the overlap is bounded by N, we have $\sum_n \chi_{E(z_n,r)} \leq N$ on $\mathbb{D}$. Therefore

$$d\mu \leq d\nu_{1/p} \leq N\, d\mu,$$

and consequently, for $f \in A^2_\omega$,

$$\langle T_\mu f, f\rangle \leq \langle T_{\nu_{1/p}} f, f\rangle \leq N\langle T_\mu f, f\rangle.$$

That is,

$$T_\mu \leq T_{\nu_{1/p}} \leq N T_\mu,$$

as positive operators. We conclude that $T_\mu \in S_p(A^2_\omega)$ and

$$\|T_\mu\|_p^p \asymp \|T_{\nu_{1/p}}\|_p^p \lesssim \sum_n a_n^p.$$

□

Example 3.8. *Let*

$$\omega_\alpha(z) = (1-|z|^2)^\alpha, \qquad \alpha \geq 0,$$

and let

$$d\mu_\beta(z) = (1-|z|^2)^\beta\, dA(z), \qquad \beta > -1.$$

For each fixed $r \in (0,1)$, the standard estimates for pseudohyperbolic disks give

$$\mu_\beta(E(z,r)) \asymp (1-|z|^2)^{\beta+2},$$

while

$$[\omega_\alpha](z) = (1-|z|^2)^{\alpha+2}.$$

Hence

$$\widehat{\mu_\beta}_{\omega_\alpha,r}(z) = \frac{\mu_\beta(E(z,r))}{[\omega_\alpha](z)} \asymp (1-|z|^2)^{\beta-\alpha}.$$

Therefore, by Theorem 3.7,

$$T_{\mu_\beta} \in S_p(A^2_{\omega_\alpha}) \iff \widehat{\mu_\beta}_{\omega_\alpha,r} \in L^p(\mathbb{D}, d\lambda).$$

Since $d\lambda(z) = dA(z)/(1-|z|^2)^2$, this is equivalent to

$$\int_{\mathbb{D}} (1-|z|^2)^{p(\beta-\alpha)}\, d\lambda(z) < \infty,$$

which holds if and only if

$$p(\beta-\alpha) > 1,$$

or, equivalently,

$$\beta > \alpha + \frac{1}{p}.$$

Thus, for the standard weights, Theorem 3.7 recovers the classical Schatten class threshold for positive Toeplitz operators:

$$T_{\mu_\beta} \in S_p(A^2_{\omega_\alpha}) \iff \beta > \alpha + \frac{1}{p}.$$

This agrees with the known characterizations of Luecking and Zhu for standard weighted Bergman spaces; see, e.g., [11, 17].

We next extend the Schatten class inclusion obtained for positive measures to Toeplitz operators induced by complex measures, obtaining a sufficient condition in terms of the total variation. For a finite complex Borel measure μ on $\mathbb{D}$, we write $|\mu|$ for its total variation measure and define

$$\widehat{|\mu|}_{\omega,r}(z) = \frac{|\mu|(E(z,r))}{[\omega](z)}, \qquad z \in \mathbb{D}.$$

Theorem 3.9. *Let ω be an almost standard weight on $\mathbb{D}$, $1 \le p < \infty$, and $0 < r < 1$. Let μ be a finite complex Borel measure on $\mathbb{D}$, and $\{z_n\}$ be an r–lattice in $\mathbb{D}$. If*

$$\sum_n \left(\widehat{|\mu|}_{\omega,r}(z_n)\right)^p < \infty,$$

then $T_\mu \in S_p(A^2_\omega)$ and

$$\|T_\mu\|_p^p \lesssim \sum_n \left(\widehat{|\mu|}_{\omega,r}(z_n)\right)^p. \tag{6}$$

Proof. Write $d\mu = \sigma\, d|\mu|$ with $|\sigma| = 1$, $|\mu|$–a.e. Then the associated Toeplitz operator admits the factorization

$$T_\mu = I^*_{|\mu|} M_\sigma I_{|\mu|},$$

where $I_{|\mu|} : A^2_\omega \to L^2(|\mu|)$ is the canonical embedding and M_σ is unitary on $L^2(|\mu|)$. Moreover, $T_{|\mu|} = I^*_{|\mu|} I_{|\mu|}$. By Theorem 3.7 applied to the positive measure $|\mu|$, the assumption

$$\sum_n (\widehat{|\mu|}_{\omega,r}(z_n))^p < \infty$$

implies that $T_{|\mu|} \in S_p(A^2_\omega)$ and

$$\|T_{|\mu|}\|_p^p \asymp \sum_n (\widehat{|\mu|}_{\omega,r}(z_n))^p.$$

Consequently $I_{|\mu|} \in S_{2p}$, and by the ideal property of Schatten classes,

$$T_\mu \in S_p(A^2_\omega) \quad \text{and} \quad \|T_\mu\|_p \lesssim \|T_{|\mu|}\|_p,$$

which yields (6). □

Example 3.10. *Let ν be a finite positive Borel measure on $\mathbb{D}$, and suppose that*

$$\sum_n (\widehat{\nu}_{\omega,r}(z_n))^p < \infty.$$

Let

$$d\mu = e^{i\theta}\, d\nu$$

for some constant $\theta \in \mathbb{R}$. Then $|\mu| = \nu$, and Theorem 3.9 implies that

$$T_\mu \in S_p(A^2_\omega)$$

with

$$\|T_\mu\|_p^p \lesssim \sum_n (\widehat{\nu}_{\omega,r}(z_n))^p \,.$$

Moreover, since $T_\mu = e^{i\theta} T_\nu$, the operators T_μ and T_ν have the same singular values. Therefore, Theorem 3.7, applied to the positive measure $\nu = |\mu|$, shows that in this constant-argument case the sufficient condition in Theorem 3.9 is also necessary. Namely,

$$T_\mu \in S_p(A^2_\omega) \quad \Longleftrightarrow \quad \sum_n \left(\widehat{|\mu|}_{\omega,r}(z_n)\right)^p < \infty,$$

and the corresponding two-sided norm estimates hold.

Remark 3.11. The sufficient condition in Theorem 3.9 is not necessary for general complex measures. Fix $0 < r < 1$, and let $\{z_n\}$ be an r–lattice in $\mathbb{D}$. If $r' \in (0, r)$ is the separation constant in the definition of the lattice, then the disks $E(z_n, r'/2)$ are pairwise disjoint. Hence, by the local comparability of almost standard weights,

$$\sum_n [\omega](z_n) \lesssim \sum_n \int_{E(z_n, r'/2)} \omega(\xi)\, dA(\xi) \le \int_{\mathbb{D}} \omega(\xi)\, dA(\xi) < \infty.$$

Moreover, since $T_{\delta_a} = K^\omega_a \otimes K^\omega_a$ and $K^\omega_b \to K^\omega_a$ in A^2_ω as $b \to a$, we have $T_{\delta_b} \to T_{\delta_a}$ in the trace norm.

For each n, choose $\zeta_n \in E(z_n, r'/4)$ such that

$$\zeta_n \notin \{z_k : k \ge 1\}$$

and

$$\|T_{\delta_{z_n}} - T_{\delta_{\zeta_n}}\|_{S_1} < \frac{2^{-n}}{1 + [\omega](z_n)}.$$

Then

$$[\omega](z_n) \|T_{\delta_{z_n}} - T_{\delta_{\zeta_n}}\|_{S_1} < 2^{-n}.$$

Define

$$\mu = \sum_n [\omega](z_n)(\delta_{z_n} - \delta_{\zeta_n}).$$

The measure μ is finite, and since the atoms are pairwise distinct,

$$|\mu| = \sum_n [\omega](z_n)(\delta_{z_n} + \delta_{\zeta_n}).$$

Since $\zeta_n \in E(z_n, r'/4)$, the atoms of $|\mu|$ remain uniformly attached to the lattice $\{z_n\}$. Thus the finite overlap property and the local comparability of $[\omega]$ imply

$$|\mu|(E(z_k, r)) \le C[\omega](z_k), \qquad k \ge 1.$$

Equivalently,

$$\sup_{k\geq 1} \widehat{|\mu|}_{\omega,r}(z_k) < \infty.$$

By Theorem 2.3 with $q = 2$, $|\mu|$ is an A^2_ω–Carleson measure, and hence the Toeplitz form induced by μ is bounded.

On the other hand,

$$T_\mu = \sum_n [\omega](z_n) \left(T_{\delta_{z_n}} - T_{\delta_{\zeta_n}}\right)$$

converges in S_1. Therefore $T_\mu \in S_p(A^2_\omega)$ for every $1 \leq p < \infty$. However,

$$\widehat{|\mu|}_{\omega,r}(z_n) = \frac{|\mu|(E(z_n, r))}{[\omega](z_n)} \geq 1$$

for every n, and consequently

$$\sum_n \left(\widehat{|\mu|}_{\omega,r}(z_n)\right)^p = \infty.$$

Thus $T_\mu \in S_p(A^2_\omega)$, whereas the discrete condition in Theorem 3.9 fails for $|\mu|$. This shows that the condition in Theorem 3.9 is sufficient, but not necessary in general.

Theorem 3.12. *Let ω be an almost standard weight on $\mathbb{D}$, $1 \leq p < \infty$, and $0 < r < 1$. Let μ and ν be finite complex Borel measures on $\mathbb{D}$, and $\{z_n\}$ be an r–lattice in $\mathbb{D}$. If*

$$\sum_n \left(\widehat{|\mu - \nu|}_{\omega,r}(z_n)\right)^p < \infty, \tag{7}$$

then $T_\mu - T_\nu \in S_p(A^2_\omega)$ and

$$\|T_\mu - T_\nu\|_p^p \lesssim \sum_n \left(\widehat{|\mu - \nu|}_{\omega,r}(z_n)\right)^p.$$

Proof. Set $\lambda = \mu - \nu$. Then $T_\mu - T_\nu = T_\lambda$. Assumption (7) and Theorem 3.7 applied to the positive measure $|\lambda| = |\mu - \nu|$ yield $T_{|\lambda|} \in S_p$ and

$$\|T_{|\lambda|}\|_p^p \asymp \sum_n \left(\widehat{|\lambda|}_{\omega,r}(z_n)\right)^p.$$

Applying Theorem 3.9 to λ gives $T_\lambda \in S_p$ and

$$\|T_\mu - T_\nu\|_p^p = \|T_\lambda\|_p^p \lesssim \|T_{|\lambda|}\|_p^p,$$

which implies the desired estimate. □

Example 3.13. *Let*

$$\omega_\alpha(z) = (1 - |z|^2)^\alpha, \qquad \alpha \geq 0,$$

and let

$$d\eta_\beta(z) = (1 - |z|^2)^\beta \, dA(z), \qquad \beta > -1.$$

Let σ be a finite complex measure for which T_σ is bounded, and set

$$d\mu = d\sigma + e^{i\theta}\, d\eta_\beta, \qquad d\nu = d\sigma,$$

where $\theta \in \mathbb{R}$ is fixed. Then

$$\mu - \nu = e^{i\theta}\eta_\beta \qquad \textit{and} \qquad |\mu - \nu| = \eta_\beta.$$

As in the Example 3.8, for each fixed $r \in (0,1)$,

$$\widehat{\eta_\beta}_{\omega_\alpha, r}(z) \asymp (1 - |z|^2)^{\beta - \alpha}.$$

Therefore condition (7) holds if and only if

$$\beta > \alpha + \frac{1}{p}.$$

In this case Theorem 3.12 gives

$$T_\mu - T_\nu = e^{i\theta} T_{\eta_\beta} \in S_p(A^2_{\omega_\alpha}).$$

Thus, for standard weights, the theorem recovers the classical Schatten class threshold for differences whose inducing measure has constant argument:

$$T_\mu - T_\nu \in S_p(A^2_{\omega_\alpha}) \quad \Longleftrightarrow \quad \beta > \alpha + \frac{1}{p}.$$

Remark 3.14. The condition in Theorem 3.12 is a sufficient condition expressed in terms of the total variation of the difference measure $\mu - \nu$. If $\mu - \nu$ has constant argument, say $d(\mu - \nu) = e^{i\theta}\, d\tau$ with $\tau \geq 0$, then

$$T_\mu - T_\nu = e^{i\theta} T_\tau.$$

Hence Theorem 3.7 applied to the positive measure τ shows that, in this constant-argument case, condition (7) is also necessary and the corresponding two-sided norm estimates hold. For general complex difference measures, however, cancellations in the phase of $\mu - \nu$ are not detected by $|\mu - \nu|$, so it is stated as a sufficient criterion.

We conclude by mentioning some possible directions for further study. It would be natural to investigate whether the Schatten class characterizations obtained here remain valid for broader classes of radial weights, or even for suitable non-radial weights. Another challenging problem is to study the quasi-Banach range $0 < p < 1$, for which the methods used in the present paper do not apply directly. One may also ask whether analogous results hold for weighted Bergman spaces over higher dimensional domains, such as the unit ball. Finally, the case of complex measures deserves further attention: while the present paper provides sufficient conditions in terms of total variation, sharper necessary and sufficient criteria would require a more precise understanding of the cancellation phenomena caused by the phase of the measure.

Acknowledgements. The author would like to thank the anonymous referees for a careful reading of the manuscript and for helpful comments that improved the presentation of the paper.

Conflict of Interest. The author declares no conflicts of interest that are relevant to the content of this article.

References

[1] Arazy, J., Fisher, S. D., & Peetre, J. (1988). Hankel operators on weighted Bergman spaces. Amer. J. Math., 110(6), 989–1053. https://doi.org/10.2307/2374685

[2] Arroussi, H., Park, I., & Pau, J. (2015). Schatten class Toeplitz operators acting on large weighted Bergman spaces. Studia Math., 229(3), 203–221. https://doi.org/10.4064/sm8345-12-2015

[3] Constantin, O. (2010). Carleson embeddings and some classes of operators on weighted Bergman spaces. J. Math. Anal. Appl., 365(2), 668–682. https://doi.org/10.1016/j.jmaa.2009.11.035

[4] Čučković, Z., & Zhao, R. (2004). Weighted composition operators on the Bergman space. J. London Math. Soc., 70(2), 499–511. https://doi.org/10.1112/S0024610704005605

[5] Duan, Y., Guo, K., Wang, S., & Wang, Z. (2022). Toeplitz operators on weighted Bergman spaces induced by a class of radial weights. J. Geom. Anal., 32, Paper No. 39, 29 pp. https://doi.org/10.1007/s12220-021-00777-z

[6] Duren, P., & Schuster, A. (2004). Bergman Spaces. Mathematical Surveys and Monographs, 100. American Mathematical Society, Providence, RI. https://doi.org/10.1090/surv/100

[7] Esmaeili, K., & Kellay, K. (2023). Weighted composition operators on weighted Bergman and Dirichlet spaces. Canad. Math. Bull., 66(1), 286–302. https://doi.org/10.4153/S0008439522000297

[8] Esmaeili, K. (2026). A new characterization for the essential norm of weighted composition operators and their differences on weighted Bergman spaces. Adv. Oper. Theory, 11(2), Paper No. 22, 27 pp. https://doi.org/10.1007/s43036-026-00497-7

[9] Huang, W., Huang, L., & Wang, X. (2024). Schatten class properties and essential norm estimates of operators on Bergman spaces induced by regular weights of annulus. Banach J. Math. Anal., 18(4), Paper No. 68, 48 pp. https://doi.org/10.1007/s43037-024-00378-2

[10] Lin, P., & Rochberg, R. (1996). Trace ideal criteria for Toeplitz and Hankel operators on the weighted Bergman spaces with exponential type weights. Pacific J. Math., 173(1), 127–146.

[11] Luecking, D. H. (1987). Trace ideal criteria for Toeplitz operators. J. Funct. Anal., 73(2), 345–368. https://doi.org/10.1016/0022-1236(87)90072-3

[12] Peláez, J. A., & Rättyä, J. (2014). Weighted Bergman spaces induced by rapidly increasing weights. Memoirs Amer. Math. Soc., 227, No. 1066. https://doi.org/10.1090/memo/1066

[13] Peláez, J. A., & Rättyä, J. (2016). Trace class criteria for Toeplitz and composition operators on small Bergman spaces. Adv. Math., 293, 606–643. https://doi.org/10.1016/j.aim.2016.02.017

[14] Wang, X., Xia, J., & Liu, Y. (2023). Schatten class operators on exponential weighted Bergman spaces. J. Inequal. Appl., 2023, Paper No. 129, 26 pp. https://doi.org/10.1186/s13660-023-03031-y

[15] Yang, W., & Liu, J. (2023). Schatten classes of Toeplitz operators on Bergman–Besov Hilbert spaces in the unit ball. J. Math. Anal. Appl., 526(2), Paper No. 127257. https://doi.org/10.1016/j.jmaa.2023.127257

[16] Zeng, Z., Hu, Z., & Wang, X. (2025). Bounded, compact and Schatten class Hankel operators on Fock-type spaces. Trans. Amer. Math. Soc., 378(2), 805–849. https://doi.org/10.1090/tran/9290

[17] Zhu, K. (1988). Positive Toeplitz operators on weighted Bergman spaces of bounded symmetric domains. J. Operator Theory, 20(2), 329–357.

[18] Zhu, K. (2007). Operator Theory in Function Spaces. Second edition. Mathematical Surveys and Monographs, 138. American Mathematical Society, Providence, RI. https://doi.org/10.1090/surv/138

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