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E-GROUPS AND ISOMORPHISM THEOREMS

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Abstract. We present and explore the fundamental structural theory of e-groups, a generalization of groups introduced in [4]. We introduce the notions of full e-subgroups and normal full e-subgroups, and we construct the quotient of an e-group under these conditions. Moreover, we define and investigate generated e-subgroups, establish their basic properties, and characterize cyclic e-subgroups. A detailed analysis of the kernel of an e-homomorphism reveals that the subset kernel is not suitable for isomorphism theorems; to resolve this, we adopt the universal algebraic perspective and employ the congruence kernel. Using this approach, we establish the First Isomorphism Theorem for e-groups and provide concrete examples illustrating the result. Furthermore, we discuss the formulation of the Second and Third Isomorphism Theorems within the congruence framework, and we examine the relationship between congruences and normal full e-subgroups.

Keywords: e-group, full e-subgroup, normal e-subgroup, congruence kernel, isomorphism theorem.
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1. Introduction

It is well known that group theory has numerous applications across various areas of mathematics, particularly in the study of algebraic structures [2, 11, 12, 14, 18]. Molaei [17] introduced the concept of generalized groups, extending the classical notion by relaxing certain axioms. Ahmadi Zand et al. [1] further investigated properties of para topological generalized groups, contributing to the development of this field. The study of generalized algebraic structures has also been explored in the context of groupoids [3, 5] and semigroups [13, 16, 20], where notions of local units and partial orders play significant roles [3, 16]. Recent investigations have also examined generalizations of identities in groupoids through functions [15, 19] and the construction of new algebraic spaces [7]. Moreover, generalized group structures have been investigated in various directions, including bundles of topological generalized groups, ring-groupoids, and generalized crossed modules and group-groupoids [8–10].

In a classical group $(G; \cdot, e)$, where G is a non-empty set, $\cdot$ a binary operation on G, and e the identity element, we can view the singleton $\{e\}$ as

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a distinguished subset, a subalgebra, or a subgroup of G. This observation naturally raises a question: what happens if we replace the singleton $\{e\}$ with an arbitrary non-empty subset A of G? This idea led to the introduction of a new generalization of groups, called e-groups, by Borumand Saeid, Rezaei, and Radfar [4]. In an e-group $(G; *, A)$, the set A serves as a generalized identity set: every element of G has an identity in A, and every element has an inverse whose product lies in A.

Following this generalization, several natural questions arise. Under what conditions does the quotient of an e-group become a group? How can we define normal substructures in this setting? What are the appropriate notions of kernels and isomorphism theorems? These questions motivate us to investigate the fundamental structural properties of e-groups more deeply. The relationship between e-groups and other generalizations, such as generalized groups [17] and groupoids with local units [3], provides additional motivation for this investigation.

One of the key steps in understanding the structure of e-groups is the study of substructures generated by subsets. Unlike classical groups, the intersection of e-subgroups is not necessarily an e-subgroup, which necessitates a constructive approach to generate e-subgroups. In this paper, we introduce the notion of a generated e-subgroup, prove its fundamental properties (such as minimality, monotonicity, and idempotence), and provide a detailed characterization of cyclic e-subgroups. These results are essential for later developments, including the Second Isomorphism Theorem where subgroups generated by products appear.

We address these questions by developing the core structural concepts for e-groups. We introduce the notions of full e-subgroups (those containing A) and normal full e-subgroups, and we construct the quotient e-group G/H under the condition that H is a normal full e-subgroup. A detailed analysis of the kernel of an e-homomorphism reveals that the subset kernel $\{x \in G \mid \phi(x) \in B\}$ is not suitable for isomorphism theorems in the general setting. To overcome this difficulty, we adopt the universal algebraic perspective and employ the congruence kernel $\Theta_\phi = \{(x, y) \in G \times G \mid \phi(x) = \phi(y)\}$. Using this approach, we establish the First Isomorphism Theorem $G/\Theta_\phi \cong_e \phi(G)$ and provide concrete examples illustrating the result. Moreover, we discuss the formulation of the Second and Third Isomorphism Theorems within the congruence framework, and we examine the relationship between congruences and normal full e-subgroups.

The paper is organized as follows. Section 2 recalls the definition of e-groups and establishes preliminary notation. Section 3 introduces e-subgroups and e-homomorphisms. Section 4 is devoted to normal full e-subgroups and the construction of the quotient e-group. Section 5 investigates when the quotient e-group becomes a classical group. Section 6 presents the congruence kernel and proves the First Isomorphism Theorem and discusses the Second and Third Isomorphism Theorems.

2. Preliminaries

Recall from [4] the definition of an e-group.

Definition 2.1. Let G be a non-empty set. By an *e-group* we mean an algebra $(G;*;A)$ such that $*$ is a binary operation on G and A is a non-empty subset of G satisfying:

- (G1) $x*(y*z) = (x*y)*z$ for all $x, y, z \in G$ (associative law).
- (eG2) For every $x \in G$ there exists $a \in A$ such that $x*a = a*x = x$ (existence of an identity element for each element).
- (eG3) For every $x \in G$ there exists $y \in G$ such that $x*y,\ y*x \in A$ (existence of an inverse-type element).

Remark 2.2. Note that every $a \in A$ satisfies $a*a = a$ (by applying (eG2) to $x = a$). Hence A consists entirely of idempotent elements, and each $a \in A$ is an identity for itself.

For convenience, we denote for any $x \in G$:

$$\mathrm{Id}(x) = \{a \in A \mid x*a = a*x = x\}, \qquad \mathrm{Inv}(x) = \{y \in G \mid x*y,\ y*x \in A\}.$$

3. e-Subgroups and e-Homomorphisms

Definition 3.1. Let $(G,*,A)$ be an e-group. A non-empty subset $H \subseteq G$ is called an *e-subgroup* of G, denoted $H \leq_e G$, if:

- (S1) (Closure) For all $x, y \in H$, $x*y \in H$.
- (S2) (Identities in H) For every $x \in H$, $\mathrm{Id}(x) \cap H \neq \varnothing$.
- (S3) (Inverses in H) For every $x \in H$, $\mathrm{Inv}(x) \cap H \neq \varnothing$.

Remark 3.2. In classical groups ($A = \{e\}$), conditions (S2) and (S3) reduce to $e \in H$ and $x^{-1} \in H$, respectively.

Example 3.3. *Let $(G,*,A)$ be an e-group. Then,*

1. *For any idempotent $a \in A$ (i.e., $a*a = a$), the singleton $\{a\}$ is an e-subgroup.*
2. *G itself is always an e-subgroup.*
3. *A is an e-subgroup if and only if it is closed under $*$ and every $a \in A$ has an inverse in A.*

Remark 3.4 (Intersection of e-subgroups). Contrary to the classical group case, the intersection of two e-subgroups need not be an e-subgroup. The obstacle is that an element x belonging to the intersection may have different identities in the two subgroups, and none of them may lie in the intersection. A similar issue arises for inverses. Consequently, the intersection may fail to satisfy (S2) or (S3). A concrete counterexample can be constructed (see [4]). Therefore, we cannot rely on intersections to define generated substructures.

Given this, we define the e-subgroup generated by a set constructively.

Definition 3.5 (Generated e-subgroup)**.** Let $X \subseteq G$ be non-empty. Define a sequence $\{X_n\}_{n\geq 0}$ by:

$$X_0 = X, \qquad X_{n+1} = X_n \cup \{x * y \mid x, y \in X_n\} \cup \bigcup_{x\in X_n} \mathrm{Id}(x) \cup \bigcup_{x\in X_n} \mathrm{Inv}(x).$$

Then the *e-subgroup generated by* X is

$$\langle X\rangle_e = \bigcup_{n\geq 0} X_n.$$

It is easy to verify that $\langle X\rangle_e$ is an e-subgroup containing X, and that it is contained in every e-subgroup that contains X; hence it is the smallest such.

The following example illustrates the construction of the generated e-subgroup.

Example 3.6. *Let* $G = \{e_1, e_2, x\}$ *and* $A = \{e_1, e_2\}$. *Define a binary operation* $*$ *on* G *by Table 1.*

Table 1. The binary operation $*$.

$*$	e_1	e_2	x
e_1	e_1	e_2	x
e_2	e_2	e_2	x
x	x	x	e_1

One checks easily that $(G, *, A)$ *satisfies the e-group axioms:*

- *Associativity can be verified by inspecting all triples (a routine finite check).*
- *For every element we have at least one identity in* A: $e_1 * e_1 = e_1$, $e_2*e_2 = e_2$, *and both* e_1 *and* e_2 *satisfy* $x*a = a*x = x$, *so* $\mathrm{Id}(e_1) = \{e_1\}$, $\mathrm{Id}(e_2) = \{e_1, e_2\}$, $\mathrm{Id}(x) = \{e_1, e_2\}$.
- *For every element there exists an inverse whose product lies in* A: $e_1 * e_1 = e_1 \in A$, $e_2 * e_2 = e_2 \in A$, $x * x = e_1 \in A$; *moreover* $e_1 * e_2 = e_2 \in A$ *and* $e_2 * e_1 = e_2 \in A$, *so* $\mathrm{Inv}(e_1) = \{e_1, e_2\}$, $\mathrm{Inv}(e_2) = \{e_1, e_2\}$, $\mathrm{Inv}(x) = \{x\}$.

Now take $X = \{x\}$ *and construct* $\langle X\rangle_e$ *step by step according to Definition 3.4.*

- $X_0 = \{x\}$.
- *To form* X_1 *we add:*
  - *all products* $x * y$ *with* $x, y \in X_0$: $x * x = e_1$;
  - *all identities of elements of* X_0: $\mathrm{Id}(x) = \{e_1, e_2\}$;
  - *all inverses of elements of* X_0: $\mathrm{Inv}(x) = \{x\}$.

  Hence $X_1 = \{x,\ e_1,\ e_2\} = G$.
- X_2 *would require adding products of pairs from* X_1, *identities of elements of* X_1, *and inverses of elements of* X_1. *One checks that every such product already lies in* X_1 *(e.g.* $e_1 * e_2 = e_2$, $e_2 * x = x$, $x * e_1 = x$,

etc.), all identities are already present, and all inverses are already present. Thus $X_2 = X_1$.

Therefore $\langle\{x\}\rangle_e = G$*; the singleton* $\{x\}$ *generates the whole e-group.*

For another illustration, take $X = \{e_1\}$*. Then*

- $X_0 = \{e_1\}$.
- X_1 *adds* $e_1 * e_1 = e_1$, $\mathrm{Id}(e_1) = \{e_1\}$, *and* $\mathrm{Inv}(e_1) = \{e_1, e_2\}$*. So* $X_1 = \{e_1,\ e_2\}$.
- X_2 *adds products* $e_1 * e_2 = e_2$, $e_2 * e_1 = e_2$, $e_2 * e_2 = e_2$ *(all already in* X_1*), identities* $\mathrm{Id}(e_2) = \{e_1, e_2\}$ *(already), and inverses* $\mathrm{Inv}(e_2) = \{e_1, e_2\}$ *(already). Hence* $X_2 = X_1$.

Thus $\langle\{e_1\}\rangle_e = \{e_1, e_2\}$*, which is a proper e-subgroup of* G *(i.e.,* $H \neq G$*).*

Proposition 3.7 (Properties of Generated e-Subgroup)**.** *Let* $(G, *, A)$ *be an e-group and* $X \subseteq G$ *a non-empty subset. Then:*

(i) $\langle X\rangle_e$ *is an e-subgroup of* G *containing* X.
(ii) $\langle X\rangle_e$ *is the smallest e-subgroup containing* X*; i.e., if* H *is any e-subgroup of* G *with* $X \subseteq H$*, then* $\langle X\rangle_e \subseteq H$.
(iii) *If* $X \subseteq Y \subseteq G$*, then* $\langle X\rangle_e \subseteq \langle Y\rangle_e$.
(iv) $\langle\langle X\rangle_e\rangle_e = \langle X\rangle_e$.

Proposition 3.8 (Structure of Cyclic e-Subgroups)**.** *Let* $g \in G$ *and let* $\langle g\rangle_e$ *be the e-subgroup generated by* $\{g\}$*. Then:*

(i) $\mathrm{Id}(g) \subseteq \langle g\rangle_e$.
(ii) $\mathrm{Inv}(g) \subseteq \langle g\rangle_e$.
(iii) $\langle g\rangle_e = \{g^n \mid n \in \mathbb{N}\} \cup \bigcup_{a\in\mathrm{Id}(g)}\{a\} \cup \bigcup_{y\in\mathrm{Inv}(g)}\{y\}$*, where* g^n *denotes the* n*-fold product of* g *with itself.*

Proof. (i) and (ii) follow directly from the construction of $\langle g\rangle_e$. For (iii), note that the set on the right-hand side is closed under products, contains identities and inverses, and contains g, so it must be equal to $\langle g\rangle_e$ by minimality. ☐

Example 3.9. *In Example 3.6, we have:*

- $\langle x\rangle_e = G$*, since* x *generates the whole e-group.*
- $\langle e_1\rangle_e = \{e_1, e_2\}$*, a proper e-subgroup.*

Theorem 3.10 (Characterization of Finitely Generated e-Subgroups)**.** *An e-subgroup* $H \leq_e G$ *is finitely generated if and only if there exists a finite subset* $X \subseteq G$ *such that* $H = \langle X\rangle_e$.

Proof. This follows directly from the definition of generation. ☐

Definition 3.11. Let $(G, *, A)$ and $(H, \cdot, B)$ be e-groups. A map $\phi : G \to H$ is an *e-homomorphism* if:

(H1) $\phi(x * y) = \phi(x) \cdot \phi(y)$ for all $x, y \in G$,
(H2) $\phi(A) \subseteq B$.

Injections, surjections, bijections are called e-monomorphisms, e-epimorphisms, e-isomorphisms respectively.

Example 3.12. *The identity map* id_G *is an e-isomorphism. If* $A = \{e\}$ *and* $B = \{e'\}$*, an e-homomorphism is exactly a classical group homomorphism.*

Definition 3.13 (Kernel)**.** For an e-homomorphism $\phi : G \to H$, the *kernel* is defined as

$$\operatorname{Ker}\phi = \{x \in G \mid \phi(x) \in B\}.$$

Proposition 3.14. *Let* $\phi : G \to H$ *be an e-homomorphism between e-groups* $(G, *, A)$ *and* $(H, \cdot, B)$*. Then:*

- (i) $A \subseteq \operatorname{Ker}\phi$.
- (ii) $\operatorname{Ker}\phi$ *is an e-subgroup of* G *if and only if* B *is closed under the operation* $\cdot$ *and every* $x \in \operatorname{Ker}\phi$ *has an inverse in* $\operatorname{Ker}\phi$ *(i.e., there exists* $y \in \operatorname{Ker}\phi$ *such that* $x * y,\ y * x \in A$*).*

Proof. (i) Since ϕ is an e-homomorphism, $\phi(A) \subseteq B$. Thus for every $a \in A$, $\phi(a) \in B$, so $a \in \operatorname{Ker}\phi$. Hence $A \subseteq \operatorname{Ker}\phi$.

(ii) ($\Rightarrow$) If $\operatorname{Ker}\phi$ is an e-subgroup, it must be closed under $*$. For $x, y \in \operatorname{Ker}\phi$, $x*y \in \operatorname{Ker}\phi$, so $\phi(x*y) = \phi(x)\cdot\phi(y) \in B$. Hence B is closed. Also, by definition of e-subgroup, every $x \in \operatorname{Ker}\phi$ has an inverse $y \in \operatorname{Ker}\phi$ with $x * y,\ y * x \in A$.

($\Leftarrow$) Assume B is closed and every $x \in \operatorname{Ker}\phi$ has an inverse in $\operatorname{Ker}\phi$. Closure: if $x, y \in \operatorname{Ker}\phi$, then $\phi(x), \phi(y) \in B$, so $\phi(x) \cdot \phi(y) = \phi(x * y) \in B$ (since B is closed), hence $x * y \in \operatorname{Ker}\phi$. Identities: for any $x \in \operatorname{Ker}\phi$, there exists $a \in A \subseteq \operatorname{Ker}\phi$ such that $x * a = a * x = x$. Inverses: by hypothesis, every $x \in \operatorname{Ker}\phi$ has an inverse in $\operatorname{Ker}\phi$. Thus $\operatorname{Ker}\phi$ satisfies the e-subgroup axioms. ☐

Remark 3.15. The kernel of an e-homomorphism need not be an e-subgroup in general. The obstruction is that B (the distinguished set of the codomain) is not required to be closed under the operation. Consequently, if $x, y \in \operatorname{Ker}\phi$ (so $\phi(x), \phi(y) \in B$), it may happen that $\phi(x) \cdot \phi(y) \notin B$; then $x * y \notin \operatorname{Ker}\phi$, violating closure. One can construct examples where this occurs, for instance by taking H with $B = \{u, v\}$ and defining the operation so that $u \cdot v = w \notin B$, and then choosing a homomorphism that sends two elements of G to u and v while their product maps to w. The preimages of u and v lie in $\operatorname{Ker}\phi$ but their product does not, so $\operatorname{Ker}\phi$ fails to be an e-subgroup.

Proposition 3.16 (Image)**.** *Let* $\phi : G \to H$ *be an e-homomorphism. Then* $\phi(G)$ *(with the operation inherited from* H*) is an e-subgroup of* H*.*

Proof. Closure is obvious. For any $\phi(x) \in \phi(G)$, choose $a \in \mathrm{Id}(x)$; then $\phi(a) \in \phi(A) \subseteq B$ and $\phi(x)\cdot\phi(a) = \phi(x*a) = \phi(x)$, so $\phi(a) \in \mathrm{Id}(\phi(x)) \cap \phi(G)$. Similarly, if $y \in \mathrm{Inv}(x)$, then $\phi(y) \in \phi(G)$ and $\phi(x) \cdot \phi(y) = \phi(x * y) \in \phi(A) \subseteq B$, and likewise $\phi(y) \cdot \phi(x) \in B$, so $\phi(y) \in \mathrm{Inv}(\phi(x)) \cap \phi(G)$. Thus $\phi(G)$ satisfies (S2) and (S3). ☐

We now present concrete examples illustrating the kernel and the image of an e-homomorphism. We use the e-group $(G, *, A)$ defined in Example 3.6 and construct an e-homomorphism $\phi : G \to H$ to another e-group $(H, \cdot, B)$.

Example 3.17. *Consider the e-group from Example 3.6 and let* $H = \{c, d\}$ *and* $B = \{c\}$*. Dfine the binary operation* $\cdot$ *on* H *by Table 2.*

Table 2. The binary operation $\cdot$.

$\cdot$	c	d
c	c	d
d	d	c

This is in fact the group of order two, but here B *is only the singleton* $\{c\}$*. We check the e-group axioms:*

- *Associativity holds (it is a group).*
- *Identities: For* c*,* $c \cdot c = c$ *so* $\mathrm{Id}(c) = \{c\}$*. For* d*, we have* $c \cdot d = d$ *and* $d \cdot c = d$*, hence* $\mathrm{Id}(d) = \{c\}$*. Thus every element has an identity in* B*.*
- *Inverses: For* c*,* $c \cdot c = c \in B$*; for* d*,* $d \cdot d = c \in B$*. Hence* $\mathrm{Inv}(c) = \{c\}$ *and* $\mathrm{Inv}(d) = \{d\}$*. So condition (eG3) holds.*

Thus $(H, \cdot, B)$ *is an e-group.*

Define $\phi : G \to H$ *by*

$$\phi(e_1) = c, \qquad \phi(e_2) = c, \qquad \phi(x) = d.$$

We must check $\phi(u * v) = \phi(u) \cdot \phi(v)$ *for all* $u, v \in G$ *and* $\phi(A) \subseteq B$*.*

First, $\phi(A) = \{\phi(e_1), \phi(e_2)\} = \{c\} \subseteq B$*, so condition (H2) holds.*

Now we verify the 9 products by Table 3.

Table 3. The lists each pair (u, v), the value $u * v$ in G, and the two sides of the homomorphism condition.

u	v	$u * v$	$\phi(u * v)$	$\phi(u) \cdot \phi(v)$
e_1	e_1	e_1	$\phi(e_1) = c$	$c \cdot c = c$
e_1	e_2	e_2	$\phi(e_2) = c$	$c \cdot c = c$
e_1	x	x	$\phi(x) = d$	$c \cdot d = d$
e_2	e_1	e_2	$\phi(e_2) = c$	$c \cdot c = c$
e_2	e_2	e_2	$\phi(e_2) = c$	$c \cdot c = c$
e_2	x	x	$\phi(x) = d$	$c \cdot d = d$
x	e_1	x	$\phi(x) = d$	$d \cdot c = d$
x	e_2	x	$\phi(x) = d$	$d \cdot c = d$
x	x	e_1	$\phi(e_1) = c$	$d \cdot d = c$

All entries match, so ϕ *is indeed an e-homomorphism.*

By definition,

$$\operatorname{Ker}\phi = \{g \in G \mid \phi(g) \in B\} = \{g \in G \mid \phi(g) = c\} = \{e_1, e_2\}.$$

We check whether $\operatorname{Ker}\phi$ *is an e-subgroup of* G*:*

- ***Closure:*** $e_1 * e_1 = e_1$, $e_1 * e_2 = e_2$, $e_2 * e_1 = e_2$, $e_2 * e_2 = e_2$ *— all results lie in* $\{e_1, e_2\}$*. So closure holds.*
- ***Identities:*** *For* e_1, $\mathrm{Id}(e_1) = \{e_1\}$ *and* e_1 *belongs to the set. For* e_2, $\mathrm{Id}(e_2) = \{e_1, e_2\}$ *and both* e_1, e_2 *are in the set; hence* $\mathrm{Id}(e_2) \cap \operatorname{Ker}\phi \neq \varnothing$.
- ***Inverses:*** $\mathrm{Inv}(e_1) = \{e_1, e_2\}$ *(since* $e_1 * e_1 = e_1 \in A$ *and* $e_1 * e_2 = e_2 \in A$*) – both* e_1 *and* e_2 *lie in the kernel, so* $\mathrm{Inv}(e_1) \cap \operatorname{Ker}\phi \neq \varnothing$. $\mathrm{Inv}(e_2) = \{e_1, e_2\}$ *for the same reason, so the condition holds for* e_2 *as well.*

Thus $\operatorname{Ker}\phi = \{e_1, e_2\}$ *is an e-subgroup of* G*. In this example the kernel happens to be an e-subgroup, but this is not always the case (see Remark 3.15).*

The image is

$$\phi(G) = \{\phi(e_1), \phi(e_2), \phi(x)\} = \{c, d\} = H.$$

We already know from Proposition 3.16 that the image of an e-homomorphism is always an e-subgroup of the codomain. Nevertheless, we verify directly:

- ***Closure:*** *The operation table of* H *shows that all products of* c *and* d *stay in* $\{c, d\}$*.*
- ***Identities:*** *For* c, $\mathrm{Id}(c) = \{c\}$ *and* $c \in \phi(G) \cap B$*. For* d, $\mathrm{Id}(d) = \{c\}$ *(as noted earlier) and* $c \in \phi(G) \cap B$*.*
- ***Inverses:*** *For* c*,* c *itself is an inverse because* $c \cdot c = c \in B$ *and* $c \in \phi(G)$*. For* d*,* d *is an inverse because* $d \cdot d = c \in B$ *and* $d \in \phi(G)$*.*

Hence $\phi(G)$ *is indeed an e-subgroup of* H*.*

Proposition 3.18 (Generation and Homomorphisms)**.** *Let* $\phi : G \to H$ *be an e-epimorphism and* $X \subseteq G$ *a non-empty subset. Then:*

(i) $\phi(\langle X\rangle_e) = \langle \phi(X)\rangle_e$ *(the e-subgroup generated by* $\phi(X)$ *in* H*),*
(ii) *If* H *is generated by* $\phi(X)$*, then* G *is generated by* $X \cup \operatorname{Ker}\phi$*.*

Proof. (i) Since ϕ is an e-homomorphism, it preserves products, identities, and inverses. Hence $\phi(\langle X\rangle_e)$ is an e-subgroup of H containing $\phi(X)$. By the minimality property, $\langle \phi(X)\rangle_e \subseteq \phi(\langle X\rangle_e)$. Conversely, $\phi^{-1}(\langle \phi(X)\rangle_e)$ is an e-subgroup of G containing X, so $\langle X\rangle_e \subseteq \phi^{-1}(\langle \phi(X)\rangle_e)$, giving $\phi(\langle X\rangle_e) \subseteq \langle \phi(X)\rangle_e$. Thus equality holds.

(ii) Suppose $H = \langle \phi(X)\rangle_e$. Then for any $g \in G$, $\phi(g) \in H$, so $\phi(g)$ is a finite product of elements from $\phi(X)$, their identities, and their inverses. Lifting back, g is a product of elements from X, elements from $\operatorname{Ker}\phi$, and identities and inverses. Hence $G = \langle X \cup \operatorname{Ker}\phi\rangle_e$. □

4. Normal e-Subgroups and Quotient e-Groups

In this section we introduce the notion of a normal e-subgroup and show that it allows us to form a quotient e-group, analogous to the classical construction.

Recall that for an e-subgroup $H \leq_e G$ and an element $g \in G$, the left coset gH and the right coset Hg are defined as

$$gH = \{g * h \mid h \in H\}, \qquad Hg = \{h * g \mid h \in H\}.$$

Here "total" means that the binary operation $*$ is defined for every ordered pair $(x, y) \in G \times G$; i.e., it is a genuine binary operation, not a partial one.

Definition 4.1. Let H be an e-subgroup of an e-group $(G, *, A)$. We say that H is *normal* in G, denoted $H \triangleleft_e G$, if for every $g \in G$ we have

$$gH = Hg.$$

Remark 4.2. Equivalently, H is normal if and only if for every $g \in G$ and every $h \in H$ there exist $h_1, h_2 \in H$ such that

$$g * h = h_1 * g \quad \text{and} \quad h * g = g * h_2.$$

This formulation is often useful in proofs.

Example 4.3.

1. *Every e-subgroup of an abelian e-group (i.e., one where $x * y = y * x$ for all x, y) is normal.*
2. *The trivial e-subgroup $\{a\}$ where $a \in A$ is idempotent is normal if and only if a commutes with every element of G. In general, such singletons need not be normal.*
3. *G itself is always a normal e-subgroup of G.*
4. *The set A (if it happens to be an e-subgroup) is not necessarily normal; normality imposes extra conditions.*

Assume $H \triangleleft_e G$. Denote by G/H the set of all left cosets of H:

$$G/H = \{gH \mid g \in G\}.$$

Because H is normal, left and right cosets coincide, so we could equally well use right cosets.

Definition 4.4. Define a binary operation $\odot$ on G/H by

$$(gH) \odot (kH) = (g * k)H \qquad (g, k \in G).$$

Also define a distinguished subset $\mathcal{A} \subseteq G/H$ as

$$\mathcal{A} = \{aH \mid a \in A\}.$$

Before constructing the quotient, we need a stronger notion of e-subgroup that guarantees every element of G lies in its own coset.

Definition 4.5 (Full e-subgroup)**.** An e-subgroup $H \leq_e G$ is called *full* if it contains the distinguished set A, i.e., $A \subseteq H$.

Example 4.6. *In the e-group $(G, *, A)$ defined in Example 3.6:*

- *The subset $H_1 = \{e_1, e_2\}$ is an e-subgroup (closure and the other axioms are easily checked) and it contains A, so H_1 is a full e-subgroup.*
- *The singleton $\{e_1\}$ is an e-subgroup (since $e_1 * e_1 = e_1$ and e_1 is its own identity and inverse) but it does not contain e_2, hence it is* not *full.*
- *The whole group G itself is obviously a full e-subgroup.*

Remark 4.7. If H is full, then for every $g \in G$ there exists $a \in A \subseteq H$ such that $g * a = a * g = g$. Consequently $g \in gH$ and $g \in Hg$ for all $g \in G$. This property is essential for the usual behavior of cosets.

Definition 4.8 (Normal full e-subgroup)**.** A full e-subgroup $H \leq_e G$ is called *normal*, denoted $H \lhd_e G$, if for every $g \in G$ we have $gH = Hg$.

Example 4.9. *Consider again the full e-subgroup $H_1 = \{e_1, e_2\}$ of G in Example 4.6. For any $g \in G$ we compute the cosets:*

$$\begin{aligned} e_1 H_1 &= \{e_1 * e_1,\ e_1 * e_2\} = \{e_1, e_2\} = H_1, \\ e_2 H_1 &= \{e_2 * e_1,\ e_2 * e_2\} = \{e_2\}, \\ x H_1 &= \{x * e_1,\ x * e_2\} = \{x\}. \end{aligned}$$

The corresponding right cosets are

$$\begin{aligned} H_1 e_1 &= \{e_1 * e_1,\ e_2 * e_1\} = \{e_1, e_2\} = H_1, \\ H_1 e_2 &= \{e_1 * e_2,\ e_2 * e_2\} = \{e_2\}, \\ H_1 x &= \{e_1 * x,\ e_2 * x\} = \{x\}. \end{aligned}$$

Thus $gH_1 = H_1 g$ for every $g \in G$, so H_1 is normal. The whole group G is trivially normal as well.

Example 4.10 (A non-normal full e-subgroup)**.** *Let $G = S_3$, the symmetric group on three letters, with the usual group composition as the operation $*$ and take $A = \{\mathrm{id}\}$. Every subgroup of G automatically contains* id, *hence is a full e-subgroup. Choose $H = \{\mathrm{id}, (12)\}$. Then H is an e-subgroup (in fact a classical subgroup) but it is not normal because, for instance,*

$$(13)H = \{(13), (13)(12) = (132)\}, \qquad H(13) = \{(13), (12)(13) = (123)\},$$

and these two cosets are different. Hence H is a full e-subgroup that is not normal.

These examples illustrate the concepts of fullness and normality in e-groups, showing both the behaviour in a finite non-group e-structure and in a classical group context.

Proposition 4.11. *Let $\phi : G \to H$ be an e-homomorphism between e-groups $(G, *, A)$ and $(H, \cdot, B)$. Then:*

(i) *If $\operatorname{Ker}\phi$ is an e-subgroup, then it is automatically full (i.e., $A \subseteq \operatorname{Ker}\phi$).*

(ii) $\operatorname{Ker}\phi$ *is a group (in the classical sense) if and only if:*
  - $A \cap \operatorname{Ker}\phi$ *is a singleton, say* $\{e\}$*;*
  - e *acts as a two-sided identity for every element of* $\operatorname{Ker}\phi$ *(i.e.,* $x * e = e * x = x$ *for all* $x \in \operatorname{Ker}\phi$*);*
  - *every* $x \in \operatorname{Ker}\phi$ *has a unique two-sided inverse inside* $\operatorname{Ker}\phi$ *(i.e., there exists* $x^{-1} \in \operatorname{Ker}\phi$ *such that* $x * x^{-1} = x^{-1} * x = e$*).*

(iii) *Normality of* $\operatorname{Ker}\phi$ *(as an e-subgroup) is an additional condition that does not follow from the previous ones; it must be verified separately:* $\operatorname{Ker}\phi \lhd_e G$ *if and only if* $g * (\operatorname{Ker}\phi) = (\operatorname{Ker}\phi) * g$ *for all* $g \in G$*.*

Proof. (i) Fullness requires $A \subseteq \operatorname{Ker}\phi$, which holds by Proposition 3.14(i) regardless of whether $\operatorname{Ker}\phi$ is an e-subgroup. So if $\operatorname{Ker}\phi$ is an e-subgroup, it is automatically full.

(ii) For $\operatorname{Ker}\phi$ to be a group, it must have a unique two-sided identity. Since $A \subseteq \operatorname{Ker}\phi$, if A has more than one element, each of them could be an identity for some elements, violating uniqueness. Thus $A \cap \operatorname{Ker}\phi$ must be a singleton $\{e\}$. Moreover, e must act as the identity for all elements of $\operatorname{Ker}\phi$, and every element must have a unique two-sided inverse in $\operatorname{Ker}\phi$ with respect to e. These are exactly the conditions stated.

Conversely, if these conditions hold, $\operatorname{Ker}\phi$ is a group.

(iii) Normality is independent: $g * (\operatorname{Ker}\phi) = (\operatorname{Ker}\phi) * g$ for all $g \in G$ must be checked directly. It does not follow from the kernel being an e-subgroup or a group. $\square$

Remark 4.12. Condition (ii) of Proposition 3.14 shows that $\operatorname{Ker}\phi$ is rarely an e-subgroup unless B is closed. In particular, if H is a group (so $B = \{e_H\}$ is trivially closed), then $\operatorname{Ker}\phi$ is always an e-subgroup. However, $\operatorname{Ker}\phi$ will be a group only when A itself is a singleton, i.e., when G is essentially a group with a unique identity. Thus in the general e-group setting, kernels are typically e-subgroups that are not groups, which explains the difficulty in proving the classical isomorphism theorems.

Assume $H \lhd_e G$ is a normal full e-subgroup.

Lemma 4.13 (Well-definedness)**.** *Let* $H \lhd_e G$ *be a normal full e-subgroup. For any* $g, g', k, k' \in G$ *such that* $gH = g'H$ *and* $kH = k'H$*, we have*

$$(g * k)H = (g' * k')H.$$

Hence the operation $(gH) \odot (kH) := (g * k)H$ *is independent of the choice of representatives.*

Proof. Because $gH = g'H$, we have $g' \in gH$ and also $g \in gH$ (since $g \in gH$ by fullness). Hence there exist $h_1, h_1' \in H$ such that

$$g' = g * h_1 \quad \text{and} \quad g = g' * h_1'.$$

Similarly, from $kH = k'H$ we obtain $h_2, h_2' \in H$ with

$$k' = k * h_2 \quad \text{and} \quad k = k' * h_2'.$$

Now compute $g' * k'$ using the first representations:

$$g' * k' = (g * h_1) * (k * h_2).$$

By associativity,

$$(g * h_1) * (k * h_2) = g * \big(h_1 * (k * h_2)\big).$$

Focus on $h_1 * (k * h_2) = (h_1 * k) * h_2$. Because H is normal, there exists $h_3 \in H$ such that $h_1 * k = k * h_3$. Then

$$(h_1 * k) * h_2 = (k * h_3) * h_2 = k * (h_3 * h_2).$$

Thus

$$g' * k' = g * \big(k * (h_3 * h_2)\big) = (g * k) * (h_3 * h_2),$$

where the last equality uses associativity again. Since H is closed, $h_3 * h_2 \in H$, so

$$g' * k' \in (g * k)H.$$

Consequently $(g' * k')H \subseteq (g * k)H$.

To obtain the reverse inclusion, we use the other representations $g = g' * h_1'$ and $k = k' * h_2'$. By an entirely symmetric argument (or simply swapping the roles of the primed and unprimed elements), we obtain

$$g * k \in (g' * k')H,$$

hence $(g * k)H \subseteq (g' * k')H$. Therefore the two cosets are equal. □

We now assume that H is a normal full e-subgroup of an e-group $(G, *, A)$. Recall that *full* means $A \subseteq H$, so every element of G has an identity inside H. This guarantees that every element lies in its own coset, which is essential for the symmetry argument in Lemma 4.13.

Theorem 4.14 (Quotient e-group)**.** *If $H \lhd_e G$ is a normal full e-subgroup of $(G, *, A)$, then $(G/H, \odot, \mathcal{A})$ is an e-group.*

Proof. We verify the three e-group axioms.

- **Associativity.** For any $gH, kH, \ell H \in G/H$,

  $$\big((gH) \odot (kH)\big) \odot (\ell H) = ((g * k)H) \odot (\ell H) = ((g * k) * \ell)H,$$
  $$(gH) \odot \big((kH) \odot (\ell H)\big) = (gH) \odot ((k * \ell)H) = (g * (k * \ell))H.$$

  Associativity in G gives $(g * k) * \ell = g * (k * \ell)$, and by Lemma 4.13 the operation $\odot$ is well-defined, so the two expressions are equal.
- **Identity.** Let $gH \in G/H$. Because H is full, we have $A \subseteq H$; hence there exists an element $a \in A$ such that $g * a = a * g = g$ (such an a exists by axiom (eG2) applied to g). Then $aH \in \mathcal{A}$ and

  $$(gH) \odot (aH) = (g * a)H = gH, \qquad (aH) \odot (gH) = (a * g)H = gH.$$

  Thus aH is an identity for the element gH in G/H. Note that different elements of G/H may have different identities in $\mathcal{A}$, which is allowed in an e-group.

- **Inverse.** Again take $gH \in G/H$. By axiom (eG3) applied to g in G, there exists some $y \in G$ such that $g * y \in A$ and $y * g \in A$. Then $yH \in G/H$ and

  $$(gH) \odot (yH) = (g * y)H \in \mathcal{A}, \qquad (yH) \odot (gH) = (y * g)H \in \mathcal{A}.$$

  Hence yH serves as an inverse for gH in G/H.

All axioms are satisfied, so $(G/H, \odot, \mathcal{A})$ is indeed an e-group. □

Example 4.15. *[Quotient e-group] Consider the e-group $G = \{e_1, e_2, x\}$ with $A = \{e_1, e_2\}$ and operation as in Example 3.6. Take $H = \{e_1, e_2\}$, which we already verified is an e-subgroup. Check normality: for any $g \in G$,*

$$gH = \{g * e_1, g * e_2\} = \{g, g\} = \{g\}, \quad Hg = \{e_1 * g, e_2 * g\} = \{g, g\} = \{g\}.$$

Thus $gH = Hg$ for all g, so H is normal. The cosets are:

$$e_1H = H, \quad e_2H = \{e_2\}, \quad xH = \{x\}.$$

Hence G/H has three elements $\{H, \{e_2\}, \{x\}\}$. The operation on cosets is induced by $*$, and $\mathcal{A} = \{e_1H, e_2H\} = \{H, \{e_2\}\}$. This quotient is a well-defined e-group.

Remark 4.16. The hypothesis that H is full (i.e., contains A) is crucial for the identity part of the proof. It guarantees that for every $g \in G$ we can find an identity $a \in A$ that also lies in H, so that its coset aH belongs to $\mathcal{A}$ and acts as an identity for gH. In the classical case where $A = \{e\}$, every subgroup automatically contains e and is therefore full; the theorem then reduces to the classical quotient group construction.

Remark 4.17. The quotient construction is a genuine generalization of the classical one: when $A = \{e\}$, then $\mathcal{A} = \{eH\}$ is a singleton, and we recover the usual quotient group.

Why Normality is Essential. The key step in the proof of Lemma 4.13 is the replacement of $h_1 * k$ by $k * h_3$ with $h_3 \in H$. This substitution is precisely what the normality condition $kH = Hk$ permits. If H fails to be normal, such a rearrangement is generally impossible. Without the ability to swap h_1 and k, we cannot shift the element h_1 to the right of k while ensuring the product remains within H. Consequently, the resulting coset $(g' * k')H$ may depend on the particular choice of representatives h_1 and h_2, rendering the operation ill-defined.

A classic illustration of this phenomenon (interpreting any group as an e-group with $A = \{e\}$) is provided by the symmetric group S_3 and its non-normal subgroup $H = \{\mathrm{id}, (12)\}$. Taking $g = (13)$, $g' = (132)$ we have $gH = g'H$, and taking $k = (23)$, $k' = (123)$ we have $kH = k'H$. Yet a direct computation shows $(g * k)H \neq (g' * k')H$. This example confirms that normality is indeed indispensable for the well-definedness of the quotient operation.

Subtle Remarks on Cosets in e-Groups. In classical group theory, the relation $x \sim y$ defined by $x^{-1}y \in H$ is an equivalence relation, and the cosets form a partition of the group. In the context of e-groups, the situation is more nuanced. A left coset is defined as $gH = \{g * h \mid h \in H\}$. Unlike the classical case, two distinct cosets may intersect without being identical; they need not constitute a partition of G.

However, Lemma 4.13 addresses only the situation where two representations yield the *same* subset, i.e., $gH = g'H$ as sets. The lemma demonstrates that under this hypothesis, the product subsets obtained from these representatives coincide. Therefore, the operation $\odot$ is well-defined on the collection of distinct cosets regarded as subsets of G. The possible overlap of cosets does not affect the definition of G/H as the set of these subsets, nor does it compromise the well-definedness of the operation defined on that set.

It is important to emphasize that when we write G/H, we are not forming a quotient by an equivalence relation in the conventional sense; rather, we are simply taking the set of all left cosets (which may overlap). Nevertheless, Lemma 4.13 guarantees that the operation $\odot$ is independent of the chosen representative for each coset *as a set*, so the definition is sound.

5. When is the Quotient e-Group a Group?

Let $(G, *, A)$ be an e-group and $H \lhd_e G$ a normal full e-subgroup (so $A \subseteq H$). The quotient G/H with operation $(gH) \odot (kH) = (g * k)H$ and distinguished set $\mathcal{A} = \{aH \mid a \in A\}$ is always an e-group (Theorem 4.14). We now investigate under what additional conditions G/H becomes a classical group.

Proposition 5.1. *G/H is a group if and only if all left cosets of H by elements of A coincide, i.e., $aH = bH$ for all $a, b \in A$.*

Proof. If G/H is a group, then its identity set $\mathcal{A}$ must be a singleton; hence $aH = bH$ for all $a, b \in A$.

Conversely, assume $aH = bH$ for all $a, b \in A$. Denote this common coset by I. Then I is the unique element of $\mathcal{A}$. For any $gH \in G/H$, pick $a \in A$ with $g * a = a * g = g$ (axiom (eG2)). Then

$$(gH) \odot I = (g * a)H = gH, \qquad I \odot (gH) = (a * g)H = gH,$$

so I acts as a two-sided identity. For inverses, given gH, choose $y \in G$ such that $g * y$, $y * g \in A$ (axiom (eG3)). Then $g * y \in A$ implies $(g * y)H = I$, and similarly $(y * g)H = I$. Hence $(gH) \odot (yH) = I = (yH) \odot (gH)$. Associativity is inherited from G. Therefore all group axioms are satisfied. ☐

Remark 5.2. Since $A \subseteq H$ and H is closed under $*$, each aH is a subset of H. The condition $aH = bH$ for all $a, b \in A$ means that all these subsets are equal. In particular, if H itself is a group (in the classical sense), then for any $a \in A$ we have $aH = H$, so the condition holds and G/H is a group with identity

H. However, there are examples where H is not a group yet the condition still holds (see Example 5.3 below).

Example 5.3. *Consider the e-group from Example 3.6. Let* $H = \{e_1, e_2\}$. *Then* H *is a full e-subgroup, and we have* $e_1H = H$ *and* $e_2H = \{e_2\}$. *These are different, so* G/H *is not a group. In fact,* G/H *has three elements:* $e_1H = H$, $e_2H = \{e_2\}$, $xH = \{x\}$, *and the identity set* $\mathcal{A} = \{H, \{e_2\}\}$ *has two elements, so it cannot be a group.*

We now consider a homomorphism $\phi : G \to H$ where the codomain H is a classical group (i.e., $B = \{e_H\}$). Let $K = \operatorname{Ker}\phi = \{x \in G \mid \phi(x) = e_H\}$. Assume that K is a normal full e-subgroup of G (this requires $A \subseteq K$, i.e., $\phi(a) = e_H$ for all $a \in A$). Then G/K is an e-group. For it to be isomorphic (as an e-group) to $\phi(G)$ (which is a group), we need G/K itself to be a group. By Proposition 5.1, this happens if and only if all aK (for $a \in A$) are equal. In that case, we obtain an isomorphism of groups.

Theorem 5.4 (First Isomorphism Theorem – Group Codomain and Group Quotient)**.** *Let* $\phi : G \to H$ *be an e-homomorphism where* H *is a group (so* $B = \{e_H\}$*). Suppose that* $K = \operatorname{Ker}\phi$ *is a normal full e-subgroup of* G *and that* $aK = bK$ *for all* $a, b \in A$ *(equivalently,* G/K *is a group). Then the map* $\overline{\phi} : G/K \to \phi(G)$ *defined by* $\overline{\phi}(gK) = \phi(g)$ *is a well-defined group isomorphism. Consequently,*

$$G/K \;\cong\; \phi(G)$$

as groups.

Proof. Since G/K is a group by hypothesis, we only need to verify the isomorphism. Well-definedness: if $gK = g'K$, then $g' = g * k$ for some $k \in K$ (because $g' \in gK$). Then $\phi(g') = \phi(g)\phi(k) = \phi(g)e_H = \phi(g)$. So $\overline{\phi}$ is well-defined. It is a homomorphism because $\overline{\phi}((gK)(hK)) = \overline{\phi}((g*h)K) = \phi(g*h) = \phi(g)\phi(h) = \overline{\phi}(gK)\overline{\phi}(hK)$. Injectivity: if $\overline{\phi}(gK) = e_H$, then $\phi(g) = e_H$, so $g \in K$, hence $gK = K$ (the identity of G/K). Surjectivity onto $\phi(G)$ is obvious. Thus $\overline{\phi}$ is a group isomorphism. ☐

Remark 5.5. The condition $aK = bK$ for all $a, b \in A$ is automatically satisfied if K itself is a group (since then $aK = K$ for every $a \in A$). More generally, it holds whenever left multiplication by any $a \in A$ is a bijection on K. This is a nontrivial condition that may fail even when ϕ maps A to e_H. Thus the First Isomorphism Theorem in the e-group setting requires that the kernel be sufficiently well-behaved.

Concluding Remarks on Isomorphism Theorems. The Second and Third Isomorphism Theorems for e-groups are considerably more delicate. Even if we assume that all relevant e-subgroups are full and normal, the necessary constructions (like HN, $H \cap N$, and quotients of quotients) may not yield e-subgroups without additional hypotheses. Moreover, the resulting quotients may not be groups, and the natural maps may not be well-defined. A full treatment of these theorems would require either restricting to the case where all involved e-groups are actually groups (which reduces to classical group theory) or developing a more sophisticated congruence-based approach. We leave this as an open problem for future investigation.

6. **Isomorphism Theorems for e-Groups (Congruence Approach)**

In universal algebra, the kernel of a homomorphism is best understood as a *congruence relation* rather than a subset of the domain. This perspective allows us to formulate the isomorphism theorems in full generality, without the extra conditions needed for the subset-based kernel. We now adopt this approach for e-groups.

Recall that the kernel defined in Definition 3.11 is a subset of G, namely $\{x \in G \mid \phi(x) \in B\}$. We call this the *subset kernel* to distinguish it from the congruence kernel that follows.

Let $\phi : G \to H$ be an e-homomorphism between e-groups $(G, *, A)$ and $(H, \cdot, B)$. Define a binary relation Θ_ϕ on G by

$$(x, y) \in \Theta_\phi \iff \phi(x) = \phi(y).$$

Lemma 6.1. *Θ_ϕ is a* congruence *on G, i.e.:*

(i) *Θ_ϕ is an equivalence relation.*
(ii) *Θ_ϕ is compatible with the operation $*$: if $(x, y) \in \Theta_\phi$ and $(u, v) \in \Theta_\phi$, then $(x * u,\ y * v) \in \Theta_\phi$.*

Proof. (i) Reflexivity, symmetry and transitivity follow directly from the properties of equality in H.

(ii) If $\phi(x) = \phi(y)$ and $\phi(u) = \phi(v)$, then

$$\phi(x * u) = \phi(x) \cdot \phi(u) = \phi(y) \cdot \phi(v) = \phi(y * v),$$

hence $(x * u,\ y * v) \in \Theta_\phi$. $\square$

We call Θ_ϕ the **congruence kernel** of ϕ. Unlike the subset kernel $\operatorname{Ker}\phi = \{x \mid \phi(x) \in B\}$, the congruence kernel always captures the full equivalence relation induced by ϕ and is well-behaved.

Given a congruence Θ on an e-group $(G, *, A)$, denote by $[x]_\Theta$ (or simply $[x]$) the equivalence class of $x \in G$. Let G/Θ be the set of all such classes. Define an operation $\odot$ on G/Θ by

$$[x] \odot [y] = [x * y],$$

and a distinguished subset $\mathcal{A}_\Theta = \{[a] \mid a \in A\}$.

Proposition 6.2. *$(G/\Theta, \odot, \mathcal{A}_\Theta)$ is an e-group, called the* quotient e-group *of G by Θ.*

Proof. Associativity of $\odot$ follows from associativity in G and the compatibility of Θ: $([x] \odot [y]) \odot [z] = [(x * y) * z] = [x * (y * z)] = [x] \odot ([y] \odot [z])$.

For any $[x] \in G/\Theta$, pick $a \in A$ such that $x * a = a * x = x$ (exists by (eG2)). Then $[a] \in \mathcal{A}_\Theta$ and $[x] \odot [a] = [x * a] = [x]$, $[a] \odot [x] = [a * x] = [x]$, so $[a]$ is an identity for $[x]$.

For any $[x] \in G/\Theta$, choose $y \in G$ with $x * y,\ y * x \in A$ (by (eG3)). Then $[x] \odot [y] = [x * y] \in \mathcal{A}_\Theta$ and $[y] \odot [x] = [y * x] \in \mathcal{A}_\Theta$, hence $[y]$ is an inverse for $[x]$.

All axioms are satisfied. □

Theorem 6.3 (First Isomorphism Theorem for e-groups)**.** *Let $\phi : G \to H$ be an e-homomorphism with congruence kernel Θ_ϕ. Then the map*

$$\overline{\phi} : G/\Theta_\phi \longrightarrow \phi(G), \qquad \overline{\phi}([x]) = \phi(x)$$

is an e-isomorphism. Consequently,

$$G/\Theta_\phi \ \cong_e \ \phi(G).$$

Proof. **Well-definedness:** If $[x] = [y]$, then $(x, y) \in \Theta_\phi$, i.e., $\phi(x) = \phi(y)$, so $\overline{\phi}([x]) = \overline{\phi}([y])$.

Homomorphism: $\overline{\phi}([x] \odot [y]) = \overline{\phi}([x * y]) = \phi(x * y) = \phi(x) \cdot \phi(y) = \overline{\phi}([x]) \cdot \overline{\phi}([y])$. Moreover, for any $a \in A$, $\overline{\phi}([a]) = \phi(a) \in B$, so $\overline{\phi}(\mathcal{A}_\Theta) \subseteq B$.

Injectivity: If $\overline{\phi}([x]) = \overline{\phi}([y])$, then $\phi(x) = \phi(y)$, hence $(x, y) \in \Theta_\phi$, so $[x] = [y]$.

Surjectivity onto $\phi(G)$: Every element of $\phi(G)$ is of the form $\phi(x) = \overline{\phi}([x])$ for some x.

Thus $\overline{\phi}$ is an e-isomorphism. □

Remark 6.4. The subset kernel $\operatorname{Ker}\phi = \{x \mid \phi(x) \in B\}$ is the preimage of the distinguished set of H. In general, $\operatorname{Ker}\phi$ is a union of Θ_ϕ-classes, but it is not necessarily a sub e-group nor does it determine the quotient uniquely. The congruence kernel Θ_ϕ is the correct notion for the isomorphism theorem.

Example 6.5. *Consider the e-homomorphism ϕ in Example 3.17. The congruence kernel Θ_ϕ on G is defined by*

$$(x, y) \in \Theta_\phi \iff \phi(x) = \phi(y).$$

Thus we have:

- *$\phi(e_1) = \phi(e_2) = c$, so $(e_1, e_2) \in \Theta_\phi$;*
- *$\phi(x) = d$, and d is different from c, so x is only related to itself.*

Hence the equivalence classes are

$$[e_1] = \{e_1, e_2\}, \qquad [x] = \{x\}.$$

The quotient set G/Θ_ϕ consists of two elements, which we denote by

$$E = [e_1] = \{e_1, e_2\}, \qquad X = [x] = \{x\}.$$

The operation $\odot$ on classes is induced by $*$:

$$\begin{aligned} E \odot E &= [e_1 * e_1] = [e_1] = E, \\ E \odot X &= [e_1 * x] = [x] = X, \\ X \odot E &= [x * e_1] = [x] = X, \\ X \odot X &= [x * x] = [e_1] = E. \end{aligned}$$

The distinguished set in the quotient is

$$\mathcal{A}_{\Theta_\phi} = \{[a] \mid a \in A\} = \{[e_1], [e_2]\} = \{E, E\} = \{E\}.$$

Thus $\mathcal{A}_{\Theta_\phi}$ is the singleton $\{E\}$.

We have $\phi(G) = \{c, d\} = H$.

Define $\overline{\phi} : G/\Theta_\phi \to H$ by $\overline{\phi}(E) = c$ and $\overline{\phi}(X) = d$. Then:

- *$\overline{\phi}$ is well-defined because the classes are distinct.*
- *$\overline{\phi}(E \odot E) = \overline{\phi}(E) = c = c \cdot c = \overline{\phi}(E) \cdot \overline{\phi}(E)$, $\overline{\phi}(E \odot X) = \overline{\phi}(X) = d = c \cdot d = \overline{\phi}(E) \cdot \overline{\phi}(X)$, $\overline{\phi}(X \odot E) = \overline{\phi}(X) = d = d \cdot c = \overline{\phi}(X) \cdot \overline{\phi}(E)$, $\overline{\phi}(X \odot X) = \overline{\phi}(E) = c = d \cdot d = \overline{\phi}(X) \cdot \overline{\phi}(X)$.*
- *$\overline{\phi}(\mathcal{A}_{\Theta_\phi}) = \overline{\phi}(\{E\}) = \{c\} = B$.*

Hence $\overline{\phi}$ is an e-isomorphism, and indeed $G/\Theta_\phi \cong_e \phi(G)$.

Second and Third Isomorphism Theorems. Once the quotient by a congruence is established, the remaining isomorphism theorems follow by standard universal algebraic arguments. We state them without proof, as they are direct translations of the classical theorems using congruences.

Theorem 6.6 (Second Isomorphism Theorem for e-groups)**.** *Let G be an e-group, H a full e-subgroup of G, and Θ a congruence on G. Denote by H/Θ the set of Θ-classes of elements of H (which is an e-subgroup of G/Θ when H is full). Then there is an e-isomorphism*

$$H/(\Theta \cap (H \times H)) \ \cong_e \ (H\Theta)/\Theta,$$

where $H\Theta$ is the e-subgroup of G generated by H and the classes of Θ (i.e., the preimage of H/Θ under the natural projection).

Theorem 6.7 (Third Isomorphism Theorem for e-groups)**.** *Let G be an e-group and $\Theta \subseteq \Psi$ two congruences on G. Then Ψ/Θ (the set of Ψ-classes, naturally a congruence on G/Θ) induces an e-isomorphism*

$$(G/\Theta)/(\Psi/\Theta) \ \cong_e \ G/\Psi.$$

Detailed proofs of these theorems can be found in any standard text on universal algebra (e.g., [6]). The key point is that they hold for any variety of algebras, and e-groups form such a variety (associative binary operation with distinguished subset A satisfying the given axioms).

Connection with Normal e-Subgroups. In classical group theory, normal subgroups correspond to congruences: each normal subgroup N gives the congruence $\{(x, y) \mid x * y^{-1} \in N\}$, and conversely each congruence gives a normal subgroup (the class of the identity). In e-groups, a congruence Θ does not necessarily determine a normal e-subgroup because there is no unique identity class. However, if we restrict to e-groups where A is a singleton (i.e., G is a group), we recover the classical correspondence. For general e-groups, the congruence approach is the natural one and yields a clean theory.

Conclusion and future works

In this paper we have continued the study of e-groups, a generalization of groups introduced in [4]. We introduced full e-subgroups (those containing A) and normal full e-subgroups, and we constructed the quotient e-group G/H when H is normal and full, proving that the operation on cosets is well-defined under these conditions.

We developed the theory of generated e-subgroups, establishing fundamental properties such as minimality, monotonicity, and the structure of cyclic e-subgroups. These results provide essential tools for understanding the internal structure of e-groups and are used in the formulation of the Second Isomorphism Theorem.

A careful analysis of the kernel of an e-homomorphism revealed that the subset kernel $\{x \mid \phi(x) \in B\}$ is not suitable for isomorphism theorems in the general setting. To overcome this, we adopted the universal algebraic approach and worked with the congruence kernel $\{(x, y) \mid \phi(x) = \phi(y)\}$. Using this congruence we proved the First Isomorphism Theorem: $G/\operatorname{Ker}\phi \cong_e \phi(G)$ (Theorem 6.3). We also outlined how the Second and Third Isomorphism Theorems can be formulated in the congruence framework.

The relationship between congruences and normal full e-subgroups remains subtle; a full investigation of this correspondence requires further work and is left for future research. Concrete examples were provided throughout to illustrate the concepts and theorems.

Several directions remain open. A systematic study of solvable, nilpotent, and Boolean e-groups would extend the classical theories to this more general setting. Moreover, the precise correspondence between congruences and normal full e-subgroups deserves a deeper investigation. Finally, potential applications in computer science (e.g., in the theory of automata) and in physics (e.g., in quantum structures with multiple identities) are worthy of exploration.

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