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BIFURCATIONS IN A DISCRETIZED ECONOMIC MODEL: ANALYTICAL AND NUMERICAL APPROACHES

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Abstract. This paper explores a continuous-time economic model and proposes a technique for discretizing it into a discrete-time system using Euler's method. By manipulating the parameter ω, which represents the savings proportion, we determine the critical coefficient of bifurcations in the normal form using the center manifold. The coefficient for flip and Neimark-Sacker bifurcations at the trivial fixed point is obtained through affine transformations, while closed forms are employed for other types of bifurcations. To analyze the conditions and coefficients of the normal forms for each bifurcation, we employ both analytical and numerical methods. Additionally, we visualize the bifurcation curve in the parameter space and generate a bifurcation diagram using the MatcontM package.

Keywords: Neimark-Sacker bifurcation, Flip bifurcation, 1:4 resonance, Chenciner (generalized Neimark-Sacker) bifurcation.
2020 MSC: 34C23, 74H60, 37M20, 37N40.

1. Introduction

The study of economic models provides valuable insights into economic phenomena and their implications. By analyzing these models, we can gain a better understanding of economic issues and their complexities. In particular, the analysis of bifurcations in economic models offers a useful framework for comprehending the behavior of these models and the potential consequences of parameter changes [3, 4, 17, 24]. Bifurcations refer to fundamental changes in the dynamics of a system resulting from variations in its parameters. Such changes can manifest as the emergence of new equilibrium points, the creation of periodic orbits, or qualitative changes in system stability. Bifurcation theory provides a rigorous mathematical framework for understanding qualitative changes in dynamical systems when parameters vary. Among the most important bifurcations in discrete-time systems are the flip bifurcation, the Neimark-Sacker bifurcation, and higher-codimension bifurcations such as the Chenciner (generalized Neimark-Sacker) bifurcation and 1:4 resonance.
The flip bifurcation (also known as period-doubling bifurcation) occurs when

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a fixed point loses stability and a stable periodic orbit of period 2 emerges. In economic terms, this corresponds to the transition from a steady equilibrium to alternating periods of expansion and contraction (boom-bust cycles). The Neimark-Sacker bifurcation is the discrete-time analogue of the Hopf bifurcation; it occurs when a pair of complex conjugate eigenvalues crosses the unit circle, leading to the birth of a quasi-periodic invariant closed curve. This bifurcation captures business cycles with varying amplitudes and frequencies.
The Chenciner bifurcation (or generalized Neimark-Sacker bifurcation) is a codimension-2 bifurcation that occurs when the first Lyapunov coefficient vanishes, leading to more complex dynamics including the appearance of multiple invariant curves and phase-locking phenomena. The 1:4 resonance is another codimension-2 bifurcation where the eigenvalues satisfy $\lambda^4 = 1$ with $\lambda = i$, giving rise to 4-periodic cycles and potentially chaotic behavior. Understanding these bifurcations is essential for predicting the response of economic systems to changes in key parameters such as the savings proportion ω considered in this paper.
In this work, we discretize a continuous-time economic model using Euler's method, analyze the bifurcations of its equilibrium points, compute the critical normal form coefficients, and validate our findings numerically. In 1967, Goodwin introduced a groundbreaking theory that challenged the prevailing belief attributing economic fluctuations solely to external factors. Goodwin argued that internal factors could also contribute to economic fluctuations [2, 5, 6, 8, 14, 22]. Subsequently, numerous economic models were proposed to explore this concept further.

In this paper, we focus on the modified Van der Pol oscillator model, originally proposed by Bouali in 1999 [5], which has recently undergone further modifications. The model is described by the following system of differential equations:

$$
\begin{cases} \dot{x} &= \mu y + \rho x(\delta - y^2), \\ \dot{y} &= \nu y + \omega x + \sigma z, \\ \dot{z} &= \gamma x - \kappa y, \end{cases} \tag{1}
$$

Specifically, we now clarify that:

- x (savings), y (GDP (Gross Domestic Product)), and z (foreign capital inflow) interact through a nonlinear system inspired by the modified Van der Pol oscillator.
- In $\dot{x} = \mu y + \rho x(\delta - y^2)$, the term μy represents the linear positive influence of GDP on savings, while $\rho x(\delta - y^2)$ introduces a nonlinear feedback where savings growth is enhanced when GDP is below the threshold $\sqrt{\delta}$ and suppressed when GDP exceeds this threshold.
- In $\dot{y} = \nu y + \omega x + \sigma z$, all terms are positive, indicating that GDP grows with its own level, savings, and foreign capital inflow.

- In $\dot{z} = \gamma x - \kappa y$, foreign capital inflow is attracted by savings (γx) and decreases with GDP ($-\kappa y$), reflecting capital outflow in times of economic expansion.

Further details can be found in [6]. To facilitate analysis, we discretize the system using Euler's method:

$$\begin{cases} x_{n+1} &= x_n + h(\mu y_n + \rho x_n(\delta - y_n^2)), \\ y_{n+1} &= y_n + h(\nu y_n + \omega x_n + \sigma z_n), \\ z_{n+1} &= z_n + h(\gamma x_n - \kappa y_n), \end{cases} \tag{2}$$

The use of Euler's method for discretization is motivated by several considerations. First, it provides the simplest discretization scheme, yielding a discrete-time system that is analytically tractable for bifurcation analysis using center manifold theory and normal form computations. Second, Euler's method preserves the equilibrium points of the original continuous-time system, allowing direct comparison between the continuous and discrete dynamics. Third, this discretization is commonly employed in the economic literature for analyzing discrete-time versions of continuous models (see, e.g., [3, 6]). Finally, the step size h is chosen sufficiently small to ensure numerical stability and accuracy, as demonstrated by the numerical simulations in Section 4. These factors justify the use of Euler's method as an appropriate discretization approach for our analysis.

The system (2) possesses three equilibrium points: $X_0 = (0,0,0)$, $X_1 = (x^*, y^*, z^*)$, and $X_2 = (-x^*, -y^*, -z^*)$, where $x^* = \frac{\sqrt{\delta\,\kappa\,\rho+\gamma\,\mu}\sqrt{\kappa}}{\sqrt{\rho}\gamma}$, $y^* = \frac{\sqrt{\delta\,\kappa\,\rho+\gamma\,\mu}}{\sqrt{\kappa}\sqrt{\rho}}$, and $z^* = \frac{\sqrt{\delta\,\kappa\,\rho+\gamma\,\mu}(\gamma\,\nu+\kappa\,\omega)}{\sqrt{\kappa}\sqrt{\rho}\gamma\,\sigma}$.

In the subsequent sections, we delve into the bifurcations of equilibrium point X_0 in Section 2, explore the bifurcations of equilibrium point X_1 in Section 3, and conduct numerical investigations of the aforementioned bifurcations in Section 4. For additional research on bifurcations in discrete-time systems, refer to [1, 10–13, 18].

2. Bifurcation of equilibrium point X_0

The Jacobian matrix of system (2) is computed as follows:

$$\tilde{J} = \begin{pmatrix} 1 + h\rho(\delta - y^2) - 2h\rho xy & h\mu & 0 \\ h\omega & 1 + h\nu & h\sigma \\ h\gamma & -h\kappa & 1 \end{pmatrix}.$$

Evaluating the Jacobian at the equilibrium point $X_0 = (0,0,0)$ gives:

$$J = \tilde{J}|_{(0,0,0)} = \begin{pmatrix} 1 + h\rho\delta & h\mu & 0 \\ h\omega & 1 + h\nu & h\sigma \\ h\gamma & -h\kappa & 1 \end{pmatrix}. \tag{3}$$

The characteristic equation of the matrix (3) is obtained as follows:

$$f(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3,$$

where

$$\begin{aligned}
a_1 &= -(\delta\rho + \nu)h - 3,\\
a_2 &= (\delta\rho\nu + \sigma\kappa - \mu\omega)h^2 + (2\delta\rho + 2\nu)h + 3,\\
a_3 &= -\sigma(\delta\rho\kappa + \mu\gamma)h^3 - (\delta\rho\nu + \sigma\kappa - \mu\omega)h^2 - (\delta\rho + \nu)h - 1.
\end{aligned}$$

Now, according to the characteristic equation, we get the bifurcations in system (2).

2.1. **Flip bifurcation.** System (2) at the equilibrium point X_0 in $\omega = \omega^*$

$$\omega^* = \frac{\sigma\delta h^3\rho\kappa + h^3\gamma\mu\sigma + 2\,\delta h^2\rho\nu + 2\,h^2\kappa\sigma + 4\,\delta h\rho + 4\,h\nu + 8}{2\mu h^2},$$

has an eigenvalue $\lambda_1 = -1$, and if $|\lambda_{2,3}| \neq 1$, there is a probability of flip bifurcation. Now with the change of coordinates

$$\begin{pmatrix} x\\ y\\ z \end{pmatrix} = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix} \begin{pmatrix} u\\ v\\ w \end{pmatrix} \Rightarrow X = QU,$$

where

$$\begin{aligned}
q_{11} &= (1, \frac{-\delta h\rho - 2}{h\mu}, \frac{-\delta h\rho\kappa - h\gamma\mu - 2\,\kappa}{2\mu})^T,\\
q_{22} &= (1, \frac{\sqrt{M} + 2 + (-\delta\rho + \nu)\,h}{2h\mu}, \frac{\sqrt{M} - 2 - \sigma h^2\kappa + (-\delta\rho - \nu)\,h}{h^2\sigma\mu})^T,\\
q_{33} &= (1, \frac{-\sqrt{M} + 2 + (-\delta\rho + \nu)\,h}{2h\mu}, \frac{-\sqrt{M} - 2 - \sigma h^2\kappa + (-\delta\rho - \nu)\,h}{h^2\sigma\mu})^T,\\
q_1 &= \frac{q_{11}}{\langle q_{11}, q_{11}\rangle}, \qquad q_2 = \frac{q_{22}}{\langle q_{22}, q_{22}\rangle}, \qquad q_3 = \frac{q_{33}}{\langle q_{33}, q_{33}\rangle},
\end{aligned}$$

and $M = 4 + 2\,\sigma\,(\delta\rho r + \mu\gamma)\,h^3 + (\delta\rho + \nu)^2\,h^2 + (4\,\delta\rho + 4\,\nu)\,h$, we convert the linear part of the system (2) into the Jordan form:

$$\begin{pmatrix} u\\ v\\ w \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} u\\ v\\ w \end{pmatrix} + \begin{pmatrix} f_1(u,v,w)\\ f_2(u,v,w)\\ f_3(u,v,w) \end{pmatrix},$$

where f_1, f_2 and f_3 are sentences of order two and above. According to the center manifold theorem [7], we can conclude the existence of a center manifold around $(u, v, w, \omega) = (0, 0, 0, \omega^*)$. So there are functions like g_1 and g_2 so that we have,

$$\begin{cases} v = & g_1(u) = b_1u^2 + b_2u^3 + O(u^4),\\ w = & g_2(u) = c_1u^2 + c_2u^3 + O(u^4), \end{cases}$$

The normal form of the flip bifurcation on the center manifold [23] is given by

$$u \to -u + f_1(u,v,w) = -u + b_{flip}u^3 + O(u^4),$$

where b_{flip} is calculated as follows:

$$b_{flip} = \frac{2\rho\,(\delta h\rho+2)^2\left(\sigma h^2\kappa + 2\,h\nu + 4\right)}{\langle q_{11}, q_{11}\rangle^2 \mu^2\left(-16+\sigma\left(\delta\rho\kappa+\mu\gamma\right)h^3 + \left(-4\,\delta\rho - 4\,\nu\right)h\right)h}.$$

According to $b_{flip} \neq 0$, the non-degeneracy condition of the flip bifurcation is established, so we go to the transversality condition, so we have:

$$\frac{d\lambda}{d\omega}|_{w=w^*} = \frac{2\,\mu h^2}{12+\left(\delta\rho\nu+\sigma\kappa-\mu\omega\right)h^2+\left(4\,\delta\rho+4\,\nu\right)h} \neq 0,$$

Therefore, the transversality condition is also established. So we can conclude that at the equilibrium point X_0 for $\omega = \omega^*$ the flip bifurcation occurs and we propose the following theorem.

Theorem 2.1. *Consider the system* (2). *At the equilibrium point* X_0 *for* $\omega = \omega^*$*, the flip bifurcation occurs. If* $b_{flip} > 0$*, we have the supercritical flip bifurcation, and if* $b_{flip} < 0$*, we have the subcritical flip bifurcation.*

2.2. **Neimark-Sacker bifurcation.** if

$$\begin{aligned}\omega = \omega_* = &\frac{1}{\mu h^2(\cos(\theta)-1)}\Big(4\,(\cos(\theta))^3 + (-6+(-2\,\delta\rho-2\,\nu)h)(\cos(\theta))^2 + \\ &h((\delta\rho\nu+\sigma\kappa)h + 2\,\delta\rho + 2\,\nu)\cos(\theta) + 2- \\ &\sigma(\delta\rho\kappa+\mu\gamma)h^3 + (-\delta\rho\nu-\sigma\kappa)h^2\Big),\end{aligned}$$

and

$$\begin{aligned}\theta = &\arccos(\frac{1}{4}\,\delta h\rho + \frac{1}{4}\,h\nu + 1 - \\ &\frac{1}{4}\,\sqrt{4\,\sigma\delta h^3\rho\kappa + 4\,\sigma h^3\mu\gamma + \delta^2h^2\rho^2 + 2\,\delta h^2\rho\nu + h^2\nu^2}).\end{aligned}$$

there are two complex eigenvalues of size one $\lambda_{1,2} = e^{\pm i\theta}$, and if $|\lambda_3| \neq 1$, there is a possibility of Neimark-Sacker bifurcation. Therefore, we examine the conditions of this bifurcation. First, we check the transversality condition, so we have:

$$\frac{d|\lambda|}{d\omega}|_{\omega=\omega_*} = \left(A+\bar{A}\right)|_{\omega=\omega_*}$$

where

$$\begin{aligned}A = &\left(\mu h^2(\mathrm{e}^{-i\theta}-1)\right)\Big/\Big(3\,\mathrm{e}^{2\,i\theta} + (-6+(-2\,\delta\rho-2\,v)h)\mathrm{e}^{i\theta} + 3 + \\ &(\delta\rho\nu+\sigma\kappa-\mu\omega)h^2 + (2\,\delta\rho+2\,\nu)h\Big).\end{aligned}$$

If $\frac{d|\lambda|}{d\omega}|_{\omega=\omega_*} \neq 0$, the transversality condition is satisfied. The second condition is the non-degeneracy condition. The normal form of the Neimark-Sacker bifurcation in complex coordinates is

$$\xi \to \xi e^{i\theta}(1 + d_{NS}|\xi|^2) + (\mathcal{O}|\xi|^4), \quad \xi \in \mathbb{C} \text{ and } e^{im_0\theta} \neq 1 \text{ for } m_0 = 1, 2, 3, 4,$$

that the critical coefficient d_{NS} is calculated as follows [19, 20]:

$$d_{NS} = \frac{1}{2} e^{-i\theta} \langle \eta, C(\mu, \mu, \bar{\mu}) \rangle,$$

where

$$\begin{aligned}
&J\mu = e^{i\theta}\mu, \qquad J^T\eta = e^{-i\theta}\eta, \qquad \langle \mu, \mu \rangle = \langle \eta, \mu \rangle = 1, \\
&\mu_1 = \left(1, \frac{-\delta h\rho + \mathrm{e}^{i\theta} - 1}{h\mu}, \frac{\delta h\rho\kappa + h\gamma\mu - \mathrm{e}^{i\theta}\kappa + \kappa}{(\mathrm{e}^{i\theta} - 1)\,\mu}\right)^T, \\
&\eta_1 = \left(\frac{\sigma h^2\kappa - \mathrm{e}^{-i\theta}h\nu + \left(\mathrm{e}^{-i\theta}\right)^2 + h\nu - 2\,\mathrm{e}^{-i\theta} + 1}{\mu h^2\sigma}, \frac{\mathrm{e}^{-i\theta} - 1}{h\sigma}, 1\right)^T, \\
&\mu = \frac{\mu_1}{\sqrt{\langle \mu_1, \mu_1 \rangle}}, \qquad \eta = \frac{\eta_1}{\langle \eta_1, \mu \rangle}.
\end{aligned}$$

So we have:

$$\begin{aligned}
d_{NS} = -\frac{1}{\langle \mu_1, \mu_1 \rangle \langle \eta_1, \mu_1 \rangle h^3 \mu^3 \sigma} \Bigg(&\rho(-\delta h\rho + \mathrm{e}^{i\theta} - 1)(-3\,\delta h\rho + \mathrm{e}^{i\theta} + \\
&2\,\mathrm{e}^{-i\theta} - 3)(\mathrm{e}^{-i\theta}\sigma h^2\kappa - h\nu + \mathrm{e}^{i\theta} + \mathrm{e}^{-i\theta}h\nu - 2 + \mathrm{e}^{-i\theta}) \Bigg).
\end{aligned}$$

Now, if the transversality condition $\frac{d|\lambda|}{d\omega}|_{\omega=\omega_*} \neq 0$ and the non-degeneracy condition $\Re(d_{NS}) \neq 0$ were satisfied, the Neimark-Sacker bifurcation occurs. Therefore, we propose the following theorem.

Theorem 2.2. *Consider the system* (2)*, at the equilibrium point* X_0 *if* $\omega = \omega_*$ *and the conditions* $\frac{d|\lambda|}{d\omega}|_{\omega=\omega_*} \neq 0$ *and* $\Re(d_{NS}) \neq 0$ *hold, the Neimark-Sacker bifurcation occurs. If* $\Re(d_{NS}) < 0$*, the Neimark-Sacker bifurcation is supercritical and the resulting closed invariant curve is stable; if* $\Re(d_{NS}) > 0$*, it is subcritical and the curve is unstable.*

2.3. **Chenciner (generalized Neimark-Sacker) bifurcation.** If $\Re(d_{NS}) = 0$, there is a probability of the generalized Neimark-Sacker bifurcation in system (2). The normal form up to fifth order is

$$\zeta \to \zeta e^{i\theta_0} + c_1\zeta|\zeta|^2 + c_2\zeta|\zeta|^4 + (\mathcal{O}|\zeta|^6), \quad \zeta \in \mathbb{C},$$

that the critical coefficients c_1 and c_2 are calculated as follows:

$$\begin{aligned}
c_1 =& \frac{1}{2}\langle \eta, C(\mu, , \mu, \bar{\mu})\rangle = e^{i\theta_0} d_{NS},\\
c_2 =& \frac{1}{12}\langle \eta, 3C(\mu,\mu,h_{12}) + 6C(\mu,\bar{\mu},h_{21}) + C(\bar{\mu},\bar{\mu},h_{30})\rangle,\\
&h_{30} = -(J - e^{3i\theta_0} I_n)^{-1} C(\mu,\mu,\mu),\\
&h_{21} = (J - e^{i\theta_0} I_n)^{-1}\big(2c_1\mu - C(\mu,\mu,\bar{\mu})\big),\\
&h_{12} = (J - e^{-i\theta_0} I_n)^{-1}\big(2\bar{c}_1\bar{\mu} - C(\mu,\bar{\mu},\bar{\mu})\big),
\end{aligned}$$

so we have:

$$\begin{aligned}
c_2 = \frac{\bar{\eta_{11}}}{12(\langle \mu_1, \mu_1\rangle)^2\langle \eta_1, \mu_1\rangle}\Bigg(& - 2\rho h\big(\bar{\mu}_{22}^2\Omega_1 + 2\bar{\mu}_{22}\Omega_2\big) -\\
& 12\rho h\big(|\mu_{22}|^2\Psi_1 + \mu_{22}\Psi_2 + \bar{\mu}_{22}\Psi_2\big)d - 6\rho h\big(\mu_{22}^2\Phi_1 + 2\mu_{22}\Phi_2\big)\Bigg),
\end{aligned}$$

where

$$\begin{aligned}
\Omega_1 &= \frac{-6\rho h}{f(3i\theta_0)}\Big((-h\nu - 2)\,\mathrm{e}^{3\,i\theta_0} + \sigma h^2\kappa + h\nu + \mathrm{e}^{6\,i\theta_0} + 1\Big)\mu_{22}^2\\
\Omega_2 &= \frac{-6\rho h}{f(3i\theta_0)}\Big(h\left(h\sigma\gamma + w\mathrm{e}^{3\,i\theta_0} - \omega\right)\Big)\mu_{22}^2\\
\Psi_1 &= \frac{1}{f(i\theta_0)}\Big[(\mathrm{e}^{2\,i\theta_0} + (-h\nu - 2)\mathrm{e}^{i\theta_0} + \sigma h^2\kappa + h\nu + 1)\times\\
&\quad \big(2c_1 + 2\rho h(2|\mu_{22}|^2 + \mu_{22}^2)\big) + 2h\mu(\mathrm{e}^{i\theta_0} - 1)c_1\mu_{22} + 2h^2\mu\sigma c_1\mu_{33}\Big],\\
\Psi_2 &= \frac{1}{f(i\theta_0)}\Big[h\left(h\sigma\gamma + \omega\mathrm{e}^{i\theta_0} - \omega\right)\left(2c_1 + 2\rho h(2|\mu_{22}|^2 + \mu_{22}^2)\right) +\\
&\quad 2\left(-\delta h\rho + \mathrm{e}^{i\theta_0} - 1\right)\left(\mathrm{e}^{i\theta_0} - 1\right) c_1\mu_{22} + 2\left(-\delta h\rho + \mathrm{e}^{i\theta_0} - 1\right) h\sigma c_1\mu_{33}\Big],\\
\Phi_1 &= \bar{\Psi}_1, \qquad \Phi_2 = \bar{\Psi}_2, \qquad \mu_{11} = 1, \qquad \mu_{22} = \frac{-\delta h\rho + \mathrm{e}^{i\theta_0} - 1}{h\mu},\\
\mu_{33} &= \frac{\delta h\rho\kappa + h\gamma\mu - \mathrm{e}^{i\theta_0}\kappa + \kappa}{(\mathrm{e}^{i\theta_0} - 1)\,\mu},\\
\eta_{11} &= \frac{\sigma h^2\kappa - \mathrm{e}^{-i\theta_0}h\nu + \left(\mathrm{e}^{-i\theta_0}\right)^2 + h\nu - 2\,\mathrm{e}^{-i\theta_0} + 1}{\mu h^2\sigma},
\end{aligned}$$

If $L_2 = \Re(e^{-i\theta_0}(c_2)) + \frac{1}{2}\big(e^{-i\theta_0}\Im(c_1)\big)^2 \neq 0$, the generalized Neimark-Sacker bifurcation occurs. Therefore, we conclude the following theorem.

Theorem 2.3. *Consider the system* (2)*, if* $\omega = \omega_*$*,* $\frac{d|\lambda|}{d\omega}|_{\omega=\omega_*} \neq 0$*,* $\Re(d_{NS}) = 0$ *and* $L_2 \neq 0$*, then the generalized Neimark-Sacker bifurcation occurs.*

2.4. 1 : 4 **resonance.** In the system (2), if we have, $\omega = \omega_*$, $\theta = \frac{\pi}{2}$ and $\gamma = \gamma_* = -\frac{\sigma\delta h^3\rho\kappa - 2\,\delta h\rho - 2\,h\nu - 4}{h^3\mu\sigma}$, there is a probability of the 1 : 4 resonance. The normal form for the 1:4 resonance is

$$w \to iw + c_0 w|w|^2 + d_0\bar{w}^3 + (\mathcal{O}|w|^4), \quad w \in \mathbb{C},$$

that c_0 and d_0 are calculated as follows:

$$c_0 = \frac{1}{2}\langle \eta, C(\mu,\mu,\bar{\mu})\rangle = id_{NS}, \qquad d_0 = \frac{1}{6}\langle \eta, C(\bar{\mu},\bar{\mu},\bar{\mu})\rangle,$$
$$J\mu = e^{\frac{\pi}{2}i}\mu, \qquad J^T\eta = e^{-\frac{\pi}{2}i}\eta, \qquad \langle \mu,\mu\rangle = \langle \eta,\mu\rangle = 1,$$

so, we have:

$$c_0 = -\frac{\rho\,(3\,\delta h\rho + 3 + i)\left(2 - 2\,i + (1+i)\,\sigma\kappa h^2 + 2\,h\nu\right)(-\delta h\rho - 1 + i)}{\langle \mu_1,\mu_1\rangle\langle \eta_1,\mu_1\rangle h^2\mu^3},$$
$$d_0 = \frac{3\rho\left(2 - 2\,i + (1+i)\,\sigma\kappa h^2 + 2\,h\nu\right)(\delta h\rho + 1 + i)^2}{\langle \mu_1,\mu_1\rangle\langle \eta_1,\mu_1\rangle h^2\mu^3}.$$

If $d_0 \neq 0$, the normal form coefficient for the 1:4 resonance is given by $A = -ic_0/|d_0|$, which determines the bifurcation scenario [19, 20]:

- $|A| > 1$: Two fold curves of period-4 cycles bifurcate from the resonance point, creating stable and unstable period-4 orbits.
- $|A| < 1$: No period-4 fold curves appear; the dynamics near the resonance are regular.
- $|A| = 1$: A degenerate case requiring higher-order analysis.

In our analysis, we compute $A = 2.4813 \times 10^{-5}$, which yields $|A| < 1$. Consequently, no period-4 fold curves emerge from the resonance point, and the local dynamics do not exhibit complex behavior such as period-4 cycles or chaotic attractors. This result is consistent with the numerical continuation performed in Section 4.

3. Bifurcations of equilibrium points X_1 and X_2

The Jacobian matrix of the system (2) at equilibrium point X_1 is as follows:

$$\tilde{J}|_{(x^*,y^*,z^*)} = J = \begin{pmatrix} 1 + h\rho(\delta - {y^*}^2) & -2h\rho x^* y^* + h\mu & 0 \\ h\omega & 1 + h\nu & h\sigma \\ h\gamma & -h\kappa & 1 \end{pmatrix}. \tag{3}$$

The characteristic equation of the relation (3) is $f(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$, where

$$a_1 = \frac{\mu h\gamma^2 + \kappa(-h\nu - 3)\gamma}{\kappa\gamma},$$

$$a_2 = \frac{-h\mu(h\nu + 2)\gamma^2 + \kappa((\sigma\kappa + \mu\omega)h^2 + 2\,h\nu + 3)\gamma + 2\,\delta h^2\rho\kappa^2\omega}{\kappa\gamma},$$

$$a_3 = \frac{1}{\kappa\gamma}\Bigg((2h^2\sigma\kappa + h\nu + 1)\mu h\gamma^2 + (2\,\sigma\delta h^3\rho\kappa^2 + (-\sigma\kappa^2 - \mu\kappa\omega)h^2 - \kappa\nu h - \kappa)\gamma - 2\,\delta h^2\rho\kappa^2\omega\Bigg).$$

Now, according to the characteristic equation above, we get the bifurcations at equilibrium point X_1 in system (2). It should be noted that similar bifurcations are assumed for the equilibrium point X_2.

3.1. **Flip bifurcation.** At the equilibrium point X_1 for

$$\omega = \omega_1 = \frac{\gamma\left(\sigma\delta h^3\rho\kappa^2 + \sigma h^3\mu\kappa\gamma - \sigma h^2\kappa^2 + h^2\mu\gamma\nu + 2\,h\mu\gamma - 2\,\kappa\nu h - 4\,\kappa\right)}{h^2\kappa\,(2\,\delta\rho\kappa + \mu\gamma)}$$

, system (2) has an eigenvalue $\lambda = -1$, so there is a possibility of flip bifurcation. Therefore, we check the flip bifurcation conditions. First we check the transversality condition:

$$\frac{d\lambda}{d\omega}|_{w=w_1} = \frac{-2\kappa h^2\,(2\,\delta\rho\kappa + \mu\gamma)}{(\sigma\delta h^3\rho\kappa^2 + \sigma h^3\mu\kappa\gamma - 2\,h\mu\gamma + 2\,\kappa\nu h + 8\,\kappa)\,\gamma} \neq 0,$$

Therefore, the transversality condition is satisfied. Now we check the non-degeneracy condition by deriving the normal form of the reduced system on the center manifold:

$$w \to -w + b_{flip}w^3 + \mathcal{O}(w^4), \quad w \in \mathbb{R}^1.$$

The left and right eigenvectors are obtained as follows:

$$Jq = -q, \qquad J^T p = -p, \qquad \langle q, q\rangle = 1, \quad \langle p, q\rangle = 1,$$

$$q_1 = \left(-\frac{h\,(2\,\delta\rho\kappa + \mu\gamma)\,\kappa}{\gamma\,(h\mu\gamma - 2\,\kappa)}, 1, \frac{h\kappa\,(\delta h\rho\kappa + h\mu\gamma - \kappa)}{h\mu\gamma - 2\,\kappa}\right)^T,$$

$$p_1 = \left(\frac{\gamma\left(h^2\sigma\kappa + 2\,h\nu + 4\right)}{2h\,(2\,\delta\rho\kappa + \mu\gamma)}, 1, \frac{-h\sigma}{2}\right)^T,$$

$$q = \frac{q_1}{\sqrt{\langle q_1, q_1\rangle}}, \qquad p = \frac{p_1}{\langle p_1, q\rangle},$$

so, the critical coefficient b_{flip} is obtained from the following relationship:

$$b_{flip} = \frac{4\left(h^2\sigma\kappa + 2\,h\nu + 4\right)\rho\kappa^3\,(\delta h\rho + 1)^2}{\langle q_1, q_1\rangle\langle p_1, q_1\rangle\,(2\,\delta\rho\kappa + \mu\gamma)\,(h\mu\gamma - 2\,\kappa)^2} \neq 0,$$

Therefore, the non-degeneracy condition is satisfied and the flip bifurcation occurs at the equilibrium point X_1 for $\omega = \omega_1$.

Theorem 3.1. *Consider the system* (2)*. At the equilibrium point X_1 for $\omega = \omega_1$ Flip bifurcation occurs.*

3.2. **Neimark-Sacker bifurcation.** Consider the system (2). If

$$\begin{aligned}\omega = \omega_2 = -\frac{\gamma}{h^2\kappa(\cos(\theta)-1)(2\,\delta\rho\kappa+\mu\gamma)}\Bigg(&4\,\kappa(\cos(\theta))^3+\\ &((2\,\mu\gamma-2\,\nu\kappa)h-6\,\kappa)(\cos(\theta))^2+((\sigma\kappa^2-\mu\gamma\nu)h-\\ &2\,\mu\gamma+2\,\nu\kappa)h\cos(\theta)+2\,\sigma\kappa(\delta\rho\kappa+\mu\gamma)h^3+\\ &(-\sigma\kappa^2+\mu\gamma\nu)h^2+2\,\kappa\Bigg),\end{aligned}$$

and

$$\begin{aligned}\theta = \arccos\Bigg(&(\frac{1}{4\kappa})\big(-h\mu\gamma+h\kappa\nu+4\,\kappa+\\ &\sqrt{-8\,\sigma\delta h^3\rho\kappa^3-8\,\sigma h^3\mu\kappa^2\gamma+h^2\mu^2\gamma^2-2\,h^2\mu\kappa\gamma\nu+h^2\kappa^2\nu^2}\big)\Bigg),\end{aligned}$$

system (2) has two complex eigenvalues on the unit circle $|\lambda_{1,2}| = 1$, and if $|\lambda_3| \neq 1$, there is a possibility of Neimark-Sacker bifurcation. Therefore, we examine the conditions of this bifurcation. First, we check the transversality condition:

$$\frac{d|\lambda|}{d\omega}|_{\omega=\omega_2} = 2\Re\Big(\frac{-2\kappa h^2\left(e^{-i\theta}-1\right)\left(\delta\rho\kappa+\frac{1}{2}\,\mu\gamma\right)}{A}\Big),$$

where

$$\begin{aligned}A =&3\,e^{2\,i\theta}\kappa\gamma+2\,\left(h\mu\gamma+(-h\nu-3)\,\kappa\right)\gamma e^{i\theta}-h\mu\left(h\nu+2\right)\gamma^2+\\ &\kappa\left(h^2\sigma\kappa+h^2\mu\omega+2\,h\nu+3\right)\gamma+2\,\delta h^2\rho\kappa^2\omega.\end{aligned}$$

If $\frac{d|\lambda|}{d\omega}|_{\omega=\omega_2} \neq 0$, the transversality condition is satisfied. So we check the non-degeneracy condition by deriving the normal form on the center manifold:

$$\xi \to \xi e^{i\theta}(1+d_{NS}|\xi|^2)+(\mathcal{O}|\xi|^4), \quad \xi\in\mathbb{C} \text{ and } e^{im_0\theta}\neq 1 \text{ for } m_0 = 1,2,3,4,$$

We obtain the critical coefficient d_{NS} as follows:

$$\begin{aligned}d_{NS} =&\frac{1}{2}e^{-i\theta}\langle p, C(q,q,\bar{q})+2B(q,(I_n-J)^{-1}B(q,\bar{q})+\\ &B(\bar{q},(e^{2i\theta}I_n-J)^{-1}B(q,q))\rangle,\end{aligned}$$

the left and right eigenvectors corresponding to the eigenvalues are obtained as follows:

$$Jq = e^{i\theta}q, \qquad J^T p = e^{-i\theta}p, \qquad \langle q, q\rangle = \langle p, q\rangle = 1,$$

$$q_1 = \left(\frac{\left(\mathrm{e}^{i\theta}-1\right)\left(2\,\delta\rho\kappa+\mu\gamma\right)}{\gamma\left(2\,\delta h\rho\kappa+2\,h\mu\gamma+\mathrm{e}^{i\theta}\kappa-\kappa\right)}, -\frac{\left(h\mu\gamma+\mathrm{e}^{i\theta}\kappa-\kappa\right)\left(\mathrm{e}^{i\theta}-1\right)}{\kappa\left(2\,\delta h\rho\kappa+2\,h\mu\gamma+\mathrm{e}^{i\theta}\kappa-\kappa\right)h}, 1\right)^T,$$

$$p_1 = \left(-\frac{\left(\left(-h\nu-2\right)\mathrm{e}^{-i\theta}+h^2\sigma\kappa+h\nu+\mathrm{e}^{-2\,i\theta}+1\right)\gamma}{h^2\left(2\,\delta\rho\kappa+\mu\gamma\right)\sigma}, \frac{\mathrm{e}^{-i\theta}-1}{h\sigma}, 1\right)^T,$$

$$q = \frac{q_1}{\sqrt{\langle q_1, q_1\rangle}}, \qquad p = \frac{p_1}{\langle p_1, q\rangle}.$$

so, we have:

$$d_{NS} = \frac{-2\rho h}{\langle q_1, q_1\rangle\langle p_1, q_1\rangle}\Big(\big(2q_{11}|q_{22}|^2 + \bar{q}_{11}q_{22}^2\big) + 2\big(x^*q_{22}\Omega_2 + y^*q_{11}\Omega_2 + y^*q_{22}\Omega_1\big) + \big(x^*\bar{q}_{22}\Phi_2 + y^*\bar{q}_{11}\Phi_2 + y^*\bar{q}_{22}\Phi_1\big)\Big),$$

where

$$\Omega_1 = \frac{\kappa}{2h\left(\delta\rho\kappa+\mu\gamma\right)}\big(-2\rho h(x^*|q_{22}|^2 + y^*q_{11}\bar{q}_{22} + y^*q_{22}\bar{q}_{11})\big),$$

$$\Omega_2 = \frac{\gamma}{2h\left(\delta\rho\kappa+\mu\gamma\right)}\big(-2\rho h(x^*|q_{22}|^2 + y^*q_{11}\bar{q}_{22} + y^*q_{22}\bar{q}_{11})\big),$$

$$\Phi_1 = \left(\left(-h\nu-2\right)\mathrm{e}^{2\,i\theta}+h^2\sigma\kappa+h\nu+\mathrm{e}^{4\,i\theta}+1\right)\big(-2\rho h(x^*q_{22}^2 + 2y^*q_{11}q_{22})\big),$$

$$\Phi_2 = h\left(h\sigma\gamma+\omega\mathrm{e}^{2\,i\theta}-\omega\right)\big(-2\rho h(x^*q_{22}^2 + 2y^*q_{11}q_{22})\big),$$

$$q_{11} = \frac{\left(\mathrm{e}^{i\theta}-1\right)\left(2\,\delta\rho\kappa+\mu\gamma\right)}{\gamma\left(2\,\delta h\rho\kappa+2\,h\mu\gamma+\mathrm{e}^{i\theta}\kappa-\kappa\right)},$$

$$q_{22} = -\frac{\left(h\mu\gamma+\mathrm{e}^{i\theta}\kappa-\kappa\right)\left(\mathrm{e}^{i\theta}-1\right)}{\kappa\left(2\,\delta h\rho\kappa+2\,h\mu\gamma+\mathrm{e}^{i\theta}\kappa-\kappa\right)h},$$

$$q_{33} = 1.$$

If the transversality condition $\frac{d|\lambda|}{d\omega}|_{\omega=\omega_2} \neq 0$ and the non-degeneracy condition $\Re(d_{NS}) \neq 0$ were satisfied , the Neimark-Sacker bifurcation occurs. Therefore, we propose the following theorem.

Theorem 3.2. *Consider the system* (2)*, at the equilibrium point* X_1 *if* $\omega = \omega_2$ *and the conditions* $\frac{d|\lambda|}{d\omega}|_{\omega=\omega_2} \neq 0$ *and* $\Re(d_{NS}) \neq 0$ *hold, the Neimark-Sacker bifurcation occurs.*

4. Numerical simulation

Now, using the MATLAB package MATCONTM, [9, 16], we numerically check the different bifurcations in X_0. We assign specific values to all parameters $\mu = 0.9, \rho = 0.2, \delta = 0.7, \nu = 0.005, \sigma = 0.1, \gamma = 0.5, \kappa = 0.1, h = 3,$

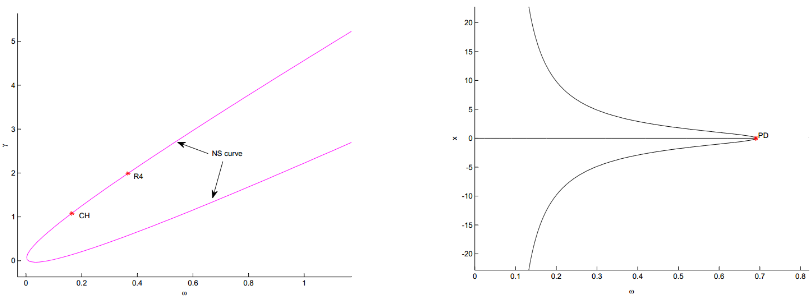
and we keep the parameter ω free, at $\omega = 0.69045679$ the flip bifurcation occurs, whose critical coefficient is $b_{flip} = -9.3169 \times 10^{-2}$. At $\omega = 0.052639263$, the Neimark-Sacker bifurcation occurs, whose critical coefficient is $\Re(d_{NS}) = -1.4548 \times 10^{-3}$. Now, by placing on the Neimark-Sacker bifurcation point and releasing the second parameter γ, and moving on the Neimark-Sacker curve, at $\omega = 0.16416496, \gamma = 1.0798782$ the generalized Neimark-Sacker bifurcation occurs, whose normal form coefficient is 2.4813×10^{-5}. At $\omega = 0.36620988, \gamma = 1.9885597$, $1:4$ resonance occurs, whose normal form coefficient is $A = 2.4813 \times 10^{-5}$.

Now we examine bifurcations at the equilibrium point X_1. In the values of $\mu = 0.9, \rho = 0.2, \delta = 0.7, \nu = 0.005, \sigma = 0.1, \gamma = 0.45480573, \kappa = 0.1, h = 0.5$, and by keeping the parameter ω, at $\omega = 0.4$ the flip bifurcation occurs, whose critical coefficient is $b_{flip} = -8.4505 \times 10^{-2}$. At $\omega = 0.07650113158$, the Neimark-Sacker bifurcation occurs, whose critical coefficient is equal to $\Re(d_{NS}) = -2.7653 \times 10^{-5}$.

The computed normal form coefficients allow us to determine the criticality of each bifurcation. For the flip bifurcation at X_0, we obtain $b_{flip} = -9.3169 \times 10^{-2} < 0$, which corresponds to a subcritical flip bifurcation. In contrast, the Neimark-Sacker bifurcation at X_0 yields $\Re(d_{NS}) = -1.4548 \times 10^{-3} < 0$, indicating a supercritical Neimark-Sacker bifurcation.
Similarly, at X_1, the flip bifurcation coefficient is $b_{flip} = -8.4505 \times 10^{-2} < 0$, confirming a subcritical flip bifurcation, while the Neimark-Sacker bifurcation gives $\Re(d_{NS}) = -2.7653 \times 10^{-5} < 0$, confirming a supercritical Neimark-Sacker bifurcation.

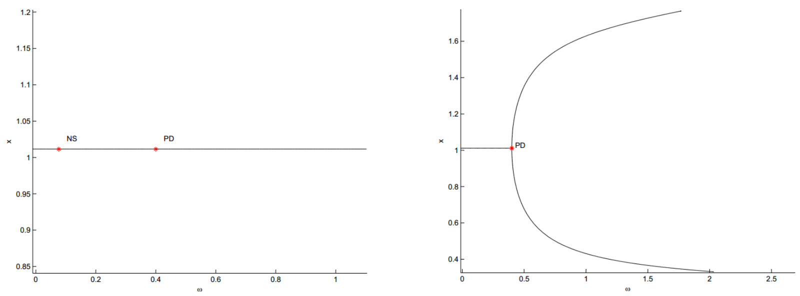
To complement the bifurcation analysis, Figures 1 and 2 illustrate the numerical results. Figure 1(a) shows the Neimark-Sacker bifurcation curve for X_0 in the (ω, γ) plane, and Figure 1(b) shows the flip bifurcation diagram for X_0. Figure 2 presents time series near the flip bifurcation at X_0: (a) for $\omega = 0.68$ (below $\omega^* = 0.69045679$), convergence to the fixed point; (b) for $\omega = 0.70$ (above ω^*), convergence to a stable 2-cycle. These results agree with the critical coefficient $b_{flip} = -9.3169 \times 10^{-2}$.

Remark (Transition to chaos). The discrete-time system (2) enters a chaotic state through two main routes. The first route is via a cascade of period-doubling (flip) bifurcations: as the bifurcation parameter ω is increased beyond the critical value ω^*, the system undergoes successive flip bifurcations, each doubling the period of the orbit, eventually leading to chaotic behavior. The second route is via the breakup of quasi-periodic invariant tori arising from the Neimark-Sacker bifurcation: for ω near ω_*, a stable invariant closed curve emerges; further increase of ω causes this curve to lose smoothness and break apart into a chaotic attractor. For the numerical parameter values used in Section 4 ($\mu = 0.9$, $\rho = 0.2$, $\delta = 0.7$, $\nu = 0.005$, $\sigma = 0.1$, $\gamma = 0.5$, $\kappa = 0.1$, $h = 3$), chaotic behavior is observed when ω exceeds approximately 0.75. The onset of chaos can be detected numerically by computing the maximum Lyapunov exponent, which becomes positive in the chaotic regime.



(a) (b)

Figure 1. (a)The Neimark-Sacker bifurcation curve for X_0.
(b) Flip bifurcation diagram for X_0.



(c) (d)

Figure 2. (c) The codimension one bifurcations curve for X_1.
(d) Flip bifurcation diagram for X_1.

5. Conclusion

In conclusion, this paper has extensively investigated a continuous-time economic model and presented a robust method for converting it into a discrete-time system using Euler's method. By manipulating the parameter ω, which represents the proportion of saving, we have successfully determined the critical coefficient of bifurcations in the normal form using the center manifold. This achievement has significantly advanced our understanding of the system's dynamics and provided valuable insights into its stability and behavior under different conditions.

The utilization of affine transformations to obtain the coefficient of the normal form for flip and Neimark-Sacker bifurcations at the trivial fixed point, in combination with the application of closed forms for other bifurcations, has allowed for a comprehensive analysis of the system's behavior across a range of scenarios. The analytical and numerical techniques employed in this study have complemented each other, providing a well-rounded analysis and enhancing the reliability of our findings.

We now explicitly explain that:

- The flip bifurcation corresponds to the transition from stable economic growth to boom-bust cycles, with the subcritical case indicating abrupt transitions that are difficult to reverse without policy intervention.
- The Neimark-Sacker bifurcation represents the emergence of business cycles, with the supercritical case indicating that these cycles appear smoothly and can be anticipated.
- The Chenciner bifurcation signals a critical transition where economic fluctuations become more complex, potentially requiring nonlinear policy approaches.
- The 1:4 resonance indicates the possibility of multiple economic regimes, with the condition $|A| < 1$ suggesting simpler dynamics in our specific case.

We discuss how policymakers can:

- Use the analytically derived bifurcation thresholds (ω^*, ω_*, ω_1, ω_2) as quantitative early warning indicators for economic instability.
- Design proactive interventions based on the criticality of bifurcations—subcritical bifurcations require stronger preventive measures, while supercritical bifurcations allow for smoother adjustments.
- Implement counter-cyclical fiscal or monetary policies to shift the system away from bifurcation boundaries.
- Recognize the limitations of linear policy tools when the system approaches codimension-2 bifurcations like the Chenciner bifurcation.

Furthermore, the visualization of the bifurcation curve in the parameter space and the generation of the bifurcation diagram using the MatcontM package have provided clear and intuitive representations of the system's complex dynamics. These visualizations have not only aided in the interpretation of the results but also facilitated the communication of the findings to a wider audience. The ability to visually explore the bifurcation behavior has been instrumental in identifying critical regions of parameter space and understanding the underlying mechanisms driving the system's behavior.

The implications of this research extend beyond the specific economic model studied in this paper. The proposed method for discretizing continuous-time systems using Euler's method can be applied to a wide range of economic models, enabling researchers to explore and analyze their dynamics in a computationally efficient manner. By understanding the effects of parameter variations and identifying bifurcation points, policymakers and economists can gain valuable insights into the behavior and stability of economic systems, aiding in the development of effective policy strategies and enhancing economic forecasting.

It is important to highlight that this study underscores the significance of integrating both analytical and numerical approaches in the analysis of economic models. The combination of these methods allows for a comprehensive exploration of the system's dynamics, capturing both the theoretical underpinnings and the practical implications of the model. Future research in this field can build upon the findings of this study by investigating more complex economic models and exploring alternative numerical techniques to further enhance our understanding of economic systems.

In summary, this paper has made substantial contributions to the field of economic modeling by presenting a systematic approach to convert continuous-time economic models into discrete-time systems using Euler's method. The analysis of bifurcations and the determination of critical coefficients have shed light on the system's stability and behavior. The integration of analytical and numerical methods, along with the visualization of results, has provided a comprehensive understanding of the system's dynamics. The implications of this research extend beyond the specific model studied and offer valuable insights for policymakers and economists. By embracing both analytical and numerical approaches, future research can continue to advance our understanding of economic systems and their complex dynamics.

Flip bifurcation represents the transition from a stable economic equilibrium to a 2-period cycle, indicating alternating periods of economic expansion and contraction (boom-bust cycles). Neimark-Sacker bifurcation leads to the emergence of quasi-periodic oscillations, which can be interpreted as business cycles with varying amplitudes and frequencies. Chenciner bifurcation indicates changes in the stability of these cycles, potentially leading to more complex economic fluctuations. 1:4 resonance corresponds to the appearance of 4-periodic cycles or more irregular economic behavior.

6. Author Contributions

J. Hadadi: conceptualization, software, validation, formal analysis, investigation, writing—original draft preparation, visualization; Z. Eskandari: conceptualization, methodology, validation, writing—review and editing, supervision, project administration. All authors have read and agreed to the published version of the manuscript.

7. Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

In this section, please provide details regarding where data supporting reported results can be found, including links to publicly archived datasets analyzed or generated during the study (see examples). If the study did not report any data, you might add "Not applicable" here.

8. Aknowledgement

Not applicable.

9. **Ethical considerations**

Not applicable. This study involves no human or animal subjects, and no personal or sensitive data were collected or analyzed.

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11. **Conflict of interest**

The authors declare no conflicts of interest.

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