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THEORETICAL MODELS FOR CONSTRUCTING H_v-NEAR FIELDS

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Abstract. In this paper, we introduce the notion of H_v-near fields as a new generalization of classical near fields within the framework of hyperstructure theory. The concept of an H_v-near field is introduced and defined here for the first time. The main contribution of this paper is the development of the foundational theory of H_v-near fields, including the investigation of their basic algebraic properties, structural characteristics, and construction methods. Several examples and construction techniques are presented to illustrate the existence and diversity of these hyperalgebraic structures. Furthermore, we define and study the fundamental relation θ^*, which plays a central role in describing equivalence classes and the internal structure of H_v-near fields. Our approach is based on methods and techniques from hyperstructure theory.

Keywords: H_v-nearring, H_v-near field, Near field, Fundamental relation.
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1. Introduction

The study of hyperstructures has seen considerable growth since Marty first introduced the concept in 1934. Marty pioneered the idea of hypergroups, which generalize the principles of classical group theory. This groundbreaking framework has expanded research opportunities in abstract algebra and found applications in diverse areas such as rational functions, algebraic functions, and non-commutative groups. Hyperstructure theory enhances classical algebraic structures by permitting operations on two elements to produce a set of outcomes instead of a single value. This introduces a layer of complexity and depth that is absent in conventional algebraic systems.

Among the most significant advancements in this field are the notions of hyperfields and hyperrings, independently developed initially by Krasner [12]. In his work, Krasner used an underlying field and a subgroup of its multiplicative group to construct what we now call hyperfields. This concept has been further explored and expanded by numerous researchers, leading to a deeper understanding of the properties and applications of hyperfields and hyperrings, particularly in the study of finite Krasner hyperstructures, for example see [14].

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The notion of a near field was originally introduced by Leonard Dickson in 1905 [9]. Initially, near fields were applied in the realm of incidence geometries, especially within projective geometries [8, 23]. They have since found a variety of applications, primarily in geometry. More recently, near fields have been notably utilized in the development of encryption ciphers, including Hill ciphers [11].

The foundational structures discussed in [4], which are the crucial point for defining the H_v, were initially introduced in [22]. The H_v-structures, a class of hyperstructures introduced by Vougiouklis in 1990 [17, 22], are defined by weak axioms in which the condition of equality is replaced by the requirement of a non-empty intersection. Hyperstructures, particularly the extensive class of H_v-structures, are utilized in a variety of scientific disciplines, including mathematics, biology, hadronic physics, leptons, linguistics, and sociology, among others. The H_v structures [18] generalize familiar algebraic hyperstructures, such as hypergroups and hyperrings. In fact, certain axioms associated with these hyperstructures have been replaced with their corresponding weaker versions. In [3], Davvaz introduced the concept of H_v-near rings, which generalizes the idea of hypernearrings. This notion has been further investigated in references [10, 15, 16, 20, 21].

Also for the first time, the concept of the H_v-field was introduced by Vougiouklis in [19]. An H_v-ring $(R, +, \cdot)$ is referred to as an H_v-field if the quotient structure R/γ^* forms a field. Here, γ^* represents the fundamental relation on R, which partitions the H_v-ring into equivalence classes. Specifically, if R is an H_v-ring and γ^* is the fundamental relation defined on R, then R is classified as an H_v-field precisely when the quotient R/γ^* satisfies the axioms of a field. This definition extends the classical notion of fields to the hyperstructures.

In what follows, we introduce the concept of H_v-near fields as a natural extension of classical near fields within the theory of hyperstructures. We investigate their fundamental algebraic properties and present several construction techniques that illustrate the variety of these generalized structures. A key element of our study is the analysis of the fundamental relation θ^*, which characterizes equivalence among elements in an H_v-near field. This investigation lays the groundwork for a broader understanding of hyperstructure-based algebraic systems and paves the way for future developments in this area.

The motivation for introducing H_v-near fields stems from the desire to extend and generalize the classical notion of near fields, a well-established structure in algebraic systems. Classical near fields are limited in certain contexts where traditional field axioms do not fully capture the required algebraic properties, especially in non-commutative or more complex settings. By introducing H_v-near fields, we aim to explore a broader class of algebraic structures that can potentially model systems with richer structural behavior, such as those found in certain cryptographic applications, coding theory, and combinatorics. It is worth noting that many chemical, biological, and physical phenomena can be modelled using algebraic H_v-structures, for example see [1, 5–7]. This raises the hope that, in the future, other natural phenomena may also be effectively modelled through this new algebraic framework.

2. Basic definitions

In this section, we will explain several fundamental definitions that are essential for further understanding. If these foundational concepts are studied and comprehended properly, they can lay the groundwork for grasping new material in this field.

Before introducing the definition of an H_v-near field, we briefly recall the classical notion of a near field in order to provide the necessary background. Near fields constitute an important generalization of fields in which one of the distributive laws is weakened, and they play a significant role in the theory of near-rings and related algebraic structures. The standard definition and basic properties of near fields can be found in the classical literature on near-rings, in particular in the works of Pilz [13].

Definition 2.1. Let M be a non-empty set equipped with two operations $+$ and $\cdot$. Then $(M, +, \cdot)$ is called a (right) near field if:

(1) $(M, +)$ is an abelian group with identity element 0;
(2) $(M^*, \cdot)$ is a group with multiplicative identity 1, where $M^* = M \setminus \{0\}$;
(3) $(a + b) \cdot c = a \cdot c + b \cdot c$, for all $a, b, c \in M$.

If condition (3) is replaced by the left distributive law, it would instead define a left near field. For the remainder of this discussion, we will refer to right near fields simply as "near fields".

Remark 2.2. To avoid ambiguity regarding multiplication by zero, we note that in general right near-field axioms do not guarantee the identity $a \cdot 0 = 0$ without an additional assumption. While the property $0 \cdot a = 0$ follows from right distributivity and the group structure of $(N, +)$, the corresponding left annihilation law $a \cdot 0 = 0$ is not derivable in the absence of left distributivity. Therefore, if required in subsequent arguments, this property may be explicitly included as an additional axiom.

Below is a complete explanation of the unique proper near field of order 9 (often called the Dickson near field of order 9) [9]. This near field is built on the set of order 9 elements of the finite field $\mathbb{F}_9$, but with a "twisted" multiplication so that one of the distributive laws fails.

Example 2.3. *Let $\mathbb{F}_9 = \{a + b\alpha \mid a, b \in \mathbb{F}_3\}$, where $\alpha^2 + 1 = 0$. The addition is exactly that of finite field $\mathbb{F}_9$, in other words,*

$$(a + b\alpha) + (c + d\alpha) = (a + c) + (b + d)\alpha.$$

In the field $\mathbb{F}_9$, multiplication is defined in the usual way:

$$(a + b\alpha) \cdot (c + d\alpha) = (ac + 2bd) + (ad + bc)\alpha.$$

However, to get a near field, we "twist" the multiplication. The idea in Dickson construction is to use the unique subfield of $\mathbb{F}_9$ *of order* 3*:*

$$\mathbb{F}_3 = \{0, 1, 2\}.$$

For any $a \in \mathbb{F}_3$ *and any non-zero* $b \in \mathbb{F}_9$*:*

- *If* $b \in \mathbb{F}_3$*, then* $a \circ b = a \cdot b$.
- *If* $b \in \mathbb{F}_9 \setminus \mathbb{F}_3$*, then set* $a \circ b = a \cdot b^3$.
- *By convention, define* $a \circ 0 = 0$*, for all* a.

Thus, the near field $(\mathbb{F}_9, +, \circ)$ *is defined by keeping the addition of* $\mathbb{F}_9$ *but "twisting" the multiplication when the right factor is not in the subfield* $\mathbb{F}_3$.

Proposition 2.4. *The operation* $\circ$ *is right distributive over addition, i.e.,*

$$(x + y) \circ z = x \circ z + y \circ z \quad \forall x, y, z \in \mathbb{F}_9.$$

Proof. Let $z \in \mathbb{F}_9$. We consider two cases.

Case 1: $z \in \mathbb{F}_3$. Then $\circ$ coincides with the usual field multiplication, hence

$$(x + y) \circ z = (x + y)z = xz + yz = x \circ z + y \circ z.$$

Case 2: $z \in \mathbb{F}_9 \setminus \mathbb{F}_3$. Then by definition of the Dickson twist, $(x + y) \circ z = (x+y) \cdot z^3$. Since multiplication in $\mathbb{F}_9$ is distributive, it follows that $(x+y) \cdot z^3 = xz^3 + yz^3$. Again using the definition of $\circ$, we have $xz^3 = x \circ z, \quad yz^3 = y \circ z$. Hence, $(x + y) \circ z = x \circ z + y \circ z$. □

Proposition 2.5. *The operation* $\circ$ *is not left distributive in general, i.e.,*

$$x \circ (y + z) \neq x \circ y + x \circ z$$

for some $x, y, z \in \mathbb{F}_9$.

Proof. Take $x \in \mathbb{F}_9 \setminus \mathbb{F}_3$ and $y, z \in \mathbb{F}_9 \setminus \mathbb{F}_3$ such that $y + z \in \mathbb{F}_3$ (this is possible since $\mathbb{F}_9$ is a 2-dimensional vector space over $\mathbb{F}_3$). Then, $x \circ (y + z) = x(y + z)$ because $y + z \in \mathbb{F}_3$. However, $x \circ y = xy^3, \quad x \circ z = xz^3$. Thus, $x \circ y + x \circ z = x(y^3 + z^3)$. In general, $y + z \neq y^3 + z^3$ in $\mathbb{F}_9$, hence $x \circ (y + z) \neq x \circ y + x \circ z$. Therefore, left distributivity fails. □

Definition 2.6. A hypergroupoid $(S, \star)$ is defined as a non-empty set S equipped with a map $\star : S \times S \to \mathcal{P}^*(S)$, referred to as a (binary) hyperoperation. Here, $\mathcal{P}^*(S)$ denotes the set of all non-empty subsets of S. The result of applying this hyperoperation to a pair (x, y) is denoted by $x \star y$.

If A and B are non-empty subsets of S and $x \in S$, then the expressions $A \star B$, $A \star x$, and $x \star B$ represent the following:

$$A \star B = \bigcup_{\substack{a \in A \\ b \in B}} a \star b, \quad A \star x = A \star \{x\}, \quad x \star B = \{x\} \star B.$$

Definition 2.7. A hypergroupoid $(S, \star)$ is called a semihypergroup if

$$(x \star y) \star z = x \star (y \star z),$$

for all $x, y, z \in S$. This means that

$$\bigcup_{a \in x \star y} a \star z = \bigcup_{b \in y \star z} x \star b.$$

A semihypergroup S is considered finite when it contains only a finite number of elements. A semihypergroup S is called commutative if it meets specific condition:

$$x \star y = y \star x,$$

for all $x, y \in S$.

The " $\star$ " in S is called weak associative if

$$(x \star y) \star z \cap x \star (y \star z) \neq \emptyset, \text{ for all } x, y, z \in S.$$

The " $\star$ " is called weak commutative if

$$x \star y \cap y \star x \neq \emptyset, \text{ for all } x, y \in S.$$

Example 2.8. *Let $S = \{a, b\}$ and define a hyperoperation "$\star$" on S by*

$$\begin{aligned} a \star a &= \{a\}, \\ a \star b &= \{a, b\}, \\ b \star a &= \{a\}, \\ b \star b &= \{a\}. \end{aligned}$$

We show that $(S, \star)$ is weakly associative but not associative.

The following table verifies the weak associativity condition

$$(x \star y) \star z \cap x \star (y \star z) \neq \emptyset$$

for all $x, y, z \in S$.

(x, y, z)	$(x \star y) \star z$	$x \star (y \star z)$	$((x \star y) \star z) \cap (x \star (y \star z))$
(a, a, a)	$\{a\}$	$\{a\}$	$\{a\}$
(a, a, b)	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$
(a, b, a)	$\{a\}$	$\{a, b\}$	$\{a\}$
(a, b, b)	$\{a, b\}$	$\{a\}$	$\{a\}$
(b, a, a)	$\{a\}$	$\{a\}$	$\{a\}$
(b, a, b)	$\{a, b\}$	$\{a\}$	$\{a\}$
(b, b, a)	$\{a\}$	$\{a\}$	$\{a\}$
(b, b, b)	$\{a\}$	$\{a\}$	$\{a\}$

Since all intersections are nonempty, $(S, \star)$ satisfies the weak associativity property.

However,

$$(a \star b) \star b = \{a, b\},$$

while

$$a \star (b \star b) = \{a\}.$$

Thus,

$$(a \star b) \star b \neq a \star (b \star b),$$

and therefore $(S, \star)$ *is not associative.*

Definition 2.9. [3, 4] A right H_v-near ring is an algebraic hyperstructure $(P, +, \cdot)$ that satisfies the following axioms:

(1) $(P, +)$ is a weak canonical hypergroup, i.e., for every $x, y, z \in P$, the following conditions hold:
   (i) For every $x, y, z \in P$, $x+(y+z)\cap(x+y)+z \neq \emptyset$ and $x+y\cap y+z \neq \emptyset$.
   (ii) There exists an element $0 \in P$ such that $x + 0 = 0 + x = x$, for all $x \in P$.
   (iii) For each $x \in P$, there exists a unique element $x' \in P$ such that $0 \in x + x' \cap x' + x$. We denote this element by $-x$, and refer to it as the opposite of x.
   (iv) If $x \in y + z$, then $z \in -y + x$ and $y \in x - z$.
(2) The structure $(P, \cdot)$ forms a semigroup under multiplication.
(3) The multiplication is weakly distributive with respect to the addition operation, on the right, i.e., for all $x, y, z \in P$,

$$(x + y) \cdot z \cap x \cdot z + y \cdot z \neq \emptyset.$$

Corollary 2.10. *The following properties hold for all* $x, y \in P$*:*

$$-(-x) = x,\ 0 = -0,\ 0 \textit{ is unique, and } -(x + y) = -y - x.$$

Definition 2.11. [3, 4] Assume that $(P, +)$ is a weak canonical hypergroup and $M \subseteq P$. Then M is called a weak canonical subhypergroup of P if $(M, +)$ is a weak canonical hypergroup.

Definition 2.12. [3,4] A weak canonical subhypergroup N of the weak canonical $(P, +)$ is called normal if for all $x \in P$, $x + N = N + x$.

Definition 2.13. [3, 4] Assume that R_1 and R_2 are two H_v-near rings. A mapping $\Psi : R_1 \longrightarrow R_2$ is an H_v-homomorphism if:

$$\Psi(x + y) \cap \Psi(x) + \Psi(y) \neq \emptyset \text{ for all } x, y \in R_1,$$

$$\Psi(x \cdot y) \cap \Psi(x) \cdot \Psi(y) \neq \emptyset \text{ for all } x, y \in R_1,$$

$$\Psi(0) = 0.$$

Ψ is called a strong homomorphism if:

$$\Psi(x + y) = \Psi(x) + \Psi(y) \text{ for all } x, y \in R_1,$$

$$\Psi(x \cdot y) = \Psi(x) \cdot \Psi(y) \text{ for all } x, y \in R_1,$$

$$\Psi(0) = 0.$$

If Ψ is one-to-one, onto, and a strong H_v-homomorphism, then it is called an isomorphism.

3. H_v-Near Fields

In this section, we introduce the concept of an H_v-near field and explore several properties of this new class of hyperstructures.

An H_v-near field is a structure similar to an H_v-field but weaker in distributivity. Depending on which distributive law is required, there are two kinds. A right H_v-near field (see Definition 3.1) satisfies right distributivity $(x+y)\cdot z \cap x\cdot z + y\cdot z \neq \emptyset$ for all elements, but left distributivity may fail. Similarly, a left H_v-near field satisfies left distributivity $x\cdot(y+z)\cap x\cdot y + x\cdot z \neq \emptyset$, while right distributivity may fail.

Definition 3.1. A right H_v-near field is a right H_v-near ring $(F, +, \cdot)$ in which $(F^*, \cdot)$ is a group with zero as a bilateral absorbing element.

If both the right and left weak distributive laws are valid, then $(F, +, \cdot)$ is referred to as a distributive H_v-near field. For simplicity, in the context of this paper, we will assume that all H_v-near fields under consideration are right H_v-near fields unless otherwise specified.

Example 3.2. *Let $F = \{0, 1, a\}$ with the following hyperoperation and multiplication:*

$+$	0	1	a
0	0	1	a
1	1	0	F
a	a	F	a

$\cdot$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	1

Then $(F, +, \cdot)$ is an H_v-near field.

Remark 3.3. It is evident that every H_v-near field can be regarded as an H_v-near ring, because, being a group, is necessarily a semigroup, and the remaining axioms are identical in both structures.

Moreover, it is possible to consider H_v-near rings that are not H_v-near fields, as illustrated in the following example.

Example 3.4. *Let $R = \{0, x, y\}$ be a set with the following hyperoperation and multiplication:*

$+$	0	x	y
0	0	x	y
x	x	R	$\{x, y\}$
y	y	$\{x, y\}$	R

$\cdot$	0	x	y
0	0	0	0
x	0	x	y
y	0	x	y

Then $(R, +, \cdot)$ is an H_v-near ring. Clearly, it is not an H_v-near field, because $R \setminus \{0\}$ is not a group.

Let $(M, +, \cdot)$ be a near field. Since a near field has two groups inside it, "normal subgroup" must always be understood as follows:

- A subgroup $(K, \cdot) \leq (M \setminus \{0\}, \cdot)$ is normal in the multiplicative group if
$$\forall x \in M \setminus \{0\},\ xKx^{-1} = K.$$

The following construction provides a method for obtaining an H_v-near field from a classical near field by means of multiplicative cosets. More precisely, starting from a near field $(M, +, \cdot)$ and a normal subgroup of its multiplicative group, we induce a hyperoperation from the additive structure of M while preserving a well-defined multiplication on the quotient space. This construction establishes a natural connection between classical near fields and hyperstructure theory, and yields a new family of H_v-near fields arising from quotient-type structures.

Construction. Assume that $(M, +, \cdot)$ is a near field and let $(G, \cdot)$ be a normal subgroup of $(M^*, \cdot)$, where $M^* = M \setminus \{0\}$. We define the quotient set

$$M/G = \{\sigma G \mid \sigma \in M\},$$

where for all $\alpha G, \beta G \in M/G$, the operations are defined as follows:

- Addition:
$$\alpha G \oplus \beta G = \{\eta G \mid \eta \in \alpha G + \beta G\},$$
- Multiplication:
$$\alpha G \odot \beta G = \alpha \cdot \beta G.$$

We claim that $(M/G, \oplus, \odot)$ is an H_v-near field.

Lemma 3.5. *The operations $\oplus$ and $\odot$ defined in Construction 3 are well-defined on M/G.*

Proof. First, we prove that the hyperoperation $\oplus$ is well-defined. Assume that $\sigma G = \sigma' G$ and $\beta G = \beta' G$. Then there exist $g_1, g_2 \in G$ such that $\sigma' = \sigma g_1$ and $\beta' = \beta g_2$. Now let $\eta \in \sigma' G + \beta' G$. Then there exist $a, b \in G$ such that $\eta = \sigma' a + \beta' b$. Using the above representations, we obtain $\eta = \sigma g_1 a + \beta g_2 b$, for some $g_1, g_2 \in G$. Since G is a subgroup of $(M^*, \cdot)$, it follows that $g_1 a \in G$ and $g_2 b \in G$. Put $a' = g_1 a$ and $b' = g_2 b$. Then $a', b' \in G$, and hence $\eta = \sigma a' + \beta b' \subseteq \sigma G + \beta G$. Therefore, $\sigma' G + \beta' G \subseteq \sigma G + \beta G$. Similarly, $\sigma G + \beta G \subseteq \sigma' G + \beta' G$. Consequently, $\sigma' G + \beta' G = \sigma G + \beta G$, which implies that $\{\eta G \mid \eta \in \sigma' G + \beta' G\} = \{\eta G \mid \eta \in \sigma G + \beta G\}$. Hence $\oplus$ is well-defined.

Next, we show that $\odot$ is well-defined. Assume again that $\sigma G = \sigma' G$ and $\beta G = \beta' G$. Then there exist $g_1, g_2 \in G$ such that $\sigma' = \sigma g_1$ and $\beta' = \beta g_2$. Therefore, $\sigma' \beta' = \sigma g_1 \beta g_2$. Since G is a normal subgroup of $(M^*, \cdot)$, it follows that $\beta^{-1} g_1 \beta \in G$. Hence there exists $g_3 \in G$ such that $g_1 \beta = \beta g_3$. Thus, $\sigma' \beta' = \sigma \beta g_3 g_2$. Because $g_3 g_2 \in G$, we obtain $(\sigma' \beta') G = (\sigma \beta) G$. Hence $\sigma' G \odot \beta' G = \sigma G \odot \beta G$, and so $\odot$ is well-defined. $\square$

Lemma 3.6. *The hyperoperations defined in Construction 3 satisfy the weak right distributive law, i.e., $\big((\sigma G \oplus \beta G) \odot \gamma G\big) \cap \big((\sigma G \odot \gamma G) \oplus (\beta G \odot \gamma G)\big) \neq \emptyset$, for all $\sigma G, \beta G, \gamma G \in M/G$.*

Proof. Let $A = (\sigma G \oplus \beta G) \odot \gamma G$ and $B = (\sigma G \odot \gamma G) \oplus (\beta G \odot \gamma G)$. By definition of $\oplus$, $\sigma G \oplus \beta G = \{\eta G \mid \eta \in \sigma G + \beta G\}$. Hence, $A = \{(\eta \cdot \gamma) G \mid \eta \in \sigma G + \beta G\}$.

Since $\sigma G+\beta G=\bigcup_{a,b\in G}\sigma a+\beta b$, we obtain $A=\{xG \mid x\in\sigma a\gamma+\beta b\gamma,\ a,b\in G\}$. Now let 1 denote the identity element of the multiplicative group $(M^*,\cdot)$. Since $1\in G$, by taking $a=b=1$, we get $\{xG \mid x\in(\sigma+\beta)\gamma\}\subseteq A$. On the other hand, $\sigma G\odot\gamma G=(\sigma\cdot\gamma)G$, and $\beta G\odot\gamma G=(\beta\cdot\gamma)G$. Thus, $B=\{xG \mid x\in(\sigma\cdot\gamma)G+(\beta\cdot\gamma)G\}$. Consequently, we have $\{xG \mid x\in\sigma\gamma+\beta\gamma\}\subseteq B$. Because $(M,+,\cdot)$ is a right near field, the right distributive law holds in M, and therefore $(\sigma+\beta)\cdot\gamma=\sigma\cdot\gamma+\beta\cdot\gamma$. Hence, $A\cap B\neq\emptyset$. Therefore, the weak right distributive law holds. □

Theorem 3.7. *$(M/G,\oplus,\odot)$ is an H_v-near field.*

Proof. First, by Lemma 3.5, the hyperoperations $\oplus$ and $\odot$ are well-defined on M^*/G. Now, we verify the axioms of an H_v-near field:

(1) Verifying that $(M/G,\oplus)$ is a weak canonical hypergroup:

(i) Weak commutativity:

$$\sigma G\oplus\beta G=\{\eta G \mid \eta\in\sigma G+\beta G\}=\{\eta G \mid \eta\in\beta G+\sigma G\}=\beta G\oplus\sigma G.$$

Thus, $\sigma G\oplus\beta G\cap\beta G\oplus\sigma G\neq\emptyset$.

(ii) Weak associativity of $\oplus$: By construction,

$$(\sigma G\oplus\beta G)\oplus\gamma G\cap\sigma G\oplus(\beta G\oplus\gamma G)\neq\emptyset,$$

as both sides contain elements of the form $(\sigma+\beta+\gamma)G$ obtained by choosing representatives from G.

(ii) Existence of neutral element: Since $0\in M$, it follows that $0G\in M/G$. For all $\sigma G\in M/G$, we have

$$\begin{aligned}\sigma G\oplus 0G &= 0G\oplus\sigma G=\{\eta G \mid \eta\in 0G+\sigma G\}=\{\eta G \mid \eta\in(0+\sigma)G\}\\ &=\{\eta G \mid \eta\in\sigma G\}=\sigma G.\end{aligned}$$

(iii) Existence of opposites: For all $\sigma\in M$, we have $-\sigma\in M$, so

$$\begin{aligned}&\sigma+(-\sigma)=0\ \Rightarrow\ 0\in\sigma G+(-\sigma)G\Rightarrow 0G\in\sigma G\oplus(-\sigma)G\ \text{ and}\\ &(-\sigma)+\sigma=0\ \Rightarrow\ 0\in(-\sigma)G+\sigma G\Rightarrow 0G\in(-\sigma)G\oplus\sigma G\end{aligned}$$

Thus, $0G\in\sigma G\oplus(-\sigma)G\cap(-\sigma)G\oplus\sigma G$.

(iv) Reversibility condition: Assume that $xG\in yG\oplus zG$. By definition of $\oplus$, $yG\oplus zG=\{\eta G \mid \eta\in yG+zG\}$. Hence there exists $\eta\in yG+zG$ such that $xG=\eta G$. Since $\eta\in yG+zG$, it follows that there exist $a,b\in G$ such that $\eta=ya+zb$. Moreover, from $xG=\eta G$, there exists $g\in G$ such that $x=\eta g$. Therefore, $x=(ya+zb)g$. Using the distributive law in M, $(ya+zb)g=yag+zbg$. Hence $x=yag+zbg$. Now, since $zbg\in zG$ and $x=yag+zbg$, we obtain $zbg=-yag+x$. Because $yag\in yG$, it follows that $zbg\in -yG+x$. Thus, $zG\cap(-yG+x)\neq\emptyset$, which implies that $zG\in -yG\oplus xG$. Similarly, from $x=yag+zbg$, we obtain $yag=x-zbg$. Since $zbg\in zG$, we have $yag\in x-zG$. Because

$yag \in yG$, it follows that $yG \cap (x - zG) \neq \emptyset$. Hence $yG \in xG - zG$. Therefore, if $xG \in yG \oplus zG$, then $zG \in -yG \oplus xG$ and $yG \in xG - zG$.

(v) Weak associativity: We show that

$$((\sigma G \oplus \beta G) \oplus \eta G) \cap (\sigma G \oplus (\beta G \oplus \eta G)) \neq \emptyset.$$

For all $\sigma, \beta, \eta \in M$, we have

$$(\sigma G \oplus \beta G) \oplus \eta G = \{kG \mid k \in (\sigma G + \beta G) + \eta G\} = \sigma G \oplus (\beta G \oplus \eta G),$$

showing the intersection is non-empty.

(2) Verifying that $(M^*, \odot)$ is a group:

(i) Associativity of multiplication: For all $\sigma G, \beta G, \gamma G \in M^*/G$, we have $(\sigma G \odot \beta G) \odot \gamma G = ((\sigma \cdot \beta) \cdot \gamma)G$, and $\sigma G \odot (\beta G \odot \gamma G) = (\sigma \cdot (\beta \cdot \gamma))G$. Since $(M^*, \cdot)$ is associative, these are equal. Hence multiplication is associative.

(ii) Existence of identity: For all $\sigma \in M$, we have $\sigma G \odot 1G = \sigma G$ and $1G \odot \sigma G = \sigma G$, so $1G$ is the identity element.

(iii) Existence of inverses: For every $\sigma \in M$, we have $\sigma G \odot \sigma^{-1}G = (\sigma \cdot \sigma^{-1}G) = 1G$, so every element has an inverse.

(3) Bilateral absorbing element: For all $\sigma \in M$, $\sigma G \odot 0G = (\sigma \cdot 0)G = 0G$ and $0G \odot \sigma G = (0 \cdot \sigma) = 0G$.

(4) Weak right distributivity: For all $\sigma G, \beta G, \gamma G \in M^*/G$,

$$((\sigma G \oplus \beta G) \odot \gamma G) \cap ((\sigma G \odot \gamma G) \oplus (\beta G \odot \gamma G)) \neq \emptyset,$$

as shown in Lemma 3.6, using the right distributivity in M.

Therefore all axioms of an H_v-near field are satisfied, and $(M^*/G, \oplus, \odot)$ is an H_v-near field. □

Example 3.8. *Let* $M = \mathbb{F}_9 = \{a + b\gamma \mid a, b \in \mathbb{F}_3,\ \gamma^2 = 2\}$, *equipped with the Dickson multiplication*

$$x \circ y = \begin{cases} xy, & y \in H_0, \\ x^3 y, & y \notin H_0, \end{cases}$$

where $H_0 = \langle \gamma \rangle = \{1, 2, \gamma, 2\gamma\}$. *Then* $(M, +, \circ)$ *is a right near field, but not a field. Now consider the subgroup* $G = H_0 = \{1, 2, \gamma, 2\gamma\}$. *Since* $|M^*| = 8$ *and* $|G| = 4$, *the subgroup* G *has index* 2 *in* $(M^*, \circ)$. *Hence* G *is normal in* $(M^*, \circ)$. *Therefore, the quotient set* $M^*/G = \{\sigma G \mid \sigma \in M^*\}$ *contains exactly two cosets:* $G = \{1, 2, \gamma, 2\gamma\}$, *and* $(1 + \gamma)G = \{1 + \gamma,\ 2 + 2\gamma,\ 1 + 2\gamma,\ 2 + \gamma\}$. *The operations are defined by*

$$\sigma G \oplus \beta G = \{\eta G \mid \eta \in \sigma G + \beta G\},$$

and

$$\sigma G \odot \beta G = (\sigma \circ \beta)G.$$

For example, $G \odot (1+\gamma)G = (1+\gamma)G$. *Moreover,* $1 + (1+\gamma) = 2 + \gamma \in (1+\gamma)G$, *while* $1 + (2 + 2\gamma) = 2\gamma \in G$. *Hence,* $G \oplus (1 + \gamma)G = \{G,\ (1 + \gamma)G\}$. *Thus,*

$$(M^*/G, \oplus, \odot)$$

is a nontrivial H_v-near field obtained from the Dickson near field of order 9.

Example 3.9. *Let $M = \mathbb{C}$, the field of complex numbers, regarded as a near field. Consider the subgroup $G = \{1, -1\}$ of the multiplicative group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Since $\mathbb{C}^*$ is abelian, G is a normal subgroup of $(\mathbb{C}^*, \cdot)$. The quotient set is $\mathbb{C}^*/G = \{zG \mid z \in \mathbb{C}^*\}$, where $zG = \{z, -z\}$. The operations are defined by $zG \oplus wG = \{\eta G \mid \eta \in zG + wG\}$, and $zG \odot wG = (zw)G$. Now, $1G = \{1, -1\}$ and $iG = \{i, -i\}$. Then $1G + iG = \{1+i,\ 1-i,\ -1+i,\ -1-i\}$. Hence, $1G \oplus iG = \{(1+i)G,\ (1-i)G\}$, since $(-1-i)G = (1+i)G$ and $(-1+i)G = (1-i)G$. Moreover, $1G \odot iG = (i)G = \{i, -i\}$. Thus, $(\mathbb{C}^*/G, \oplus, \odot)$ is an H_v-near field induced by Construction 1.*

Definition 3.10. Let F be an H_v-near field. Assume that N is a normal weak canonical subhypergroup of F, with $0 \in N$. We define the relation $\sigma \equiv \beta \mod (N)$ if and only if there exists a sequence of elements $\{\eta_0, \eta_1, \ldots, \eta_{k+1}\} \subseteq F$, where $\eta_0 = \sigma$ and $\eta_{k+1} = \beta$, such that the following conditions hold:

$$(\sigma - \eta_1) \cap N \neq \emptyset,\ (\eta_1 - \eta_2) \cap N \neq \emptyset,\ \ldots,\ (\eta_k - \beta) \cap N \neq \emptyset.$$

This relation is referred to as the chain relation, and it is denoted by $\sigma \varrho^* \beta$, if and only if $\sigma \equiv \beta \mod (N)$.

Lemma 3.11. *The chain relation ϱ^* is an equivalence relation.*

Proof. (1) Reflexivity: Since $0 \in (\sigma - \sigma) \cap N$ for all $\sigma \in F$, it follows that $\sigma \varrho^* \sigma$. Hence, ϱ^* is reflexive.

(2) Symmetry: Suppose that $\sigma \varrho^* \beta$. Then, there exists a sequence $\{\eta_0, \eta_1, \ldots, \eta_{k+1}\} \subseteq F$, where $\eta_0 = \sigma$ and $\eta_{k+1} = \beta$, such that

$$(\sigma - \eta_1) \cap N \neq \emptyset,\ (\eta_1 - \eta_2) \cap N \neq \emptyset,\ \ldots,\ (\eta_k - \beta) \cap N \neq \emptyset.$$

Thus, there exists $a_i \in (\eta_i - \eta_{i+1}) \cap N$ for each $i = 1, \ldots, k$, which implies $-a_i \in \eta_{i+1} - \eta_i$ and $-a_i \in N$, since N is a weak canonical subhypergroup, it is closed under the inverse operation. This shows that $\beta \varrho^* \sigma$, establishing that ϱ^* is symmetric.

(3) Transitivity: Let $\sigma \varrho^* \beta$ and $\beta \varrho^* \eta$, where $\sigma, \beta, \eta \in F$. Then, there exist sequences $\{\eta_0, \eta_1, \ldots, \eta_{k+1}\} \subseteq F$ and $\{t_0, t_1, \ldots, t_{r+1}\} \subseteq F$, with $\eta_0 = \sigma$, $\eta_{k+1} = \beta$, $t_0 = \beta$, and $t_{r+1} = \eta$, such that:

$$(\sigma - \eta_1) \cap N \neq \emptyset,\ (\eta_1 - \eta_2) \cap N \neq \emptyset,\ \ldots,\ (\eta_k - \beta) \cap N \neq \emptyset,$$

and

$$(\beta - t_1) \cap N \neq \emptyset,\ (t_1 - t_2) \cap N \neq \emptyset,\ \ldots,\ (t_r - \eta) \cap N \neq \emptyset.$$

By combining these sequences, we obtain $\{\eta_0, \eta_1, \ldots, \eta_{k+1}, t_1, t_2, \ldots, t_{r+1}\} \subseteq F$, which satisfies the condition $\sigma \varrho^* \eta$. Hence, ϱ^* is transitive.

Therefore, ϱ^* is an equivalence relation.

We denote $\varrho^*(\sigma)$ as the equivalence class of σ. □

Remark 3.12. The notion of a homomorphism for H_v-near fields is the same as that for H_v-near rings with an additional condition $1 \mapsto 1$.

Definition 3.13. If Ψ is a strong H_v-homomorphism from F_1 into F_2, then the kernel of Ψ is defined by $\ker\Psi = \{\sigma \in F_1 | \Psi(\sigma) = 0\}$. It is easy to see that $\ker\Psi$ is a normal weak subhypergroup of F_1.

4. Fundamental near fields

In this section, we establish the fundamental relations on H_v-near fields and explore some of the associated theorems. Consider an H_v-near field $(F, +, \cdot)$. We define the relation θ^* as the smallest equivalence relation on F such that the quotient F/θ^*, which represents the set of all equivalence classes, forms a near field. Here, θ^* is referred to as the fundamental equivalence relation on F, and F/θ^* is called the fundamental near field. This relation has been studied by Corsini [2] in the context of hypergroups, with additional references found in [4, 16, 17].

By using a certain type of equivalence relations, we can connect H_v-near rings to near rings and H_v-near fields to near fields. These equivalence relations are called fundamental relations. More exactly, starting with an H_v-near field and using the fundamental relation, we can construct a near field structure on the quotient set.

So, by strongly regular relations we obtain a group from a hypergroup (see Figure 1).

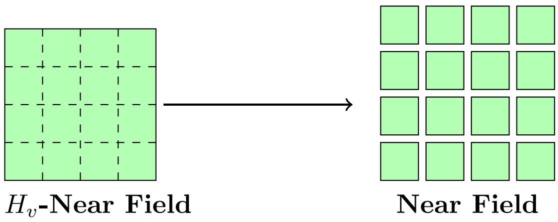

Figure 1. Fundamental equivalence relation

Definition 4.1. Let F be an H_v-near field. We define θ^* as the smallest equivalence relation such that the quotient F/θ^* is a near field. The relation θ^* is called the fundamental equivalence relation and F/θ^* is called the fundamental near field.

Remark 4.2. The fundamental equivalence relation θ^* introduced above is analogous to the fundamental relation γ^* defined by Vougiouklis [17] for H_v-rings. Recall that γ^* is the smallest equivalence relation such that the quotient of an H_v-ring becomes an ordinary ring. Similarly, θ^* is defined as the smallest equivalence relation for which the quotient of an H_v-near field becomes a classical near field. Therefore, θ^* may be regarded as the near-field counterpart of the relation γ^*. The main difference arises from the algebraic structure of near fields, where only one-sided distributivity is required.

Let $U_1 = F$. For $n > 1$, define U_n as the set of all elements obtained by either adding two elements from U_{n-1} or multiplying two elements from U_{n-1}. Then $U = \bigcup_{n\geq 1} U_n$. We define the relation θ on F as follows:

$$\sigma\theta\beta \iff \{\sigma, \beta\} \subseteq u \text{ for some } u \in U.$$

Clearly, θ is reflexive and symmetric. But it is not easy to see that θ is transitive or not. Let $\hat{\theta}$ denote the transitive closure of θ. The definition of $\hat{\theta}$ on F can then be stated as follows:

$\sigma\hat{\theta}\beta \iff$ there exist $\eta_1, \eta_2, \ldots, \eta_{n+1} \in F$ with $\eta_1 = \sigma, \eta_{n+1} = \beta$ and $u_1, \ldots, u_n \in U$ such that $\{\eta_i, \eta_{i+1}\} \subseteq u_i$ for $i = 1, \ldots, n$.

Theorem 4.3. *The fundamental relation θ^* represents the transitive closure of the relation θ.*

Proof. Let $\hat{\theta}$ denote the transitive closure of the relation θ. We will denote the equivalence class of an element σ as $\hat{\theta}(\sigma)$. First, we establish that the quotient set $F/\hat{\theta}$ forms a near field. The operations of addition $\oplus$ and multiplication $\odot$ in $F/\hat{\theta}$ are defined as follows:

$$\hat{\theta}(\sigma) \oplus \hat{\theta}(\beta) = \{\hat{\theta}(\eta) \mid \eta \in \hat{\theta}(\sigma) + \hat{\theta}(\beta)\},$$

$$\hat{\theta}(\sigma) \odot \hat{\theta}(\beta) = \{\hat{\theta}(\vartheta) \mid \vartheta \in \hat{\theta}(\sigma) \cdot \hat{\theta}(\beta)\},$$

for all $\sigma, \beta \in F$.

Consider $\sigma' \in \hat{\theta}(\sigma)$ and $\beta' \in \hat{\theta}(\beta)$. We have $\sigma'\hat{\theta}(\sigma)$ if there exists a sequence of elements $\delta_1, \ldots, \delta_m$ where $\delta_1 = \sigma'$, $\delta_{m+1} = \sigma$, and $u_1, \ldots, u_m \in U$ such that $\{\delta_i, \delta_{i+1}\} \subset u_i$ for $i = 1, \ldots, m$. Similarly, $\beta'\hat{\theta}(\beta)$ holds if and only if there is a sequence $\varepsilon_1, \ldots, \varepsilon_{n+1} \in F$ with $\varepsilon_1 = \beta'$, $\varepsilon_{n+1} = \beta$, and $v_1, \ldots, v_n \in U$ such that $\{\varepsilon_j, \varepsilon_{j+1}\} \subset v_j$ for $j = 1, \ldots, n$.

Now, we derive:

$$\{\delta_i, \delta_{i+1}\} + \varepsilon_1 \in u_i + v_1 \quad (i = 1, \ldots, m-1),$$

$$\delta_{m+1} + \{\varepsilon_j, \varepsilon_{j+1}\} \subset u_m + v_j \quad (j = 1, \ldots, n).$$

The sums $u_i + v_1$ (for $i = 1, \ldots, m-1$) and $u_m + v_j = t_{m+j-1}$ (for $j = 1, \ldots, n$) are polynomials, and thus $t_k \in U$ for all $k \in \{1, \ldots, m+n-1\}$.

Now, let us define the elements $\eta_1, \ldots, \eta_{m+n}$ such that $\eta_i \in \delta_i + \varepsilon_1$ (for $i = 1, \ldots, m$) and $\eta_{m+j} \in \delta_{m+1} + \varepsilon_{j+1}$ (for $j = 1, \ldots, n$). Thus, we obtain $\{\eta_k, \eta_{k+1}\} \subset t_k$ (for $k = 1, \ldots, m+n-1$). Therefore, every element η_1 from the sum $\delta_1 + \varepsilon_1 = \sigma' + \beta'$ is $\hat{\theta}$ equivalent to the element η_{m+n} from the sum $\delta_{m+1} + \varepsilon_{n+1} = \sigma + \beta$. This shows that:

$$\hat{\theta}(\sigma) \oplus \hat{\theta}(\beta) = \hat{\theta}(\eta)$$

for all $\eta \in \hat{\theta}(\sigma) + \hat{\theta}(\beta)$. A similar argument shows that:

$$\hat{\theta}(\sigma) \odot \hat{\theta}(\beta) = \hat{\theta}(\vartheta)$$

for all $\vartheta \in \hat{\theta}(\sigma) \cdot \hat{\theta}(\beta)$.

Next, we prove the existence of the identity element and multiplicative inverses in $F/\hat{\theta}$.

Let 1 be the multiplicative identity of the near field F. We show that the class $\hat{\theta}(1)$ is the identity element of $F/\hat{\theta}$. Indeed, for every $\hat{\theta}(\sigma) \in F/\hat{\theta}$,

$$\hat{\theta}(1) \odot \hat{\theta}(\sigma) = \hat{\theta}(1 \cdot \sigma) = \hat{\theta}(\sigma),$$

and similarly,

$$\hat{\theta}(\sigma) \odot \hat{\theta}(1) = \hat{\theta}(\sigma \cdot 1) = \hat{\theta}(\sigma).$$

Therefore,

$$I = \hat{\theta}(1)$$

is the multiplicative identity of $F/\hat{\theta}$.

Now let $\hat{\theta}(\sigma) \in F/\hat{\theta}$ with $\sigma \neq 0$. Since F is a near field, there exists $\sigma^{-1} \in F$ such that

$$\sigma \cdot \sigma^{-1} = \sigma^{-1} \cdot \sigma = 1.$$

Hence,

$$\hat{\theta}(\sigma) \odot \hat{\theta}(\sigma^{-1}) = \hat{\theta}(\sigma \cdot \sigma^{-1}) = \hat{\theta}(1) = I,$$

and similarly,

$$\hat{\theta}(\sigma^{-1}) \odot \hat{\theta}(\sigma) = \hat{\theta}(\sigma^{-1} \cdot \sigma) = \hat{\theta}(1) = I.$$

Thus,

$$\hat{\theta}(\sigma^{-1}) = \hat{\theta}(\sigma)^{-1}.$$

Therefore, every nonzero element of $F/\hat{\theta}$ has a multiplicative inverse.

Let $\varphi : F \to F/\hat{\theta}$ be the natural map such that $\varphi(\delta) = \hat{\theta}(\delta)$ for all $\delta \in F$. Let:

$$\omega_F = \{\delta \in F \mid \varphi(\delta) = I\}.$$

For any $\delta, \varepsilon \in F$, if $\varphi(\delta) = \varphi(\varepsilon) = I$, then $\hat{\theta}(\delta) = \hat{\theta}(\varepsilon) = I$, implying that δ and ε belong to the same equivalence class. Hence, $\omega_F \subseteq \hat{\theta}(\delta)$.

We claim that $\omega_F = \hat{\theta}(\delta)$. If $\sigma \in \hat{\theta}(\delta)$, then $\hat{\theta}(\delta) = \hat{\theta}(\sigma)$, and since $\hat{\theta}(\delta) = I$, we have $\hat{\theta}(\sigma) = I$, implying $\sigma \in \omega_F$. Therefore, $\hat{\theta}(\delta) \subseteq \omega_F$, and consequently, $\hat{\theta}(\delta) = \omega_F$. Thus, $\omega_F = I$.

Since F is a near field, we observe that for all $\sigma \in F$, $\sigma \cdot F = F \cdot \sigma = F$, implying that $\varphi(\sigma F) = \varphi(F)$, and hence $\varphi(\sigma) \odot \varphi(F) = \varphi(F)$. This establishes that:

$$\hat{\theta}(\sigma) \odot F/\hat{\theta} = F/\hat{\theta}.$$

For all $\hat{\theta}(\sigma) \in F/\hat{\theta}$, we have $\omega_F \in F/\hat{\theta}$, so $\omega_F \in \hat{\theta}(\sigma) \odot F/\hat{\theta}$. Thus, there exists $\hat{\theta}(\beta) \in F/\hat{\theta}$ such that $\omega_F = \hat{\theta}(\sigma) \odot \hat{\theta}(\beta)$, implying that $\hat{\theta}(\sigma) \odot \hat{\theta}(\beta) = I$. Therefore, $\hat{\theta}(\beta) = \hat{\theta}(\sigma)^{-1}$.

The weak associativity and distributivity of F ensure that associativity and distributivity hold in the quotient $F/\hat{\theta}$. Hence, $F/\hat{\theta}$ forms a near field.

Finally, let ϱ be an equivalence relation on F such that F/ϱ is a near field. Denoting the equivalence class of σ by $\varrho(\sigma)$, we find that both $\varrho(\sigma) \oplus \varrho(\beta)$ and $\varrho(\sigma) \odot \varrho(\beta)$ are singletons for any $\sigma, \beta \in F$. Specifically:

$$\varrho(\sigma) \oplus \varrho(\beta) = \varrho(\eta), \;\; \varrho(\sigma) \odot \varrho(\beta) = \varrho(d)$$

for any $\eta \in \varrho(\sigma) + \varrho(\beta)$ and $d \in \varrho(\sigma) \cdot \varrho(\beta)$. This implies for every $\sigma, \beta \in F$ and $M \subseteq \varrho(\sigma), N \subseteq \varrho(\beta)$:

$$\varrho(\sigma) \oplus \varrho(\beta) = \varrho(\sigma + \beta) = \varrho(M + N), \;\; \varrho(\sigma) \odot \varrho(\beta) = \varrho(\sigma \cdot \beta) = \varrho(M \cdot N).$$

By induction, these equalities hold for finite sums and products. Thus, for any $u \in U$ and $\delta \in u$, we have $\varrho(\delta) = \varrho(u)$. Therefore, for every $\sigma \in F$,

$$\delta \in \theta(\sigma) \implies \delta \in \varrho(\sigma).$$

Since ϱ is transitive, we conclude that

$$\delta \in \hat{\theta}(\sigma) \implies \delta \in \varrho(\sigma).$$

This means that the relation $\hat{\theta}$ is the smallest equivalence relation on F such that $F\hat{\theta}$ is a near field, thus $\hat{\theta} = \theta^*$.

So, the relation θ^* is the transitive closure of θ. □

Remark 4.4. The degeneracy of the fundamental equivalence relation θ^* is not restricted to a single pair of elements. In general, if the hyperaddition is sufficiently large in the sense that it connects elements through relations of the form $x + y = H$ for some (or several) pairs $x, y \in H$, then the induced connectivity generated by the hyperoperation may force all elements to belong to the same equivalence class. Consequently, the fundamental quotient H/θ^* may reduce to a trivial one-element structure.

Example 4.5. *Consider Example 3.2. We have $\{1, a\} \subseteq 1 + a$ and $\{0, 1\} \subseteq 1 + a$. This yields that $\theta^*(0) = \theta^*(1) = \theta^*(a)$. Therefore $F/\theta^* = \{\theta^*(0)\}$.*

Corollary 4.6. *Assume that $(F, +, \cdot)$ is an H_v-near field, and θ^* represent the fundamental equivalence relation on F. The following properties hold:*

(1) $\omega_F = \theta^*(0)$,
(2) $\theta^*(-\eta) = -\theta^*(\eta)$ *for all* $\eta \in F$.

Proof. (1) We know that $\omega_F = \{\eta \in F \mid \varphi(\eta) = I\}$, it follows that $\omega_F = I$. For any $\eta \in F$, we observe that $\eta + 0 = 0 + \eta = \eta$. Consequently,

$$\theta^*(0) \oplus \theta^*(\eta) = \{\theta^*(\eta) \mid \eta \in \theta^*(0) + \theta^*(\eta)\} = \{\theta^*(0)\}.$$

Since the set contains only one element, we conclude that $\theta^*(0)$ is the identity in the quotient. Moreover, for any $\eta \in F$, we have

$$\theta^*(\eta + 0) = \theta^*(\eta) \oplus \theta^*(0) = \theta^*(0).$$

Since $(F/\theta^*, +)$ is a group and the identity element is unique, from the uniqueness of the zero element in the quotient field we conclude that $\omega_F = I = \theta^*(0)$.

(2) For any $\eta \in F$, we have that $0 \in (\eta - \eta) \cap (-\eta + \eta)$. Using the properties of the equivalence relation, we find

$$\theta^*(\eta) \oplus \theta^*(-\eta) = \{\theta^*(\sigma) \mid \sigma \in \theta^*(\eta) + \theta^*(-\eta)\}.$$

Since $\eta \in \theta^*(\eta)$ and $-\eta \in \theta^*(-\eta)$, it follows that $\eta - \eta \in \theta^*(\eta) + \theta^*(-\eta)$. Hence,

$$\theta^*(0) \in \theta^*(\eta) \oplus \theta^*(-\eta).$$

Moreover, the set $\theta^*(\eta) \oplus \theta^*(-\eta)$ contains only one element, so we conclude that

$$\theta^*(0) = \theta^*(\eta) \oplus \theta^*(-\eta).$$

By symmetry, we also have

$$\theta^*(0) = \theta^*(-\eta) \oplus \theta^*(\eta).$$

Since the addition operation in $(F/\theta^*, +)$ is group-like and the inverse of every element is unique, we conclude that

$$-\theta^*(\eta) = \theta^*(-\eta).$$

□

5. Conclusion

In this work, we have extended the classical notion of near fields by introducing and developing the theory of H_v-near fields within the framework of hyperstructures. Through a detailed examination of their algebraic properties and the role of the fundamental relation θ^*, we have provided a deeper understanding of the internal structure of these generalized systems. The construction methods presented not only confirm the existence of various H_v-near fields but also illustrate their flexibility and richness. These results open new avenues for further exploration in hyperalgebraic systems, particularly in identifying applications of H_v-near fields in areas such as cryptography, coding theory, and algebraic combinatorics.

6. Author Contributions

Conceptualization, M.S. and B.D.; methodology, B.D.; formal analysis, B.D.; investigation, M.S. and B.D.; writing—original draft preparation, M.S.; writing—review and editing, B.D.; visualization, B.D.; supervision, B.D. All authors have read and agreed to the published version of the manuscript.

7. Data Availability Statement

Not applicable.

8. Aknowledgement

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9. Conflict of interest

The authors declare no conflict of interest.

References

[1] Al Tahan, M., & Davvaz, B. (2017). Hyperstructures associated to Biological inheritance. Mathematical Biosciences, 285, 112-118. https://doi.org/10.1016/j.mbs.2017.01.002
[2] Corsini, P. (1993). Prolegomena of hypergroup theory. Ssecond edition, Aviani editor.
[3] Davvaz, B. (2000). H_v-near rings. Math. Japonica, 52(2), 387-392.
[4] Davvaz, B., & Vougiouklis, T. (2019). A walk through weak hyperstructures: H_v-structures. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ. https://doi.org/10.1142/11229
[5] Davvaz, B., Dehghan Nezad, A., & Heidari, M.M. (2013). Inheritance examples of algebraic hyperstructures. Information Sciences, 224, 180-187. https://doi.org/10.1016/j.ins.2012.10.023
[6] Davvaz, B., Dehghan Nezad, A., Mazloum-Ardakani, M., & Sheikh-Mohseneib, M.A. (2014). Describing the algebraic hyperstructure of all elements in radiolytic processes in cement medium. MATCH Commun. Math. Comput. Chem., 72(2), 375-388.
[7] Dehghan Nezhad, A., Moosavi Nejad, S.M., Nadjafikhah, M., & Davvaz, B. (2012). A physical example of algebraic hyperstructures: Leptons. Indian Journal of Physics, 86(11), 1027-1032. https://doi.org/10.1007/s12648-012-0151-x
[8] Dembrowski, P. (1968). Finite geometries. Springer, Berlin. https://doi.org/10.1007/BFb0094449
[9] Dickson, L.E. (1905). On finite algebras. Nachr. Gesellsch. Wissensch. Gött. Math.-Phys. Klasse, 358-393.
[10] Dramalidis, A. (1996). Dual H_v-rings. Rivista di Matematica Pura Appl., 17, 55-62.
[11] Farag, M. (2007). Hill Ciphers over near-fields. Mathematics and Computer Education, 41(1), 46-54.
[12] Krasner, M. (1983). A class of hyperrings and hyperfields. Int. J. Math. and Math. Sci., 2, 307-312. https://doi.org/10.1155/S0161171283000265
[13] Pilz, G. (1983). Near-rings: the theory and its applications. North-Holland, Amsterdam.
[14] Rota, R. (1996). Hyperaffine planes over hyperrings. Discrete Math., 155, 215-223. https://doi.org/10.1016/0012-365X(94)00385-V
[15] Spartaliies, S. (1996). On the number of H_v-rings with P-hyperoperations. Discrete Math., 155, 225-231. https://doi.org/10.1016/0012-365X(94)00386-W
[16] Spartalies, S., & Vongiouklis, T. (1994). The fundamental relation on H_v-rings. Rivista di Matematica Pura Appl., 7-20.
[17] Vougiouklis, T. (1994). Hyperstructures and their Representations. Monographs in Math., Hadronic, Palm Harbor, United States.
[18] Vougiouklis, T. (2020). Minimal H_v-fields. Ratio Mathematica, 38, 313-328.
[19] Vougiouklis, T. (2017). H_v-fields, h/v-fields. Ratio Mathematica, 33, 181-201.
[20] Vougiouklis, T. (1995). A new class of hyperstructures. J. Combin. Inform. System Sci., 20, 229-235.

[21] Vougiouklis, T. (1997). Convolutions on WASS hyperstructures. Discrete Math., 174, 347-355. https://doi.org/10.1016/S0012-365X(96)00348-2
[22] Vougiouklis, T. (1991). The fundamental relation in hyperrings, The general hyperfield. 4th AHA, Xanthi (1990), World Scientific, Singapore.
[23] Veblen, O., & Wedderburn, J.H. (1907). Non-desarguesian and non-pascalian geometrie. Trans. Amer. Math. Soc., 8, 379-388. https://doi.org/10.2307/1988781

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