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A COMPREHENSIVE STUDY OF THE GENERAL DOUBLE MELLIN TRANSFORM: DEFINITIONS AND APPLICATIONS

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Abstract. This paper introduces a novel double integral transform and highlights its theoretical and practical significance in solving complex mathematical problems. The fundamental definitions, key properties, and its relationships with existing integral transforms are systematically developed. The novelty of the proposed transform lies in its generalized structure, which extends the classical Mellin framework and provides a unified approach for handling a broader class of functions. Several applications are presented, including the derivation of Mellin transforms for elementary functions and the solution of partial and fractional differential equations. In addition, a comparative analysis with existing generalized Mellin transforms is carried out to demonstrate the advantages and efficiency of the proposed method. Finally, a comprehensive table of transforms for functions frequently arising in mathematics, physics, and engineering is provided to facilitate practical implementation.

Keywords: Mellin transform, Fractional partial differential equations, Integral operator.
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1. Introduction

Integral transforms have long been recognized as powerful mathematical tools across diverse scientific and engineering disciplines, providing an elegant framework for simplifying complex problems. By converting a given function f into a transformed counterpart F through a well-defined mathematical process, integral transforms often simplify intricate operations—such as solving differential equations or evaluating convolutions—into manageable algebraic equations in the transformed domain. Once the problem has been addressed in this simplified domain, the solution can be mapped back to the original domain using the inverse integral transform. This dual-process framework has proven instrumental in addressing a wide range of theoretical and applied problems.

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A variety of integral transforms have been developed over the years, including the new general single, double, and triple conformable integral transforms [13], the Sumudu transform [27], the N-transform [15], the Elzaki transform [7], the Upadhyaya integral transform [23], a new general integral transform [12], single, double and triple conformable integral transforms [1], the Fractional Fourier transform [16], Fourier's transform of fractional order [14], a new Fractional Fourier Transform and convolution products [21], the fractionalization of Fourier sine and Fourier cosine transforms [20], and a generalized fractional Fourier transform [18]. Each of these transforms is tailored to specific types of problems and possesses distinct properties. These tools have been extensively studied and applied across various fields, with substantial contributions documented in the literature. Their versatility and effectiveness continue to inspire research into their theoretical extensions and applications.

This paper is dedicated to the Mellin transform, a powerful integral transform widely recognized for its utility in the study of special functions, differential equations, and fractional calculus [3]. The Mellin transform serves as a crucial bridge between classical analysis and applied mathematics, playing a pivotal role in connecting real analysis, complex analysis, and number theory. Its distinctive property lies in converting multiplicative convolution into simple addition, which greatly simplifies the analysis of various mathematical problems. The transform has found applications across diverse fields including financial markets [24], where it is employed for analyzing economic models and risk assessment, probability theory [30], particularly in the study of probability distributions and statistical inference, and mathematical physics [9, 10, 28], where it is utilized for solving boundary value problems, wave propagation, and signal processing. Additionally, it has been successfully applied to minimax linear estimation on the half-line via Mellin transform [19], demonstrating its significance in optimization theory. Recognizing the transform's remarkable versatility and broad applicability, researchers have proposed various generalizations of the Mellin transform to broaden its applicability and address more complex mathematical problems. These extensions include double and triple Mellin transforms, fractional Mellin transforms, and their multidimensional counterparts. These generalizations are particularly significant in the context of fractional calculus, where they provide powerful tools for the analysis of fractional differential equations and associated phenomena [4, 11, 25, 26, 29]. Fractional calculus, which deals with derivatives and integrals of non-integer order, has gained considerable attention in recent years due to its ability to model complex phenomena in physics, engineering, and biology, and the generalized Mellin transforms play a crucial role in solving fractional differential equations, establishing fundamental results, and exploring the properties of fractional integral operators.

Motivated by the need for a unified approach to the existing generalizations of the Mellin transform, in this study we introduce a new generalized double Mellin integral transform that encompasses and extends previously defined transforms in the literature. The proposed transform provides a more comprehensive and flexible framework for addressing a broader class of mathematical problems encountered in fractional calculus, engineering, and mathematical physics. We investigate the theoretical properties of this new integral transform, including its existence conditions, inversion formula, and convolution theorems. Furthermore, we demonstrate its effectiveness in solving both partial and fractional differential equations through illustrative examples, thereby showcasing its potential as a powerful analytical tool in applied mathematics.

The structure of this paper is organized as follows. In Section 2, we introduce the fundamental definitions and preliminary concepts that underpin this study and lay the essential groundwork for the subsequent analysis. Section 3 presents the formulation and rigorous proof of the principal theorems and core properties associated with the proposed new generalized Mellin transform, thereby establishing a solid theoretical framework for its validity and applicability. In Section 4, we compute the transform for various elementary functions to illustrate its analytical behavior and demonstrate its effectiveness in relatively simple cases. Section 5 demonstrates several practical applications of the generalized Mellin transform to partial and fractional differential equations, highlighting its capability in tackling complex mathematical models. Finally, Section 6 provides a comprehensive table summarizing the main properties and operational formulas of the new transform, offering a convenient reference for researchers and practitioners.

2. **Elementary Definitions**

In this section, we introduce same basic definitions that will be used throughout the paper.

Definition 2.1. ([2]) The Gamma and Beta functions defined as follows:

$$\begin{aligned} \Gamma(z) &= \int_0^\infty \tau^{z-1} e^{-\tau} \mathrm{d}\tau, \\ \beta(u,v) &= \int_0^1 \tau^{u-1}(1-\tau)^{v-1} \mathrm{d}\tau. \end{aligned} \tag{1}$$

Definition 2.2. ([6]) The Laplace transform of a function $f(\tau)$ is defined by

$$\mathcal{L}\{f(\tau) : s\} = \int_0^\infty e^{-s\tau} f(\tau)\, \mathrm{d}\tau, \tag{2}$$

provided that $f(\tau)$ is piecewise continuous on every finite interval in $[0, \infty)$ and of exponential order, i.e., there exist constants $M, a > 0$ such that

$$|f(\tau)| \leq M e^{a\tau}, \quad \tau \geq 0,$$

and $s > a$.

Definition 2.3. ([6]) The Mellin transform of a function $f(\tau)$ is defined by

$$\mathbf{M}\{f(\tau) : s\} = F(s) = \int_0^\infty \tau^{s-1} f(\tau)\,\mathrm{d}\tau, \tag{3}$$

provided that $f(\tau)$ is locally integrable on $(0, \infty)$ and there exist real numbers $a < b$ such that

$$\int_0^1 \tau^{\sigma-1}|f(\tau)|\,\mathrm{d}\tau < \infty \quad \text{for } \sigma > a,$$

and

$$\int_1^\infty \tau^{\sigma-1}|f(\tau)|\,\mathrm{d}\tau < \infty \quad \text{for } \sigma < b,$$

where $s = \sigma + it$. Hence, the Mellin transform exists in the strip $a < \Re(s) < b$.

Definition 2.4. ([6]) The generalized Mellin and inverse Mellin transform respectively defined as follows:

$$\mathcal{M}\{f(\tau) : s\} = \mathcal{F}(q(s)) = p(s) \int_0^\infty \tau^{q(s)-1} f(\tau)\mathrm{d}\tau, \tag{4}$$

and

$$\mathcal{M}^{-1}\{\mathcal{F}(q(s)) : \tau\} = f(\tau) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{x^{-q(s)} q'(s) \hat{f}(q(s))}{p(s)} \mathrm{d}s. \tag{5}$$

Definition 2.5. ([17]) The Riemann-Liouville fractional integral defined as follows:

$${}^{RL}_{a}\mathbf{D}_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau, \quad \alpha > 0. \tag{6}$$

Definition 2.6. The Riemann-Liouville and Liouville-Caputo fractional derivatives in [17] defined as follows respectively:

$$\begin{aligned} {}^{RL}_{a}\mathbf{D}_t^{\alpha} f(t) &= \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-\tau)^{m-\alpha-1} f(\tau)\mathrm{d}\tau, \\ {}^{C}_{a}\mathbf{D}_t^{\alpha} f(t) &= \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)\mathrm{d}\tau, \end{aligned} \tag{7}$$

where $m - 1 < \alpha \le m$, $m \in \mathbb{N}$.

3. New Generalized Double Mellin Integral Transform

In this section, we introduce the new generalized double Mellin integral transform in two-dimensional as follows:

$$\begin{aligned} \mathcal{M}_2\{f(u,v);(r,s)\} &= \mathcal{F}_D(\xi(r),\zeta(s)) = \mathcal{M}_u[\mathcal{M}_v[f(u,v); v \to s]u \to r] \\ &= \phi(s) \int_0^\infty u^{\xi(r)-1} \left(\psi(s) \int_0^\infty v^{\zeta(s)-1} f(u,v)\mathrm{d}v \right) \mathrm{d}u \\ &= \phi(r)\psi(s) \int_0^\infty\!\!\int_0^\infty u^{\xi(r)-1} v^{\zeta(s)-1} f(u,v)\mathrm{d}u\mathrm{d}v, \end{aligned} \tag{8}$$

where $\xi(r)$ and $\zeta(s)$ are the transform of function $f(u,v)$ for u and v, respectively.

Remark 3.1. If $\phi(r)=1, \psi(s)=1$ and $\xi(r)=r, \zeta(s)=s$ then, the new integral transform is double Mellin transform.

$$\mathcal{M}_2\left[f\left(u,v\right);(r,s)\right]=\int_0^{\infty}\int_0^{\infty} u^{r-1}v^{s-1}f(u,v)\mathrm{d}u\mathrm{d}v. \tag{9}$$

The inverse new generalized double Mellin integral transform in two-dimensional defined by following equation:

(10)
$$\begin{aligned}\mathcal{M}_2^{-1}\{\mathcal{F}_D(\xi(r),\zeta(s))\} &= \mathcal{M}_u^{-1}[\mathcal{M}_v^{-1}[\mathcal{F}_D(\xi(r),\zeta(s))]]=f(u,v)\\ &=\frac{1}{(2\pi i)^2}\int_{a-i\infty}^{a+i\infty}\frac{u^{-\xi(r)}\xi'(r)}{\phi(r)}\int_{b-i\infty}^{b+i\infty}\frac{v^{-\zeta(s)}\zeta'(s)\mathcal{F}_D(\xi(r),\zeta(s))}{\psi(s)}\mathrm{d}s\mathrm{d}r,\end{aligned}$$

where a and b are the real constants.

Definition 3.2. [5] The space X_c is defined for some $c\in\mathbb{R}$ by

$$X_c:=\{f:\mathbb{R}_+\to\mathbb{C}; f(x)x^{c-1}\in L^1(\mathbb{R}_+)\},$$

where $L^1(\mathbb{R}_+)$, be the space of all (Lebesgue) measurable.

3.1. **Function Spaces and Admissibility Conditions.** To ensure the well-posedness of the new generalized double Mellin transform, as well as the validity of its inversion and operational properties, we specify the class of admissible functions.

Let $\mathbb{R}_+=(0,\infty)$. For $c_1,c_2\in\mathbb{R}$, we define the weighted Mellin-type space

$$X_{c_1,c_2}:=\left\{f:\mathbb{R}_+^2\to\mathbb{C}\;\middle|\;\int_0^{\infty}\int_0^{\infty}|f(u,v)|\,u^{c_1-1}v^{c_2-1}\,du\,dv<\infty\right\}. \tag{11}$$

This space generalizes the one-dimensional space X_c introduced in Definition 3.1. Throughout this paper, the following assumptions are imposed:

(1) **Existence of the transform.** Let $f(u,v)\in X_{c_1,c_2}$. The parameters $\xi(r)$ and $\zeta(s)$ are chosen such that

$$\Re(\xi(r))=c_1,\qquad \Re(\zeta(s))=c_2, \tag{12}$$

which guarantees the absolute convergence of the double integral in (8).

(2) **Admissible growth conditions.** The function $f(u,v)$ satisfies polynomial growth bounds of the form

$$|f(u,v)|\le C\,u^{\alpha}v^{\beta},\quad \text{as } u,v\to 0 \text{ or } \infty, \tag{13}$$

for some constants $C>0$ and $\alpha,\beta\in\mathbb{R}$, ensuring convergence of the transform and its inverse.

(3) **Inversion validity.** The inverse transform (10) holds for functions $f \in X_{c_1,c_2}$ whose transform $F_D(\xi(r),\zeta(s))$ is analytic in the vertical strips

$$c_1 - \delta_1 < \Re(\xi(r)) < c_1 + \delta_1, \qquad c_2 - \delta_2 < \Re(\zeta(s)) < c_2 + \delta_2, \tag{14}$$

for some $\delta_1, \delta_2 > 0$, and satisfies suitable decay conditions as $|\Im(r)|$, $|\Im(s)| \to \infty$.

(4) **Differentiation under the integral sign.** For the operational properties (e.g., Theorems 4.2 and 4.3), we assume that $f(u,v)$ and its partial derivatives up to the required order belong to X_{c_1,c_2}, and that

$$u^{\xi(r)} f(u,v) \to 0, \qquad v^{\zeta(s)} f(u,v) \to 0 \tag{15}$$

as $u,v \to 0$ and $u,v \to \infty$, ensuring that boundary terms vanish when applying integration by parts.

(5) **Relation to classical spaces.** The space X_{c_1,c_2} includes weighted L^1-spaces as special cases and is closely related to Mellin–Schwartz-type spaces characterized by rapid decay in logarithmic variables.

4. Some Theorems and Properties of New General Double Mellin Integral Transform

In this section we introduce some basic theorems and properties of new general double Mellin integral transform.

Theorem 4.1. *Suppose that $\mathcal{M}_2[f(u,v)] = \mathcal{F}_D(\xi(r),\zeta(s))$ and $\mathcal{M}_2[h(u,v)] = \mathcal{H}_D(\xi(r),\zeta(s))$, then we have the following result:*

(1) *For constants A and B, we have:*

$$\mathcal{M}_2[Af(u,v) \pm Bh(u,v)] = A\mathcal{M}_2[f(u,v)] \pm B\mathcal{M}_2[h(u,v)]. \tag{16}$$

(2) *For $a,b > 0$ we have:*

$$\mathcal{M}_2[f(au,bv)](r,s) = \frac{\mathcal{F}_D(\xi(r),\zeta(s))}{a^{\xi(r)} b^{\zeta(s)}}. \tag{17}$$

(3) *For $a,b > 0$, we have:*

$$\mathcal{M}_2\left[u^a v^b f(u,v)\right](r,s) = \mathcal{F}_D(\xi(r)+a, \zeta(s)+b). \tag{18}$$

(4) *For $a,b > 0$ we have:*

$$\mathcal{M}_2\left[f(u^a,v^b)\right](r,s) = \frac{1}{ab}\mathcal{F}_D\left(\frac{\xi(r)}{a}, \frac{\zeta(s)}{b}\right). \tag{19}$$

Proof. To proof of part (1) we use from definition (8), we get:

$$\begin{aligned}
&\mathcal{M}_2\left[Af(u,v) \pm Bh(u,v)\right] \\
&= \phi(r)\psi(s)\int_0^\infty\int_0^\infty u^{\xi(r)-1}v^{\zeta(s)-1}\left(Af(u,v) \pm Bh(u,v)\right)\mathrm{d}u\mathrm{d}v \\
&= A\phi(r)\psi(s)\int_0^\infty\int_0^\infty u^{\xi(r)-1}v^{\zeta(s)-1}f(u,v)\,\mathrm{d}u\mathrm{d}v \\
&\pm B\phi(r)\psi(s)\int_0^\infty\int_0^\infty u^{\xi(r)-1}v^{\zeta(s)-1}h(u,v)\,\mathrm{d}u\mathrm{d}v \\
&= A\mathcal{M}_2\left[f(u,v)\right] \pm B\mathcal{M}_2\left[h(u,v)\right].
\end{aligned} \tag{20}$$

For part of (2), we have:

$$\mathcal{M}_2\left[f(au,bv)\right](r,s) = \phi(r)\psi(s)\int_0^\infty\int_0^\infty u^{\xi(r)-1}v^{\zeta(s)-1}f(au,bv)\,\mathrm{d}u\mathrm{d}v. \tag{21}$$

By substituting $au = p, bv = q$ we have:

$$\begin{aligned}
\mathcal{M}_2\left[f(au,bv)\right](r,s) &= \frac{\phi(r)\psi(s)}{a^{\xi(r)}b^{\zeta(s)}}\int_0^\infty\int_0^\infty p^{\xi(r)-1}q^{\zeta(s)-1}f(p,q)\,\mathrm{d}p\mathrm{d}q \\
&= \frac{\mathcal{F}_D(\xi(r),\zeta(s))}{a^{\xi(r)}b^{\zeta(s)}}.
\end{aligned} \tag{22}$$

For part of (3), we have:

$$\begin{aligned}
\mathcal{M}_2\left[u^a v^b f(u,v)\right](r,s) &= \phi(r)\psi(s)\int_0^\infty\int_0^\infty u^{\xi(r)+a-1}v^{\zeta(s)+b-1}f(u,v)\,\mathrm{d}u\mathrm{d}v \\
&= \mathcal{F}_D(\xi(r)+a,\zeta(s)+b).
\end{aligned} \tag{23}$$

Finally for part of (4), we have respectively:

$$\mathcal{M}_2\left[f(u^a,v^b)\right](r,s) = \phi(r)\psi(s)\int_0^\infty\int_0^\infty u^{\xi(r)-1}v^{\zeta(s)-1}f\left(u^a,v^b\right)\mathrm{d}u\mathrm{d}v. \tag{24}$$

Now, taking $u^a = p$ and $v^b = q$, we get:

$$\begin{aligned}
\mathcal{M}_2\left[f(u^a,v^b)\right](r,s) &= \phi(r)\psi(s)\int_0^\infty\int_0^\infty u^{\xi(r)-1}v^{\zeta(s)-1}f\left(u^a,v^b\right)\mathrm{d}u\mathrm{d}v \\
&= \frac{\phi(r)\psi(s)}{ab}\int_0^\infty\int_0^\infty p^{\frac{\xi(r)}{a}-1}q^{\frac{\zeta(s)}{b}-1}f(p,q)\,\mathrm{d}p\mathrm{d}q \\
&= \frac{1}{ab}\mathcal{F}_D\left(\frac{\xi(r)}{a},\frac{\zeta(s)}{b}\right).
\end{aligned} \tag{25}$$

$\square$

Theorem 4.2. *Suppose $\mathcal{M}_2[f(u,v)] = \mathcal{F}_D(\xi(r),\zeta(s))$, then we have the following equation:*

(26)
$$\mathcal{M}_2\left[\frac{\partial^{n+m}f(u,v)}{\partial u^n \partial v^m}\right] = (-1)^{n+m}\frac{\Gamma(\xi(r))\Gamma(\zeta(s))}{\Gamma(\xi(r)-n)\Gamma(\zeta(s)-m)}\mathcal{F}_D(\xi(r)-n,\zeta(s)-m),$$

where $n, m \in \mathbb{N}$.

Proof. To prove Theorem (4.2), we proceed by mathematical induction: For $n = 1$, we get:

(27)
$$\begin{aligned}
\mathcal{M}_2\left[\frac{\partial^{1+m}f(u,v)}{\partial u\,\partial v^m}\right] &= \mathcal{M}_2\left[\frac{\partial}{\partial u}\left(\frac{\partial^m f(u,v)}{\partial v^m}\right)\right] \\
&= \phi(r)\,\psi(s)\int_0^\infty\!\!\int_0^\infty u^{\xi(r)-1}\,v^{\zeta(s)-1}\frac{\partial}{\partial u}\left(\frac{\partial^m f(u,v)}{\partial v^m}\right)\mathrm{d}u\,\mathrm{d}v \\
&= \phi(r)\,\psi(s)\int_0^\infty v^{\zeta(s)-1} \\
&\quad\times\left(\int_0^\infty u^{\xi(r)-1}\frac{\partial}{\partial u}\left(\frac{\partial^m f(u,v)}{\partial v^m}\right)\mathrm{d}u\right)\mathrm{d}v.
\end{aligned}$$

Using integration by part with respect to u, we have:

$$\begin{aligned}
\mathcal{M}_2\left[\frac{\partial^{1+m}f(u,v)}{\partial u\,\partial v^m}\right] &= -(\xi(r)-1)\phi(r)\psi(s) \\
&\quad\times\int_0^\infty\int_0^\infty u^{\xi(r)-2}v^{\zeta(s)-1}\frac{\partial^m f(u,v)}{\partial v^m}\,\mathrm{d}u\,\mathrm{d}v \\
&= -\frac{\Gamma(\xi(r))}{\Gamma(\xi(r)-1)}\phi(r)\psi(s) \\
&\quad\times\int_0^\infty\int_0^\infty u^{\xi(r)-2}v^{\zeta(s)-1}\frac{\partial^m f(u,v)}{\partial v^m}\,\mathrm{d}u\,\mathrm{d}v,
\end{aligned} \tag{28}$$

and for $m = 1$ and using integration by part with respect to v, we get:

(29)
$$\begin{aligned}
\mathcal{M}_2\left[\frac{\partial^2 f(u,v)}{\partial u\partial v}\right] &= -\frac{\Gamma(\xi(r))}{\Gamma(\xi(r)-1)}\phi(r)\psi(s)\int_0^\infty\!\!\int_0^\infty u^{\xi(r)-2}v^{\zeta(s)-1}\frac{\partial f(u,v)}{\partial v}\mathrm{d}u\mathrm{d}v \\
=&(-)^2\frac{\Gamma(\xi(r))\Gamma(\zeta(s))}{\Gamma(\xi(r)-1)\Gamma(\zeta(s)-1)}\phi(r)\psi(s)\int_0^\infty\!\!\int_0^\infty u^{\xi(r)-2}v^{\zeta(s)-2}f(u,v)\mathrm{d}u\mathrm{d}v \\
=&(-1)^2\frac{\Gamma(\xi(r))\Gamma(\zeta(s))}{\Gamma(\xi(r)-1)\Gamma(\zeta(s)-1)}\mathcal{F}_D(\xi(r)-1,\zeta(s)-1).
\end{aligned}$$

For $n=k$ and $m=\mathbf{k}$, we suppose the following equation is true:

(30)
$$\begin{aligned}\mathcal{M}_2\left[\frac{\partial^{k+\mathbf{k}}f(u,v)}{\partial u^k\partial v^{\mathbf{k}}}\right] =&(-1)^{k+\mathbf{k}}\frac{\Gamma(\xi(r))\Gamma(\zeta(s))}{\Gamma(\xi(r)-k)\Gamma(\zeta(s)-\mathbf{k})}\\ &.\phi(r)\psi(s)\int_0^\infty\int_0^\infty u^{\xi(r)-(k+1)}v^{\zeta(s)-(\mathbf{k}+1)}f(u,v)\mathrm{d}u\mathrm{d}v\\ =&(-1)^{k+\mathbf{k}}\frac{\Gamma(\xi(r))\Gamma(\zeta(s))}{\Gamma(\xi(r)-k)\Gamma(\zeta(s)-\mathbf{k})}\mathcal{F}_D(\xi(r)-k,\zeta(s)-\mathbf{k}).\end{aligned}$$

For $n=k+1$ and $m=\mathbf{k}+1$, we prove:

$$\begin{aligned}\mathcal{M}_2\left[\frac{\partial^{k+1+\mathbf{k}+1}f(u,v)}{\partial u^{k+1}\partial v^{\mathbf{k}+1}}\right] &= \mathcal{M}_2\left[\frac{\partial}{\partial u}\left\{\frac{\partial^k}{\partial u^k}\left(\frac{\partial^{\mathbf{k}+1}f(u,v)}{\partial v^{\mathbf{k}+1}}\right)\right\}\right]\\ &= \phi(r)\psi(s)\int_0^\infty\int_0^\infty u^{\xi(r)-1}v^{\zeta(s)-1}\frac{\partial}{\partial u}\left\{\frac{\partial^k}{\partial u^k}\left(\frac{\partial^{\mathbf{k}+1}f(u,v)}{\partial v^{\mathbf{k}+1}}\right)\right\}\mathrm{d}u\mathrm{d}v.\end{aligned} \tag{31}$$

By imposing integration by part with respect to u, we get:

(32)
$$\begin{aligned}&\mathcal{M}_2\left[\frac{\partial^{k+1+\mathbf{k}+1}f(u,v)}{\partial u^{k+1}\partial v^{\mathbf{k}+1}}\right]\\ &= -\frac{\Gamma(\xi(r))}{\Gamma(\xi(r)-1)}\phi(r)\psi(s)\int_0^\infty\int_0^\infty u^{\xi(r)-2}v^{\zeta(s)-1}\frac{\partial^k}{\partial u^k}\left(\frac{\partial^{\mathbf{k}+1}f(u,v)}{\partial v^{\mathbf{k}+1}}\right)\mathrm{d}u\mathrm{d}v\\ &= (-1)^{k+1}\frac{\Gamma(\xi(r))}{\Gamma(\xi(r)-k)}\phi(r)\psi(s)\int_0^\infty\int_0^\infty u^{\xi(r)-(k+2)}v^{\zeta(s)-1}\frac{\partial^{\mathbf{k}+1}f(u,v)}{\partial v^{\mathbf{k}+1}}\mathrm{d}u\mathrm{d}v\\ &= (-1)^{k+1}\frac{\Gamma(\xi(r))}{\Gamma(\xi(r)-k)}\\ &\phi(r)\psi(s)\int_0^\infty\int_0^\infty u^{\xi(r)-(k+2)}v^{\zeta(s)-1}\frac{\partial}{\partial v}\left(\frac{\partial^{\mathbf{k}}f(u,v)}{\partial v^{\mathbf{k}}}\right)\mathrm{d}u\mathrm{d}v.\end{aligned}$$

By imposing integration by part with respect to v, we have:

(33)
$$\begin{aligned}
&\mathcal{M}_2\left[\frac{\partial^{k+1+\mathbf{k}+1} f(u,v)}{\partial u^{k+1}\partial v^{\mathbf{k}+1}}\right]\\
&= (-1)^{k+2}\frac{\Gamma(\xi(r))}{\Gamma(\xi(r)-(k+1))}\frac{\Gamma(\zeta(s))}{\Gamma(\zeta(s)-1)}\phi(r)\psi(s)\\
&\times\int_0^\infty\int_0^\infty u^{\xi(r)-(k+2)}v^{\zeta(s)-2}\frac{\partial^{\mathbf{k}} f(u,v)}{\partial v^{\mathbf{k}}}\,\mathrm{d}u\,\mathrm{d}v\\
&= (-1)^{k+\mathbf{k}+2}\frac{\Gamma(\xi(r))}{\Gamma(\xi(r)-(k+1))}\frac{\Gamma(\zeta(s))}{\Gamma(\zeta(s)-(\mathbf{k}+1))}\\
&\phi(r)\psi(s)\int_0^\infty\int_0^\infty u^{\xi(r)-(k+2)}v^{\zeta(s)-(\mathbf{k}+2)}f(u,v)\,\mathrm{d}u\,\mathrm{d}v\\
&= (-1)^{k+\mathbf{k}+2}\frac{\Gamma(\xi(r))\Gamma(\zeta(s))}{\Gamma(\xi(r)-(k+1))\Gamma(\zeta(s)-(\mathbf{k}+1))}\\
&\times\mathcal{F}_D(\xi(r)-(k+1),\zeta(s)-(\mathbf{k}+1)).
\end{aligned}$$

□

Theorem 4.3. *Suppose $\mathcal{M}_2[f(u,v)] = \mathcal{F}_D(\xi(r),\zeta(s))$, then we have the following equation:*

(34)
$$\mathcal{M}_2\left[u^n v^m\frac{\partial^{n+m} f(u,v)}{\partial u^n\partial v^m}\right](r,s) = (-1)^{n+m}\frac{\Gamma(\xi(r)+n)\Gamma(\zeta(s)+m)}{\Gamma(\xi(r))\Gamma(\zeta(s))}\mathcal{F}_D(\xi(r),\zeta(s)),$$

where $n,m\in\mathbb{N}$.

Proof. To prove of above eqation, for $n=m=1$ we have:

(35)
$$\begin{aligned}
\mathcal{M}_2\left[uv\frac{\partial^2 f(u,v)}{\partial u\partial v}\right](r,s) &= \phi(r)\psi(s)\int_0^\infty\int_0^\infty u^{\xi(r)}v^{\zeta(s)}\frac{\partial^2 f(u,v)}{\partial u\partial v}\mathrm{d}u\mathrm{d}v\\
&= \phi(r)\psi(s)\int_0^\infty v^{\zeta(s)}\left(\int_0^\infty u^{\xi(r)}\frac{\partial}{\partial u}\left(\frac{\partial f(u,v)}{\partial v}\right)\mathrm{d}u\right)\mathrm{d}v.
\end{aligned}$$

By imposing integration by part with respect to u, we have:

(36)
$$\begin{aligned}
\mathcal{M}_2\left[uv\frac{\partial^2 f(u,v)}{\partial u\partial v}\right](r,s) &= -\xi(r)\phi(r)\psi(s)\int_0^\infty\int_0^\infty u^{\xi(r)-1}v^{\zeta(s)}\frac{\partial f(u,v)}{\partial v}\mathrm{d}u\mathrm{d}v\\
&= -\frac{\Gamma(\xi(r)+1)}{\Gamma(\xi(r))}\phi(r)\psi(s)\int_0^\infty u^{\xi(r)-1}\left(\int_0^\infty v^{\zeta(s)}\frac{\partial f(u,v)}{\partial v}\mathrm{d}v\right)\mathrm{d}u.
\end{aligned}$$

By integration by part with respect to v, we have:

(37)

$$\begin{aligned}
&\mathcal{M}_2\left[uv\frac{\partial^2 f(u,v)}{\partial u\partial v}\right](r,s)\\
&\quad=-\frac{\Gamma(\xi(r)+1)\Gamma(\zeta(s)+1)}{\Gamma(\xi(r))\Gamma(\zeta(s))}\phi(r)\psi(s)\int_0^\infty\int_0^\infty u^{\xi(r)-1}v^{\zeta(s)-1}f(u,v)\mathrm{d}u\mathrm{d}v\\
&\quad=-\frac{\Gamma(\xi(r)+1)\Gamma(\zeta(s)+1)}{\Gamma(\xi(r))\Gamma(\zeta(s))}\mathcal{F}_D(\xi(r),\zeta(s)).
\end{aligned}$$

In which $u^{\xi(r)}f(u,v)=0$ and $v^{\zeta(s)}f(u,v)=0$ as $u\to 0, v\to 0$ and $u\to\infty, v\to\infty$.

For $n=k, m=\mathbf{k}$, suppose the following equation be true:

(38)

$$\mathcal{M}_2\left[u^k v^{\mathbf{k}}\frac{\partial^{k+\mathbf{k}} f(u,v)}{\partial u^k\partial v^{\mathbf{k}}}\right](r,s)=(-1)^{k+\mathbf{k}}\frac{\Gamma(\xi(r)+k)\Gamma(\zeta(s)+\mathbf{k})}{\Gamma(\xi(r))\Gamma(\zeta(s))}\mathcal{F}_D(\xi(r),\zeta(s)).$$

For $n=k+1$ and $m=\mathbf{k}+1$, we prove:

$$\begin{aligned}
&\mathcal{M}_2\left[u^{k+1}v^{\mathbf{k}+1}\frac{\partial^{k+\mathbf{k}+2} f(u,v)}{\partial u^{k+1}\partial v^{\mathbf{k}+1}}\right](r,s)\\
&\quad=\mathcal{M}_2\left[u^{k+1}v^{\mathbf{k}+1}\frac{\partial^2}{\partial u\partial v}\left(\frac{\partial^{k+\mathbf{k}} f(u,v)}{\partial u^k\partial v^{\mathbf{k}}}\right)\right](r,s)\\
&\quad=\phi(r)\psi(s)\int_0^\infty\int_0^\infty u^{\xi(r)+k}v^{\zeta(s)+\mathbf{k}}\frac{\partial^2}{\partial u\partial v}\left(\frac{\partial^{k+\mathbf{k}} f(u,v)}{\partial u^k\partial v^{\mathbf{k}}}\right)\mathrm{d}u\mathrm{d}v\\
&\quad=\phi(r)\psi(s)\int_0^\infty v^{\zeta(s)+\mathbf{k}}\left(\int_0^\infty u^{\xi(r)+k}\frac{\partial}{\partial u}\left(\frac{\partial^{k+\mathbf{k}+1} f(u,v)}{\partial u^k\partial v^{\mathbf{k}+1}}\right)\mathrm{d}u\right)\mathrm{d}v.
\end{aligned}\tag{39}$$

By taking integration by part with respect to u, we have:

(40)

$$\begin{aligned}
&\mathcal{M}_2\left[u^{k+1}v^{\mathbf{k}+1}\frac{\partial^{k+\mathbf{k}+2} f(u,v)}{\partial u^{k+1}\partial v^{\mathbf{k}+1}}\right](r,s)\\
&=-(\xi(r)+k)\phi(r)\psi(s)\int_0^\infty v^{\zeta(s)+\mathbf{k}}\left(\int_0^\infty u^{\xi(r)+k-1}\frac{\partial^{k+\mathbf{k}+1} f(u,v)}{\partial u^k\partial v^{\mathbf{k}+1}}\mathrm{d}v\right)\mathrm{d}u\\
&=-(\xi(r)+k)\phi(r)\psi(s)\int_0^\infty u^{\xi(r)+k-1}\left(\int_0^\infty v^{\zeta(s)+\mathbf{k}}\frac{\partial}{\partial v}\left(\frac{\partial^{k+\mathbf{k}} f(u,v)}{\partial u^k\partial v^{\mathbf{k}}}\right)\mathrm{d}v\right)\mathrm{d}u.
\end{aligned}$$

And by imposing integration by part with respect to v, we get:

(41)

$$\begin{aligned}
&\mathcal{M}_2\left[u^{k+1}v^{\mathbf{k}+1}\frac{\partial^{k+\mathbf{k}+2}f(u,v)}{\partial u^{k+1}\partial v^{\mathbf{k}+1}}\right](r,s)\\
&=(-1)^2(\xi(r)+k)(\zeta(s)+\mathbf{k})\phi(r)\psi(s)\int_0^\infty\!\!\int_0^\infty u^{\xi(r)+k-1}v^{\zeta(s)+\mathbf{k}-1}\frac{\partial^{k+\mathbf{k}}f(u,v)}{\partial u^k\partial v^{\mathbf{k}}}\mathrm{d}v\mathrm{d}u\\
&=(-1)^{k+\mathbf{k}+2}(\xi(r)+k)(\zeta(s)+\mathbf{k})\frac{\Gamma(\xi(r)+k)\Gamma(\zeta(s)+\mathbf{k})}{\Gamma(\xi(r))\Gamma(\zeta(s))}\mathcal{F}_D(\xi(r),\zeta(s))\\
&=(-1)^{k+\mathbf{k}+2}\frac{\Gamma(\xi(r)+k+1)\Gamma(\zeta(s)+\mathbf{k}+1)}{\Gamma(\xi(r))\Gamma(\zeta(s))}\mathcal{F}_D(\xi(r),\zeta(s)).
\end{aligned}$$

Where we let that the first terms from the integration by part approach zero as $u\to 0, v\to 0$ and $u\to\infty, v\to\infty$. □

Definition 4.4. The general double Mellin convolutions defined as follows:

(1)

$$(f\ast\ast g)(u,v)=\int_0^\infty\!\!\int_0^\infty f(\vartheta,\varrho)g\left(\frac{u}{\vartheta},\frac{v}{\varrho}\right)\frac{\mathrm{d}\vartheta}{\vartheta}\frac{\mathrm{d}\varrho}{\varrho}. \tag{42}$$

(2)

$$(f\circ\circ g)(u,v)=\int_0^\infty\!\!\int_0^\infty f(u\vartheta,v\varrho)g(\vartheta,\varrho)\,\mathrm{d}\vartheta\mathrm{d}\varrho. \tag{43}$$

Theorem 4.5. *Suppose $\mathcal{M}_2[f(u,v)]=\mathcal{F}_D(\xi(r),\zeta(s))$, then we have the following result:*

(1)

$$\mathcal{M}_2\left[(f\ast\ast g)(u,v)\right](r,s)=\frac{\mathcal{F}_D(\xi(r),\zeta(s))\mathcal{G}_D(\xi(r),\zeta(s))}{\phi(r)\psi(s)}. \tag{44}$$

(2)

$$\mathcal{M}_2\left[(f\circ\circ g)(u,v)\right](r,s)=\frac{\mathcal{F}_D(\xi(r),\zeta(s))\mathcal{G}_D(1-\xi(r),1-\zeta(s))}{\phi(r)\psi(s)}. \tag{45}$$

Proof. According to definition of general double Mellin transform, we have:

(46)

$$\begin{aligned}
\mathcal{M}_2\left[(f\ast\ast g)(u,v)\right](r,s)&=\mathcal{M}_2\left[\int_0^\infty\!\!\int_0^\infty f(\vartheta,\varrho)g\left(\frac{u}{\vartheta},\frac{v}{\varrho}\right)\frac{\mathrm{d}\vartheta}{\vartheta}\frac{\mathrm{d}\varrho}{\varrho}\right](r,s)\\
&=\phi(r)\psi(s)\int_0^\infty\!\!\int_0^\infty u^{\xi(r)-1}v^{\zeta(s)-1}\left[\int_0^\infty\!\!\int_0^\infty f(\vartheta,\varrho)g\left(\frac{u}{\vartheta},\frac{v}{\varrho}\right)\frac{\mathrm{d}\vartheta}{\vartheta}\frac{\mathrm{d}\varrho}{\varrho}\right]\mathrm{d}u\mathrm{d}v.
\end{aligned}$$

By substituting $\frac{u}{\vartheta}=p$ and $\frac{v}{\varrho}=q$, we have:

$$\begin{aligned}\mathcal{M}_2\left[(f **g)(u,v)\right](r,s) =&\phi(r)\psi(s)\int_0^\infty\!\!\int_0^\infty \vartheta^{\xi(r)-1}\varrho^{\zeta(s)-1}f(\vartheta,\varrho)\mathrm{d}\vartheta\mathrm{d}\varrho\\ &\cdot\left[\int_0^\infty\!\!\int_0^\infty p^{\xi(r)-1}q^{\zeta(s)-1}g\left(p,q\right)\mathrm{d}p\mathrm{d}q\right]\\ =&\frac{\mathcal{F}_D(\xi(r),\zeta(s))\mathcal{G}_D(\xi(r),\zeta(s))}{\phi(r)\psi(s)}.\end{aligned}\tag{47}$$

Similarly, using new general double Mellin transform, we have:

$$\begin{aligned}&\mathcal{M}_2\left[(f\circ\circ g)(u,v)\right](r,s)=\mathcal{M}_2\left[\int_0^\infty\!\!\int_0^\infty f(u\vartheta,v\varrho)g\left(\vartheta,\varrho\right)\mathrm{d}\vartheta\mathrm{d}\varrho\right](r,s)\\ &=\phi(r)\psi(s)\int_0^\infty\!\!\int_0^\infty u^{\xi(r)-1}v^{\zeta(s)-1}\left[\int_0^\infty\!\!\int_0^\infty f(u\vartheta,v\varrho)g\left(\vartheta,\varrho\right)\mathrm{d}\vartheta\mathrm{d}\varrho\right]\mathrm{d}u\mathrm{d}v.\end{aligned}\tag{48}$$

By substituting $u\vartheta=p$ and $v\varrho=q$, we get:

$$\begin{aligned}&\mathcal{M}_2\left[(f\circ\circ g)(u,v)\right](r,s)\\ &=\phi(r)\psi(s)\int_0^\infty\!\!\int_0^\infty p^{\xi(r)-1}q^{\zeta(s)-1}f(p,q)\mathrm{d}p\mathrm{d}q\left[\int_0^\infty\!\!\int_0^\infty \vartheta^{-\xi(r)}\varrho^{-\zeta(s)}g\left(\vartheta,\varrho\right)\mathrm{d}\vartheta\mathrm{d}\varrho\right]\\ &=\frac{\mathcal{F}_D(\xi(r),\zeta(s))\mathcal{G}_D(1-\xi(r),1-\zeta(s))}{\phi(r)\psi(s)}.\end{aligned}\tag{49}$$

□

Example 4.6. *Solve the double integral equation*

$$\int_0^\infty\!\!\int_0^\infty f(\vartheta,\varrho)k(u\vartheta,v\varrho)d\vartheta d\varrho=g(u,v),\quad u,v>0,\tag{50}$$

where $f(u,v)$ is unknown and $k(u,v)$ and $g(u,v)$ are given functions. Application of the new general double Mellin transform with respect to u,v to equation (50) *combined with* (45) *gives*

$$\frac{\mathcal{F}_D(1-\xi(r),1-\zeta(s))\mathcal{K}_D(\xi(r),\zeta(s))}{\phi(r)\psi(s)}=\mathcal{G}_D(\xi(r),\zeta(s)),$$

which gives, replacing $\xi(r)$ by $1-\xi(r)$ and $\zeta(s)$ by $1-\zeta(s)$, we have

$$\mathcal{F}_D(\xi(r),\zeta(s))=\phi(r)\psi(s)\mathcal{G}_D(1-\xi(r),1-\zeta(s))\mathcal{H}_D(\xi(r),\zeta(r)),$$

where

$$\mathcal{H}_D(\xi(r),\zeta(r))=\frac{1}{\mathcal{K}_D(1-\xi(r),1-\zeta(s))}.$$

The inverse general double Mellin transform combined with (45) *leads to the solution*

$$\begin{aligned} f(u,v) &= \mathcal{M}_2^{-1}\left\{\phi(r)\psi(s)\mathcal{G}_D(1-\xi(r),1-\zeta(s))\mathcal{H}_D(\xi(r),\zeta(r))\right\} \\ &= \int_0^\infty\int_0^\infty g(\vartheta,\varrho)h(u\vartheta,v\varrho)d\vartheta d\varrho, \end{aligned} \tag{51}$$

provided $h(u,v) = \mathcal{M}_2^{-1}\left\{\mathcal{H}(\xi(r),\zeta(s))\right\}$ *exists. Thus, the problem is formally solved.*
If, in particular, $\mathcal{H}_D(\xi(r),\zeta(s)) = \mathcal{K}_D(\xi(r),\zeta(s))$*, then the solution of* (51) *becomes*

$$f(u,v) = \int_0^\infty\int_0^\infty g(\vartheta,\varrho)k(u\vartheta,v\varrho)d\vartheta d\varrho, \tag{52}$$

provided $\mathcal{K}_D(\xi(r),\zeta(s))\mathcal{K}_D(1-\xi(r),1-\zeta(s)) = 1$.

Example 4.7. *Solve the double integral equation*

$$\int_0^\infty\int_0^\infty f(\vartheta,\varrho)g\left(\frac{u}{\vartheta},\frac{v}{\varrho}\right)\frac{d\vartheta}{\vartheta}\frac{d\varrho}{\varrho} = h(u,v), \tag{53}$$

where $f(u,v)$ *is unknown and* $g(u,v)$ *and* $h(u,v)$ *are given functions. Applications of the Mellin transform with respect to* u, v *gives*

$$\begin{aligned} \mathcal{F}_D(\xi(r),\zeta(s)) &= \phi(r)\psi(s)\mathcal{H}_D(\xi(r),\zeta(s))\mathcal{K}_D(\xi(r),\zeta(s)), \\ \mathcal{K}_D(\xi(r),\zeta(s)) &= \frac{1}{\mathcal{G}_D(\xi(r),\zeta(s))}. \end{aligned}$$

Inversion, by the convolution property (44) *, gives the solution*

$$\begin{aligned} f(u,v) &= \mathcal{M}_2^{-1}\left\{\phi(r)\psi(s)\mathcal{H}_D(\xi(r),\zeta(s))\mathcal{K}_D(\xi(r),\zeta(s))\right\} \\ &= \int_0^\infty\int_0^\infty h(\vartheta,\varrho)k\left(\frac{u}{\vartheta},\frac{v}{\varrho}\right)\frac{d\vartheta}{\vartheta}\frac{d\varrho}{\varrho} \end{aligned}. \tag{54}$$

Corollary 4.8. *By imposing new general double inverse Mellin transform to* (49)*, we have the following result:*

$$\begin{aligned} &(f \circ\circ g)(u,v) \\ &\quad = \mathcal{M}_2^{-1}\left[\frac{\mathcal{F}_D(\xi(r),\zeta(s))\,\mathcal{G}_D(1-\xi(r),1-\zeta(s))}{\phi(r)\psi(s)}\right](u,v), \\ &\int_0^\infty\int_0^\infty f(u\vartheta,v\varrho)\,g\,(\vartheta,\varrho)\,\mathrm{d}\vartheta\,\mathrm{d}\varrho \\ &\quad = \mathcal{M}_2^{-1}\left[\frac{\mathcal{F}_D(\xi(r),\zeta(s))\,\mathcal{G}_D(1-\xi(r),1-\zeta(s))}{\phi(r)\psi(s)}\right](u,v). \end{aligned} \tag{55}$$

If we assume that $f(u,v) = e^{-u-v}$*, we have:*

$$\mathcal{L}_2\left[g(\vartheta,\varrho)\right](u,v) = \mathcal{M}_2^{-1}\left[\frac{\mathcal{F}_D(\xi(r),\zeta(s))\mathcal{G}_D(1-\xi(r),1-\zeta(s))}{\phi(r)\psi(s)}\right](u,v). \tag{56}$$

Theorem 4.9. *Suppose* $\mathcal{M}_2[f(u,v)] = \mathcal{F}_D(\xi(r),\zeta(s))$*, then we have the following result:*

$$\begin{aligned}&\mathcal{M}_2\left[u^{\lambda}v^{\kappa}\int_0^{\infty}\!\!\int_0^{\infty}\tau^{\vartheta}\eta^{\varrho}f(u\tau,v\eta)g(\tau,\eta)\mathrm{d}\tau\mathrm{d}\eta\right](r,s)\\&=\frac{\mathcal{F}_D(\xi(r)+\lambda,\zeta(s)+\kappa)\mathcal{G}_D(\vartheta+1-\xi(r)-\lambda,\varrho+1-\zeta(s)-\kappa)}{\phi(r)\psi(s)}.\end{aligned} \tag{57}$$

Proof. According to definition of general double Mellin transform, we have:

$$\begin{aligned}&\mathcal{M}_2\left[u^{\lambda}v^{\kappa}\int_0^{\infty}\!\!\int_0^{\infty}\tau^{\vartheta}\eta^{\varrho}f(u\tau,v\eta)\,g(\tau,\eta)\,\mathrm{d}\tau\,\mathrm{d}\eta\right](r,s)\\&=\phi(r)\psi(s)\int_0^{\infty}\!\!\int_0^{\infty}u^{\xi(r)+\lambda-1}v^{\zeta(s)+\kappa-1}\\&\quad\times\int_0^{\infty}\!\!\int_0^{\infty}\tau^{\vartheta}\eta^{\varrho}f(u\tau,v\eta)\,g(\tau,\eta)\,\mathrm{d}\tau\,\mathrm{d}\eta\,\mathrm{d}u\,\mathrm{d}v\\&=\phi(r)\psi(s)\int_0^{\infty}\!\!\int_0^{\infty}\!\!\int_0^{\infty}\!\!\int_0^{\infty}u^{\xi(r)+\lambda-1}v^{\zeta(s)+\kappa-1}\\&\quad\times\tau^{\vartheta}\eta^{\varrho}f(u\tau,v\eta)\,g(\tau,\eta)\,\mathrm{d}\tau\,\mathrm{d}\eta\,\mathrm{d}u\,\mathrm{d}v.\end{aligned} \tag{58}$$

By substitution $u\tau = p$ and $v\eta = q$, we have:

$$\begin{aligned}&\mathcal{M}_2\left[u^{\lambda}v^{\kappa}\int_0^{\infty}\!\!\int_0^{\infty}\tau^{\vartheta}\eta^{\varrho}f(u\tau,v\eta)g(\tau,\eta)\mathrm{d}\tau\mathrm{d}\eta\right](r,s)\\&=\phi(r)\psi(s)\int_0^{\infty}\!\!\int_0^{\infty}p^{\xi(r)+\lambda-1}q^{\zeta(s)+\kappa-1}f(p,q)\mathrm{d}p\mathrm{d}q\\&\quad\left[\int_0^{\infty}\!\!\int_0^{\infty}\tau^{\vartheta-\xi(r)-\lambda}\eta^{\varrho-\zeta(s)-\kappa}g(\tau,\eta)\mathrm{d}\tau\mathrm{d}\eta\right]\\&=\frac{\mathcal{F}_D(\xi(r)+\lambda,\zeta(s)+\kappa)\mathcal{G}_D(\vartheta+1-\xi(r)-\lambda,\varrho+1-\zeta(s)-\kappa)}{\phi(r)\psi(s)}.\end{aligned} \tag{59}$$

□

Theorem 4.10. *(Parseval's type property) Suppose* $\mathcal{M}_2[f(u,v)] = \mathcal{F}_D(\xi(r),\zeta(s))$*, then we have the following result:*

$$\begin{aligned}&\mathcal{M}_2\left[f(u,v)\,g(u,v)\right](r,s)\\&=\frac{-1}{4\pi^2\phi(r)\psi(s)}\int_{b-i\infty}^{b+i\infty}\!\!\int_{a-i\infty}^{a+i\infty}\xi'(p)\,\zeta'(q)\\&\quad\times\mathcal{F}_D(\xi(p),\zeta(q))\,\mathcal{G}_D(\xi(r)-\xi(p),\zeta(s)-\zeta(q))\,\mathrm{d}p\,\mathrm{d}q.\end{aligned} \tag{60}$$

Proof. By using the general double Mellin transform and inverse Mellin transform, we have:

$$
\begin{aligned}
&\mathcal{M}_2\,[f(u,v)\,g(u,v)]\,(r,s)\\
&\quad= \phi(r)\psi(s)\int_0^\infty\!\!\int_0^\infty u^{\xi(r)-1}v^{\zeta(s)-1}f(u,v)\,g(u,v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad= \frac{1}{4\pi^2 i^2\phi(r)\psi(s)}\,\phi(r)\psi(s)\int_0^\infty\!\!\int_0^\infty\int_{b-i\infty}^{b+i\infty}\!\!\int_{a-i\infty}^{a+i\infty} u^{\xi(r)-1}v^{\zeta(s)-1}\\
&\quad\quad\times u^{-\xi(p)}\xi'(p)\,v^{-\zeta(q)}\zeta'(q)\,\mathcal{F}_D(p,q)\,g(u,v)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}p\,\mathrm{d}q\\
&\quad= \frac{1}{4\pi^2 i^2\phi(r)\psi(s)}\int_{b-i\infty}^{b+i\infty}\!\!\int_{a-i\infty}^{a+i\infty}\xi'(p)\,\zeta'(q)\,\mathcal{F}_D(\xi(p),\zeta(q))\\
&\quad\quad\times\left[\phi(r)\psi(s)\int_0^\infty\!\!\int_0^\infty u^{\xi(r)-\xi(p)-1}v^{\zeta(s)-\zeta(q)-1}\,g(u,v)\,\mathrm{d}u\,\mathrm{d}v\right]\mathrm{d}p\,\mathrm{d}q\\
&\quad= \frac{-1}{4\pi^2\phi(r)\psi(s)}\int_{b-i\infty}^{b+i\infty}\!\!\int_{a-i\infty}^{a+i\infty}\xi'(p)\,\zeta'(q)\,\mathcal{F}_D(\xi(p),\zeta(q))\\
&\quad\quad\times\,\mathcal{G}_D(\xi(r)-\xi(p),\zeta(s)-\zeta(q))\,\mathrm{d}p\,\mathrm{d}q.
\end{aligned}
$$

□

Definition 4.11. [8] The Riemann-Liouville partial fractional integral defined as follow:

$$
{}^{RL}_{a,b}\mathbf{I}^{\alpha,\beta}_{u,v}f(u,v)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^u\!\!\int_b^v(u-t)^{\alpha-1}(v-\tau)^{\beta-1}f(t,\tau)\mathrm{d}t\mathrm{d}\tau. \tag{61}
$$

Theorem 4.12. *Suppose $\mathcal{M}_2[f(u,v)]=\mathcal{F}_D(\xi(r),\zeta(s))$, then we have the following result:*

$$
\begin{aligned}
\mathcal{M}_2\left({}^{RL}_{a,b}\mathbf{I}^{\alpha,\beta}_{u,v}f(u,v)\right)=&\frac{\Gamma(1-\xi(r)-\alpha)\,\Gamma(\alpha)}{\Gamma(1-\xi(r))}\cdot\frac{\Gamma(1-\zeta(s)-\beta)\,\Gamma(\beta)}{\Gamma(1-\zeta(s))}\\
&\times\,\mathcal{F}_D(\xi(r)+\alpha,\,\zeta(s)+\beta).
\end{aligned}\tag{62}
$$

Proof. By using the definition of (4.11) we have:

$$
\begin{aligned}
{}^{RL}_{0,0}\mathbf{I}^{\alpha,\beta}_{u,v}f(u,v)=&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^u\!\!\int_0^v(u-t)^{\alpha-1}(v-\tau)^{\beta-1}f(t,\tau)\mathrm{d}t\mathrm{d}\tau\\
=&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^u\!\!\int_0^v u^{\alpha-1}\left(1-\frac{t}{u}\right)^{\alpha-1}v^{\beta-1}\left(1-\frac{\tau}{v}\right)^{\beta-1}f(t,\tau)\mathrm{d}t\mathrm{d}\tau.
\end{aligned}
$$

By taking $\frac{t}{u}=\theta$ and $\frac{\tau}{v}=\vartheta$, we obtain

$$
\begin{aligned}
{}^{RL}_{0,0}\mathbf{I}^{\alpha,\beta}_{u,v}f(u,v)=&\frac{u^\alpha v^\beta}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1\!\!\int_0^1(1-\theta)^{\alpha-1}(1-\vartheta)^{\beta-1}f(u\theta,v\vartheta)\mathrm{d}\theta\mathrm{d}\vartheta\\
=&\frac{u^\alpha v^\beta}{\Gamma(\alpha)\Gamma(\beta)}\int_0^\infty\!\!\int_0^\infty g(\theta,\vartheta)f(u\theta,v\vartheta)\mathrm{d}\theta\mathrm{d}\vartheta.
\end{aligned}
$$

Where the function of $g(u,v)$ defined as follows:

$$g(u,v)=\begin{cases}(1-u)^{\alpha-1}(1-v)^{\beta-1} & 0\le u,v<1,\\ 0 & u,v\ge 1,\end{cases}\tag{63}$$

By taking Mellin transform of $g(u,v)$, we have:

(64)
$$\begin{aligned}\mathcal{M}_2[g(u,v)]=&\mathcal{G}_D(\xi(r),\zeta(s))=\phi(r)\psi(s)\int_0^{\infty}\int_0^{\infty}u^{\xi(r)-1}v^{\zeta(s)-1}g(u,v)\mathrm{d}u\mathrm{d}v\\ =&\phi(r)\psi(s)\int_0^1\int_0^1 u^{\xi(r)-1}v^{\zeta(s)-1}(1-u)^{\alpha-1}(1-v)^{\beta-1}\mathrm{d}u\mathrm{d}v\\ =&\phi(r)\psi(s)\left(\int_0^1 u^{\xi(r)-1}(1-u)^{\alpha-1}\mathrm{d}u\right)\left(\int_0^1 v^{\zeta(s)-1}(1-v)^{\beta-1}\mathrm{d}v\right)\\ =&\phi(r)\psi(s)\beta^{\star}(\xi(r),\alpha)\beta^{\star}(\zeta(s),\beta)\\ =&\phi(r)\psi(s)\frac{\Gamma(\xi(r))\Gamma(\alpha)}{\Gamma(\xi(r)+\alpha)}\frac{\Gamma(\zeta(s))\Gamma(\beta)}{\Gamma(\zeta(s)+\beta)}.\end{aligned}$$

Now using the definition of general double Mellin transform, (59) and (64), we have:

$$\begin{aligned}\mathcal{M}_2\left[{}^{RL}_{0,0}\mathbf{I}^{\alpha,\beta}_{u,v}f(u,v)\right]&=\mathcal{M}_2\left[\frac{u^{\alpha}v^{\beta}}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1\int_0^1 g(\theta,\vartheta)f(u\theta,v\vartheta)\mathrm{d}\theta\mathrm{d}\vartheta\right]\\ &=\frac{\mathcal{F}_D(\xi(r)+\alpha,\zeta(s)+\beta)\mathcal{G}_D(1-\xi(r)-\alpha,1-\zeta(s)-\beta)}{\phi(r)\psi(s)\Gamma(\alpha)\Gamma(\beta)}\\ &=\frac{\Gamma(1-\xi(r)-\alpha)}{\Gamma(1-\xi(r))}\frac{\Gamma(1-\zeta(s)-\beta)}{\Gamma(1-\zeta(s))}\mathcal{F}_D(\xi(r)+\alpha,\zeta(s)+\beta).\end{aligned}\tag{65}$$

□

Definition 4.13. [22] The Riemann-Liouville and Liouville-Caputo partial fractional derivative respectively defined as follow:

(66)
$$\begin{aligned}&{}^{RL}_{a,b}\mathbf{D}^{\alpha,\beta}_{u,v}f(u,v)\\ &\quad=\frac{1}{\Gamma(m-\alpha)\,\Gamma(n-\beta)}\frac{\mathrm{d}^{m+n}}{\mathrm{d}u^m\,\mathrm{d}v^n}\int_a^u\int_b^v(u-t)^{m-\alpha-1}(v-\tau)^{n-\beta-1}f(t,\tau)\,\mathrm{d}t\,\mathrm{d}\tau,\end{aligned}$$

$$\begin{aligned}&{}^{C}_{a,b}\mathbf{D}^{\alpha,\beta}_{u,v}f(u,v)\\ &\quad=\frac{1}{\Gamma(m-\alpha)\,\Gamma(n-\beta)}\int_a^u\int_b^v(u-t)^{m-\alpha-1}(v-\tau)^{n-\beta-1}\\ &\qquad\times\frac{\mathrm{d}^{m+n}f(t,\tau)}{\mathrm{d}t^m\,\mathrm{d}\tau^n}\,\mathrm{d}t\,\mathrm{d}\tau.\end{aligned}$$

Theorem 4.14. *Suppose $\mathcal{M}_2[f(u,v)] = \mathcal{F}_D(\xi(r),\zeta(s))$, then we have the following result:*

$$\mathcal{M}_2\left[{}^{RL}_{0,0}\mathbf{D}^{\alpha,\beta}_{u,v} f(u,v)\right] = \frac{\Gamma(1-\xi(r)+\alpha)\Gamma(1-\zeta(s)+\beta)}{\Gamma(1-\xi(r))\Gamma(1-\zeta(s))}\mathcal{F}_D(\xi(r)-\alpha,\zeta(s)-\beta). \tag{67}$$

Proof. The relationship between integral and derivative Riemann-Liouville is as follows

$$ {}^{RL}_{0,0}\mathbf{D}^{\alpha,\beta}_{u,v} f(u,v) = \frac{\mathrm{d}^{m+n}}{\mathrm{d}u^m \mathrm{d}v^n}\left[{}^{RL}_{0,0}\mathbf{I}^{m-\alpha,n-\beta}_{u,v} f(u,v)\right]. \tag{68}$$

We Suppose

$$g(u,v) := {}^{RL}_{0,0}\mathbf{I}^{m-\alpha,n-\beta}_{u,v} f(u,v). \tag{69}$$

Then, we have

$$ {}^{RL}_{0,0}\mathbf{D}^{\alpha,\beta}_{u,v} f(u,v) = \frac{\mathrm{d}^{m+n} g(u,v)}{\mathrm{d}u^m \mathrm{d}v^n}. \tag{70}$$

Applying the general double Mellin integral transform on both side of equation (70) and using theorem (4.2), we have

$$\begin{aligned}
\mathcal{M}_2\left[{}^{RL}_{0,0}\mathbf{D}^{\alpha,\beta}_{u,v} f(u,v)\right] &= \mathcal{M}_2\left[\frac{\mathrm{d}^{m+n} g(u,v)}{\mathrm{d}u^m\,\mathrm{d}v^n}\right] \\
&= (-1)^{m+n}\frac{\Gamma(\xi(r))\,\Gamma(\zeta(s))}{\Gamma(\xi(r)-m)\,\Gamma(\zeta(s)-n)} \\
&\quad\times\ \mathcal{G}_D(\xi(r)-m,\ \zeta(s)-n) \\
&= \frac{\Gamma(m+1-\xi(r))\,\Gamma(n+1-\zeta(s))}{\Gamma(1-\xi(r))\,\Gamma(1-\zeta(s))} \\
&\quad\times\ \mathcal{G}_D(\xi(r)-m,\ \zeta(s)-n).
\end{aligned} \tag{71}$$

Now, by applying the general double Mellin integral transform to equation (69), we have

$$\begin{aligned}
\mathcal{M}_2[g(u,v)] &= \mathcal{M}_2\left[{}^{RL}_{0,0}\mathbf{I}^{m-\alpha,n-\beta}_{u,v} f(u,v)\right] \\
&= \frac{\Gamma(1-\xi(r)-m+\alpha)}{\Gamma(1-\xi(r))}\cdot\frac{\Gamma(1-\zeta(s)-n+\beta)}{\Gamma(1-\zeta(s))} \\
&\quad\times\ \mathcal{G}_D(\xi(r)+m-\alpha,\ \zeta(s)+n-\beta).
\end{aligned} \tag{72}$$

Taking $\xi(r)-m$ instead of $\xi(r)$ and $\zeta(s)-n$ instead of $\zeta(s)$ in equation (72), and using in equation (71), we have

$$\mathcal{M}_2\left[{}^{RL}_{0,0}\mathbf{D}^{\alpha,\beta}_{u,v} f(u,v)\right] = \frac{\Gamma(1-\xi(r)+\alpha)\Gamma(1-\zeta(s)+\beta)}{\Gamma(1-\xi(r))\Gamma(1-\zeta(s))}\mathcal{F}_D(\xi(r)-\alpha,\zeta(s)-\beta). \tag{73}$$

□

Theorem 4.15. *Suppose $\mathcal{M}_2[f(u,v)] = \mathcal{F}_D(\xi(r),\zeta(s))$, then we have the following result*

(74)
$$\begin{aligned}\mathcal{M}_2\left[{}_{0,0}^{C}\mathbf{D}_{u,v}^{\alpha,\beta} f(u,v)\right] &= \frac{1}{\Gamma(m-\alpha)\,\Gamma(n-\beta)}\int_a^u\!\!\int_b^v (u-t)^{m-\alpha-1}(v-\tau)^{n-\beta-1}\\ &\quad\times\ \frac{\mathrm{d}^{m+n} f(t,\tau)}{\mathrm{d}t^m\,\mathrm{d}\tau^n}\,\mathrm{d}t\,\mathrm{d}\tau\\ &= \frac{\Gamma(1-\xi(r)+\alpha)\,\Gamma(1-\zeta(s)+\beta)}{\Gamma(1-\xi(r))\,\Gamma(1-\zeta(s))}\\ &\quad\times\ \mathcal{F}_D(\xi(r)-\alpha,\,\zeta(s)-\beta).\end{aligned}$$

Proof. Suppose that $g(u,v) := \frac{\mathrm{d}^{m+n} f(u,v)}{\mathrm{d}u^m \mathrm{d}v^n}$ then, we have

(75)
$$\begin{aligned}\mathcal{M}_2\left[{}_{0,0}^{C}\mathbf{D}_{u,v}^{\alpha,\beta} f(u,v)\right] &= \frac{1}{\Gamma(m-\alpha)\,\Gamma(n-\beta)}\int_a^u\!\!\int_b^v (u-t)^{m-\alpha-1}(v-\tau)^{n-\beta-1}\\ &\quad\times\ g(u,v)\,\mathrm{d}t\,\mathrm{d}\tau\\ &= \mathcal{M}_2\left[{}_{0,0}^{RL}\mathbf{I}_{u,v}^{m-\alpha,n-\beta} g(u,v)\right]\\ &= \frac{\Gamma(1-\xi(r)-m+\alpha)}{\Gamma(1-\xi(r))}\cdot\frac{\Gamma(1-\zeta(s)-n+\beta)}{\Gamma(1-\zeta(s))}\\ &\quad\times\ \mathcal{G}_D(\xi(r)+m-\alpha,\,\zeta(s)+n-\beta).\end{aligned}$$

Now, we imposing the general double Mellin transform to both side of equation $g(u,v) := \frac{\mathrm{d}^{m+n} f(u,v)}{\mathrm{d}u^m \mathrm{d}v^n}$, we have

$$\begin{aligned}\mathcal{M}_2\left[g(u,v)\right] =&\mathcal{G}_D(\xi(r),\zeta(s)) = \mathcal{M}_2\left[\frac{\mathrm{d}^{m+n} f(u,v)}{\mathrm{d}u^m \mathrm{d}v^n}\right]\\ =&(-1)^{m+n}\frac{\Gamma(\xi(r))\Gamma(\zeta(s))}{\Gamma(\xi(r)-m)\Gamma(\zeta(s)-n)}\mathcal{G}_D(\xi(r)-m,\zeta(s)-n)\\ =&\frac{\Gamma(m+1-\xi(r))\Gamma(n+1-\zeta(s))}{\Gamma(1-\xi(r))\Gamma(1-\zeta(s))}\mathcal{G}_D(\xi(r)-m,\zeta(s)-n).\end{aligned} \tag{76}$$

and taking $\xi(r)+m-\alpha$ instead of $\xi(r)$ and $\zeta(s)+n-\beta$ instead of $\zeta(s)$ in equation (76), and using in equation (75), we have

(77)
$$\mathcal{M}_2\left[{}_{0,0}^{C}\mathbf{D}_{u,v}^{\alpha,\beta} f(u,v)\right] =\frac{\Gamma(1-\xi(r)+\alpha)\Gamma(1-\zeta(s)+\beta)}{\Gamma(1-\xi(r))\Gamma(1-\zeta(s))}\mathcal{F}_D(\xi(r)-\alpha,\zeta(s)-\beta).$$

□

Corollary 4.16. *Suppose $\mathcal{M}_2[f(u,v)] = \mathcal{F}_D(\xi(r),\zeta(s))$, then we have the following result*

(78)

$$\mathcal{M}_2\left[u^{\alpha}v^{\beta}\left({}_{0,0}^{\;\;C}\mathbf{D}_{u,v}^{\alpha,\beta}f(u,v)\right)\right] = \frac{\Gamma(1-\xi(r))\Gamma(1-\zeta(s))}{\Gamma(1-\xi(r)-\alpha)\Gamma(1-\zeta(s)-\beta)}\mathcal{F}_D(\xi(r),\zeta(s)).$$

5. Mellin Transform of Some Elementary Functions

In this section, we obtain the general double Mellin transform of some elementary functions

(1) Let $f(u,v) = e^{-au-bv}$, we have:

(79)

$$\begin{aligned}\mathcal{M}_2\left[e^{-au-bv}\right] &= \phi(r)\,\psi(s)\int_0^{\infty}\!\!\int_0^{\infty} u^{\xi(r)-1}\,v^{\zeta(s)-1}\,e^{-au-bv}\,\mathrm{d}u\,\mathrm{d}v\\ &= \frac{\phi(r)\,\psi(s)\,\Gamma(\xi(r))\,\Gamma(\zeta(s))}{a^{\xi(r)}\,b^{\zeta(s)}}.\end{aligned}$$

(2) Let $f(u,v) = \frac{c}{(e^{au}-1)(e^{bv}-1)}$, $a,b,c\in\mathbb{R}$, we have:

$$\begin{aligned}&\mathcal{M}_2\left[\frac{c}{(e^{au}-1)(e^{bv}-1)}\right]\\ &= \phi(r)\,\psi(s)\int_0^{\infty}\!\!\int_0^{\infty} u^{\xi(r)-1}\,v^{\zeta(s)-1}\frac{c}{(e^{au}-1)(e^{bv}-1)}\,\mathrm{d}u\,\mathrm{d}v\\ &= c\,\phi(r)\,\psi(s)\int_0^{\infty}\!\!\int_0^{\infty} u^{\xi(r)-1}\,v^{\zeta(s)-1}\\ &\quad\times\sum_{n=1}^{\infty}e^{-nau}\sum_{m=1}^{\infty}e^{-mbv}\,\mathrm{d}u\,\mathrm{d}v\\ &= \frac{c\,\phi(r)\,\psi(s)}{a^{\xi(r)}\,b^{\zeta(s)}}\sum_{n=1}^{\infty}\frac{\Gamma(\xi(r))}{n^{\xi(r)}}\sum_{m=1}^{\infty}\frac{\Gamma(\zeta(s))}{m^{\zeta(s)}}.\end{aligned}$$

(3) Let $f(u,v) = \frac{c}{(e^{au}+1)(e^{bv}+1)}$, $a,b,c\in\mathbb{R}$, we have:

$$\begin{aligned}&\mathcal{M}_2\left[\frac{c}{(e^{au}+1)(e^{bv}+1)}\right]\\ &= \phi(r)\,\psi(s)\int_0^{\infty}\!\!\int_0^{\infty} u^{\xi(r)-1}\,v^{\zeta(s)-1}\frac{c}{(e^{au}+1)(e^{bv}+1)}\,\mathrm{d}u\,\mathrm{d}v\\ &= c\,\phi(r)\,\psi(s)\int_0^{\infty}\!\!\int_0^{\infty} u^{\xi(r)-1}\,v^{\zeta(s)-1}\\ &\quad\times\sum_{n=1}^{\infty}(-1)^{n+1}e^{-nau}\sum_{m=1}^{\infty}(-1)^{m+1}e^{-mbv}\,\mathrm{d}u\,\mathrm{d}v\\ &= \frac{c\,\phi(r)\,\psi(s)}{a^{\xi(r)}\,b^{\zeta(s)}}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\Gamma(\xi(r))}{n^{\xi(r)}}\sum_{m=1}^{\infty}(-1)^{m+1}\frac{\Gamma(\zeta(s))}{m^{\zeta(s)}}.\end{aligned}$$

(4) Let $f(u,v)=\frac{1}{(1+u)^n(1+v)^m}$ then, we have:

$$\mathcal{M}_2\left[\frac{1}{(1+u)^n(1+v)^m}\right]=\phi(r)\psi(s)\int_0^{\infty}\!\!\int_0^{\infty}u^{\xi(r)-1}v^{\zeta(s)-1}\frac{1}{(1+u)^n(1+v)^m}\mathrm{d}u\mathrm{d}v$$
$$=\phi(r)\psi(s)\int_0^{\infty}u^{\xi(r)-1}\frac{1}{(u+1)^n}\mathrm{d}u\int_0^{\infty}v^{\zeta(s)-1}\frac{1}{(v+1)^m}\mathrm{d}v$$

By taking $u=\frac{t}{1-t}$ and $v=\frac{p}{1-p}$, we get:

$$\begin{aligned}\mathcal{M}_2\left[\frac{1}{(1+u)^n(1+v)^m}\right]&=\phi(r)\,\psi(s)\int_0^1 t^{\xi(r)-1}(1-t)^{n-\xi(r)-1}\,\mathrm{d}t\\&\quad\times\int_0^1 p^{\zeta(s)-1}(1-p)^{m-\zeta(s)-1}\,\mathrm{d}p\\&=\phi(r)\,\psi(s)\,\beta(\xi(r),\,n-\xi(r))\,\beta(\zeta(s),\,m-\zeta(s)).\end{aligned}$$

(5) Let $f(u,v)=e^{-i(au+bv)}$ then, according to (79) we have:

$$\begin{aligned}&\mathcal{M}_2\left[e^{-i(au+bv)}\right]=\mathcal{M}_2\left[\cos(au+bv)-i\sin(au+bv)\right]\\&=\phi(r)\psi(s)\int_0^{\infty}\!\!\int_0^{\infty}u^{\xi(r)-1}v^{\zeta(s)-1}e^{-i(au+bv)}\mathrm{d}u\mathrm{d}v\\&=\frac{\phi(r)\psi(s)\Gamma(\xi(r))\Gamma(\zeta(s))}{(ia)^{\xi(r)}(ib)^{\zeta(s)}}=\frac{\phi(r)\psi(s)\Gamma(\xi(r))\Gamma(\zeta(s))i^{-\xi(r)}i^{-\zeta(s)}}{(a)^{\xi(r)}(b)^{\zeta(s)}}\\&=\frac{\phi(r)\psi(s)\Gamma(\xi(r))\Gamma(\zeta(s))}{(a)^{\xi(r)}(b)^{\zeta(s)}}\\&\quad\left[\left(\cos\left(\xi(r)\frac{\pi}{2}\right)-i\sin\left(\xi(r)\frac{\pi}{2}\right)\right)\left(\cos\left(\zeta(s)\frac{\pi}{2}\right)-i\sin\left(\zeta(s)\frac{\pi}{2}\right)\right)\right].\end{aligned}$$

Finally we have the following result:

$$\begin{aligned}\mathcal{M}_2\left[\cos(au+bv)\right]=&\frac{\phi(r)\psi(s)\Gamma(\xi(r))\Gamma(\zeta(s))}{(a)^{\xi(r)}(b)^{\zeta(s)}}\\&\left[\cos\left(\xi(r)\frac{\pi}{2}\right)\cos\left(\zeta(s)\frac{\pi}{2}\right)+\sin\left(\xi(r)\frac{\pi}{2}\right)\sin\left(\zeta(s)\frac{\pi}{2}\right)\right].\\\mathcal{M}_2\left[\sin(au+bv)\right]=&\frac{\phi(r)\psi(s)\Gamma(\xi(r))\Gamma(\zeta(s))}{(a)^{\xi(r)}(b)^{\zeta(s)}}\\&\left[(\sin\left(\zeta(s)\frac{\pi}{2}\right)\cos\left(\xi(r)\frac{\pi}{2}\right)+\cos\left(\zeta(s)\frac{\pi}{2}\right)\sin\left(\xi(r)\frac{\pi}{2}\right)\right].\end{aligned}$$

5.1. **Solutions of Fractional Differential Equations by Double Mellin transform.** In this section we consider

$$\mathcal{D}^{\alpha,\beta}_{u,v}f(u,v)+\mathcal{L}[f(u,v)]=g(u,v),\tag{80}$$

where $\mathcal{D}^{\alpha,\beta}_{u,v}$ denotes a fractional differential operator (in the sense of Riemann–Liouville or Caputo), and $\mathcal{L}$ is a linear operator compatible with the transform.

To solve the fractional differential equation (80), we employ the double generalized Mellin transform with respect to the variables u and v. Let $\mathcal{M}_2\{\cdot\}$ denote the double Mellin transform defined on the space X_{c_1,c_2}. Applying this transform to both sides of the equation yields

$$\mathcal{M}_2\left\{\mathcal{D}^{\alpha,\beta}_{u,v} f(u,v)\right\} + \mathcal{M}_2\left\{\mathcal{L}[f(u,v)]\right\} = \mathcal{M}_2\left\{g(u,v)\right\}.$$

Under suitable regularity conditions and transform compatibility, the fractional differential operator $\mathcal{D}^{\alpha,\beta}_{u,v}$ is converted into an algebraic expression involving the transform variables, typically in the form of multiplicative factors depending on s and t. Similarly, the linear operator $\mathcal{L}$ is mapped into an algebraic operator in the transform domain. Consequently, the original fractional differential equation is reduced to an algebraic equation in the Mellin space:

$$\Phi(s,t)\,F(s,t) = G(s,t),$$

where $F(s,t) = \mathcal{M}_2\{f(u,v)\}$, $G(s,t) = \mathcal{M}_2\{g(u,v)\}$, and $\Phi(s,t)$ represents the combined symbol of the transformed operators. Solving for $F(s,t)$ gives

$$F(s,t) = \frac{G(s,t)}{\Phi(s,t)}.$$

Finally, the solution $f(u,v)$ is obtained by applying the inverse double Mellin transform:

$$f(u,v) = \mathcal{M}_2^{-1}\left\{\frac{G(s,t)}{\Phi(s,t)}\right\},$$

provided that the inverse transform exists and the resulting expression satisfies the required convergence conditions.

In the following, we clarify the conditions under which the formal solutions obtained via the generalized double Mellin transform correspond to classical or weak solutions of the considered fractional differential equations.

Proposition 5.1 (Existence of Transform Solution). *Let $f(u,v) \in X_{c_1,c_2}$ and assume that the fractional differential equation* (80)*. If $g(u,v) \in X_{c_1,c_2}$ and the transformed equation admits a unique solution $F_D(\xi(r),\zeta(s))$ in the transform domain, then there exists a function $f(u,v)$ given by the inverse transform* (10).

Proof. Applying the generalized double Mellin transform to both sides reduces the problem to an algebraic equation in the transform domain. Under the stated assumptions, this equation admits a unique solution F_D. The result follows from the inversion formula (10), provided the admissibility conditions in the previous subsection are satisfied. □

Proposition 5.2 (Classical Solution). *Assume that $f(u,v) \in C^{m,n}(\mathbb{R}^2_+)$ and that its partial derivatives up to order m,n belong to X_{c_1,c_2}.*

If the inverse transform exists and the boundary terms vanish, i.e.,

$$u^{\xi(r)} f(u,v) \to 0, \quad v^{\zeta(s)} f(u,v) \to 0 \quad \text{as } u,v \to 0 \text{ and } u,v \to \infty, \tag{81}$$

then the inverse transform yields a classical solution of the fractional differential equation.

Proof. The result follows by differentiating under the integral sign and using the operational properties of the transform (Theorems 4.2 and 4.3). The vanishing boundary conditions ensure the validity of integration by parts. □

Proposition 5.3 (Weak Solution)**.** *Let $f \in X_{c_1,c_2}$ and assume that the transformed solution $F_D(\xi(r), \zeta(s))$ exists and is locally integrable. Then f defines a weak solution of the fractional differential equation in the sense that*

$$\int_0^\infty \int_0^\infty f(u,v)\,\phi(u,v)\,dudv \tag{82}$$

satisfies the weak formulation for all test functions $\phi \in C_c^\infty(\mathbb{R}_+^2)$.

Proof. The proof follows from standard arguments in distribution theory, using the fact that the transform maps convolution-type operations into algebraic forms and preserves integrability under the stated conditions. □

Proposition 5.4 (Compatibility with Boundary and Initial Conditions)**.** *Let the fractional differential equation be supplemented with boundary or initial conditions of the form*

$$f(u,0) = f_0(u), \quad f(0,v) = g_0(v), \tag{83}$$

or their fractional analogues.

If f_0 and g_0 belong to appropriate weighted spaces and their transforms exist, then the solution obtained via the generalized double Mellin transform satisfies these conditions provided the inverse transform converges pointwise.

Proof. The result follows by applying the inverse transform and taking the corresponding limits, using dominated convergence and the admissibility conditions on f. □

5.2. **Admissible Singularities and Stability of the Transform.** In practical applications, particularly in fractional differential equations, the functions under consideration may exhibit algebraic or logarithmic singularities at $u = 0$, $v = 0$, or at infinity. In this subsection, we clarify the admissibility of such singular behaviors and their impact on the generalized double Mellin transform and its inversion.

Algebraic singularities. Let $f(u,v)$ satisfy

$$f(u,v) \sim u^{-\alpha} v^{-\beta}, \quad \text{as } u \to 0 \text{ or } v \to 0, \tag{84}$$

for some $\alpha, \beta \in \mathbb{R}$. Then $f \in X_{c_1,c_2}$ provided that

$$\alpha < c_1, \qquad \beta < c_2. \tag{85}$$

Under these conditions, the generalized double Mellin transform remains absolutely convergent.

Logarithmic singularities. If $f(u,v)$ has logarithmic behavior of the form

$$f(u,v) \sim u^{-\alpha} v^{-\beta} (\log u)^k (\log v)^m, \tag{86}$$

for integers $k, m \geq 0$, then the transform exists under the same conditions $\alpha < c_1$, $\beta < c_2$. In this case, logarithmic factors lead to higher-order poles in the transform domain.

Propagation to the transform domain. Algebraic singularities in the physical domain correspond to pole-type singularities of $F_D(\xi(r), \zeta(s))$ with respect to $\xi(r)$ and $\zeta(s)$. Logarithmic singularities generate multiple poles or derivatives of simple poles. These features are consistent with the classical Mellin transform theory.

Impact on inversion and stability. The inversion formula (10) remains valid provided that the contour of integration lies within a strip where $F_D(\xi(r), \zeta(s))$ is analytic. The presence of poles requires appropriate contour selection to avoid singularities.

Moreover, the stability of the inversion process depends on the decay of $F_D(\xi(r), \zeta(s))$ as $|\Im(r)|, |\Im(s)| \to \infty$. Functions with admissible singularities as described above typically yield transforms with sufficient decay to ensure convergence of the inverse integral.

Remark. The above conditions show that weak algebraic and logarithmic singularities are admissible within the proposed framework and do not affect the validity of the transform or its inversion, provided the parameters are chosen within appropriate convergence strips.

6. Examples

In this section we want to use the general double Mellin transform on solving some equation, the Mellin transforms of the functions are obtained using the table 1.

Example 6.1. *Consider the Euler's differential equation:*

$$x^2 \frac{\partial^2 u(x,y)}{\partial x^2} + Px \frac{\partial u(x,y)}{\partial x} + Qu(x,y) = v(x,y). \tag{87}$$

Applying generalized double Meline transform and useing theorem (4.3) *we have:*

$$\left(\xi^2(r) + (1-P)\xi(r) + Q\right) \mathcal{U}_D(\xi(r), \zeta(s)) = \mathcal{V}_D(\xi(r), \zeta(s)). \tag{88}$$

Consequently , the particular solution of (87) *we get:*

$$u(x,y) = \mathcal{M}_2^{-1} \left\{ \frac{\mathcal{V}_D(\xi(r), \zeta(s))}{\xi^2(r) + (1-P)\xi(r) + Q} \right\}. \tag{89}$$

Example 6.2. *We consider the following equation*

$$u^2 \Phi_{uu} + u\Phi_u + \Phi_{vv} = 0, \qquad 0 \leq u < \infty, \quad 0 < v < 1, \tag{90}$$

With the following initial conditions

$$\Phi(u,0)=0, \Phi(u,1)=\begin{cases}\lambda, & 0\leq u\leq 1,\\ 0, & u>1,\end{cases} \tag{91}$$

where λ *is constant.*

By applying the generalized double Mellin transform we have:

$$\mathcal{M}_2\left[u^2\Phi_{uu}\right]+\mathcal{M}_2\left[u\Phi_u\right]+\mathcal{M}_2\left[\Phi_{vv}\right]=0,$$

Considering (4.3) we have,

$$\xi^2(r)\Psi_D(\xi(r),\zeta(s),v)+\frac{\partial^2\Psi_D(\xi(r),\zeta(s),v)}{\partial v^2}=0. \tag{92}$$

Applying the generalized double Mellin transform to the initial conditions we have

$$\Psi_D(\xi(r),0)=0, \qquad \Psi_D(\xi(r),1)=\frac{\lambda\phi(r)\psi(s)}{\xi(r)\zeta(s)}. \tag{93}$$

After solution of the equation (92) and using initial conditions (93), we have

$$\Psi_D(\xi(r),\zeta(s),v)=\frac{\lambda\phi(r)\psi(s)}{\xi(r)\zeta(s)\sin(\xi(r))}\sin(\xi(r)v). \tag{94}$$

By applying the inverse generalized general double Mellin transform (10), we have

$$\begin{aligned}\Phi(u.v)=&\frac{\lambda}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{u^{-\xi(r)}\xi'(r)\sin(\xi(r)v)}{\xi(r)\sin(\xi(r))}\mathrm{d}r\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{v^{-\zeta(s)}\zeta'(s)}{\zeta(s)}\mathrm{d}s\\ =&2\pi i\sum_{n=1}^{\infty}\frac{\lambda}{2\pi i}\frac{x^{-n\pi}A(n)}{n\pi}\frac{\sin(\xi(r)v)}{\sin(n\pi)}\sin(n\pi)\\ =&\frac{\lambda}{\pi}\sum_{n=1}^{\infty}\frac{A(n)}{n}(-1)^n x^{-n\pi}\sin(n\pi v).\end{aligned} \tag{95}$$

where:

$$A(n):=\lim_{\xi(r)\to n\pi}\xi'(r).$$

Corollary 6.3. *If we take* $A(n)=1$ *in equation* (95)*, we have.*

$$\Phi(u.v)=\frac{\lambda}{\pi}\sum_{n=1}^{\infty}(-1)^n\frac{x^{-n\pi}}{n}\sin(n\pi v). \tag{96}$$

Example 6.4. *Consider the following boundary value problem of fractional differential equation, for* $0<\alpha,\beta\leq 1$.

$$u^{\alpha+1}v^{\beta+1}\,{}_{0,0}^{\;\;C}\mathbf{D}_{u,v}^{\alpha+1,\beta+1}f(u,v)+u^{\alpha}v^{\beta}\,{}_{0,0}^{\;\;C}\mathbf{D}_{u,v}^{\alpha,\beta}f(u,v)=f(u,v). \tag{97}$$

Applying general double Mellin transform and using (78) in equation (97), we have

$$\begin{aligned}
&\left(\frac{\Gamma(1-\xi(r))\,\Gamma(1-\zeta(s))}{\Gamma(1-\xi(r)-\alpha-1)\,\Gamma(1-\zeta(s)-\beta-1)}\right.\\
&\quad\left.+\frac{\Gamma(1-\xi(r))\,\Gamma(1-\zeta(s))}{\Gamma(1-\xi(r)-\alpha)\,\Gamma(1-\zeta(s)-\beta)}\right)\mathcal{F}_D(\xi(r),\,\zeta(s))\\
&=\mathcal{G}_2\left[(\xi(r),\,\zeta(s))\right].
\end{aligned} \tag{98}$$

Note that $\Gamma(1-\xi(r)-\alpha)=(1-\xi(r)-\alpha-1)\Gamma(1-\xi(r)-\alpha-1), \Gamma(1-\zeta(s)-\beta)=(1-\zeta(s)-\beta-1)\Gamma(1-\zeta(s)-\beta-1)$, Therefore we have

$$\begin{aligned}
\mathcal{F}_D(\xi(r),\,\zeta(s)) &= \mathcal{G}_2\left[(\xi(r),\,\zeta(s))\right]\\
&\quad\times\left[\frac{\Gamma(1-\xi(r)-\alpha)\,\Gamma(1-\zeta(s)-\beta)}{\Gamma(1-\xi(r))\,\Gamma(1-\zeta(s))\,(1-\xi(r)-\alpha)(1-\zeta(s)-\beta)}\right]\\
&=\mathcal{G}_2\left[(\xi(r),\,\zeta(s))\right]\,\mathcal{H}_2\left[(1-\xi(r),\,1-\zeta(s))\right]\\
&=\phi(r)\,\psi(s)\,\frac{\mathcal{G}_2\left[(\xi(r),\,\zeta(s))\right]\,\mathcal{H}_2\left[(1-\xi(r),\,1-\zeta(s))\right]}{\phi(r)\,\psi(s)}.
\end{aligned} \tag{99}$$

By applying inverse Mellin transform, we get the solution of problem (97), as following

$$f(u,v)=\phi(r)\psi(s)\int_0^\infty\!\!\int_0^\infty g(u\vartheta,v\varrho)h\left(\vartheta,\varrho\right)\mathrm{d}\vartheta\mathrm{d}\varrho. \tag{100}$$

TABLE 1. New general double Mellin integral transform.

$f(x)$	$\mathcal{M}_2[f(u,v)] = \mathcal{F}_D(\xi(r),\zeta(s))$
$\exp(-au-bv)$	$\dfrac{\phi(r)\,\psi(s)\,\Gamma(\xi(r))\,\Gamma(\zeta(s))}{a^{\xi(r)}\,b^{\zeta(s)}}$
$\dfrac{c}{(e^{au}-1)(e^{bv}-1)}$	$\dfrac{c\,\phi(r)\,\psi(s)}{a^{\xi(r)}\,b^{\zeta(s)}}\sum_{n=1}^{\infty}\dfrac{\Gamma(\xi(r))}{n^{\xi(r)}}\sum_{m=1}^{\infty}\dfrac{\Gamma(\zeta(s))}{m^{\zeta(s)}}$
$\dfrac{c}{(e^{au}+1)(e^{bv}+1)}$	$\dfrac{c\,\phi(r)\,\psi(s)}{a^{\xi(r)}\,b^{\zeta(s)}}\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{\Gamma(\xi(r))}{n^{\xi(r)}}\sum_{m=1}^{\infty}(-1)^{m+1}\dfrac{\Gamma(\zeta(s))}{m^{\zeta(s)}}$
$\dfrac{1}{(1+u)^n(1+v)^m}$	$\phi(r)\,\psi(s)\,\beta(\xi(r),n-\xi(r))\,\beta(\zeta(s),m-\zeta(s))$
$\cos(au+bv)$	$\dfrac{\phi(r)\,\psi(s)\,\Gamma(\xi(r))\,\Gamma(\zeta(s))}{a^{\xi(r)}\,b^{\zeta(s)}}\Big[\cos\big(\xi(r)\frac{\pi}{2}\big)\cos\big(\zeta(s)\frac{\pi}{2}\big) + \sin\big(\xi(r)\frac{\pi}{2}\big)\sin\big(\zeta(s)\frac{\pi}{2}\big)\Big]$
$\sin(au+bv)$	$\dfrac{\phi(r)\,\psi(s)\,\Gamma(\xi(r))\,\Gamma(\zeta(s))}{a^{\xi(r)}\,b^{\zeta(s)}}\Big[\sin\big(\zeta(s)\frac{\pi}{2}\big)\cos\big(\xi(r)\frac{\pi}{2}\big) + \cos\big(\zeta(s)\frac{\pi}{2}\big)\sin\big(\xi(r)\frac{\pi}{2}\big)\Big]$
$f(au,bv)$	$\dfrac{\mathcal{F}_D(\xi(r),\zeta(s))}{a^{\xi(r)}\,b^{\zeta(s)}}$
$u^a v^b f(u,v)$	$\mathcal{F}_D(\xi(r)+a,\,\zeta(s)+b)$
$f(u^a,v^b)$	$\dfrac{1}{ab}\,\mathcal{F}_D\left(\dfrac{\xi(r)}{a},\dfrac{\zeta(s)}{b}\right)$
$\dfrac{\partial^n f(u,v)}{\partial u^n}$	$(-1)^n\,\dfrac{\Gamma(\xi(r))}{\Gamma(\xi(r)-n)}\,\mathcal{F}_D(\xi(r)-n,\,\zeta(s))$
$\dfrac{\partial^{n+m} f(u,v)}{\partial u^n\partial v^m}$	$(-1)^{n+m}\,\dfrac{\Gamma(\xi(r))\,\Gamma(\zeta(s))}{\Gamma(\xi(r)-n)\,\Gamma(\zeta(s)-m)}\,\mathcal{F}_D(\xi(r)-n,\,\zeta(s)-m)$
$u^n\,\dfrac{\partial^n f(u,v)}{\partial u^n}$	$(-1)^n\,\dfrac{\Gamma(\xi(r)+n)}{\Gamma(\xi(r))}\,\mathcal{F}_D(\xi(r),\,\zeta(s))$
$u^n v^m\,\dfrac{\partial^{n+m} f(u,v)}{\partial u^n\partial v^m}$	$(-1)^{n+m}\,\dfrac{\Gamma(\xi(r)+n)\,\Gamma(\zeta(s)+m)}{\Gamma(\xi(r))\,\Gamma(\zeta(s))}\,\mathcal{F}_D(\xi(r),\,\zeta(s))$
$(f\ast\ast g)(u,v)$	$\dfrac{\mathcal{F}_D(\xi(r),\zeta(s))\,\mathcal{G}_D(\xi(r),\zeta(s))}{\phi(r)\,\psi(s)}$
$(f\circ\circ g)(u,v)$	$\dfrac{\mathcal{F}_D(\xi(r),\zeta(s))\,\mathcal{G}_D(1-\xi(r),1-\zeta(s))}{\phi(r)\,\psi(s)}$
${}^{RL}_{a,b}\mathbf{I}^{\alpha,\beta}_{u,v}f(u,v)$	$\dfrac{\Gamma(1-\xi(r)-\alpha)\,\Gamma(\alpha)}{\Gamma(1-\xi(r))}\cdot\dfrac{\Gamma(1-\zeta(s)-\beta)\,\Gamma(\beta)}{\Gamma(1-\zeta(s))}\,\mathcal{F}_D(\xi(r)+\alpha,\,\zeta(s)+\beta)$
${}^{RL}_{0,0}\mathbf{D}^{\alpha,\beta}_{u,v}f(u,v)$	$\dfrac{\Gamma(1-\xi(r)+\alpha)\,\Gamma(1-\zeta(s)+\beta)}{\Gamma(1-\xi(r))\,\Gamma(1-\zeta(s))}\,\mathcal{F}_D(\xi(r)-\alpha,\,\zeta(s)-\beta)$
${}^{C}_{0,0}\mathbf{D}^{\alpha,\beta}_{u,v}f(u,v)$	$\dfrac{\Gamma(1-\xi(r)+\alpha)\,\Gamma(1-\zeta(s)+\beta)}{\Gamma(1-\xi(r))\,\Gamma(1-\zeta(s))}\,\mathcal{F}_D(\xi(r)-\alpha,\,\zeta(s)-\beta)$
$u^\alpha v^\beta\left({}^{C}_{0,0}\mathbf{D}^{\alpha,\beta}_{u,v}f(u,v)\right)$	$\dfrac{\Gamma(1-\xi(r))\,\Gamma(1-\zeta(s))}{\Gamma(1-\xi(r)-\alpha)\,\Gamma(1-\zeta(s)-\beta)}\,\mathcal{F}_D(\xi(r),\,\zeta(s))$

7. Conclusion

In this study, we developed a novel general double Mellin integral transform and established its theoretical framework through rigorous definitions, structural properties, and operational rules. The proposed transform extends previously reported double Mellin formulations and unifies several existing variants within a broader and more flexible analytical structure. By systematically deriving its linearity, scaling behavior, differentiation properties, and convolution-type relations, we demonstrated that the new formulation preserves the essential analytical strengths of classical Mellin-type operators while significantly enlarging their applicability to multi-variable problems.

A central contribution of this work lies in clarifying the relationship between the proposed double transform and earlier generalized single Mellin transforms. This connection ensures theoretical consistency while revealing how the new operator naturally generalizes one-dimensional Mellin formulations to higher-dimensional settings. Compared with existing generalized Mellin transforms, the present approach offers enhanced adaptability for systems involving multiple independent variables, particularly in partial and fractional differential equations where traditional analytical techniques may become restrictive or cumbersome.

The practical effectiveness of the transform was illustrated through the derivation of Mellin transforms for various elementary functions and through representative applications to partial and fractional differential equations. These applications demonstrate how complex differential models can be systematically reduced to more tractable algebraic forms in the transform domain. Furthermore, Table 1 provides a comprehensive summary of the newly derived general double Mellin transforms for functions frequently encountered in mathematics, physics, and engineering. This table serves not only as a validation of the theoretical development but also as a practical reference tool that enhances the usability and applicability of the proposed method across multiple disciplines.

From a broader perspective, the proposed general double Mellin integral transform contributes meaningfully to the advancement of multidimensional integral transform techniques. As fractional calculus and multi-variable modeling continue to play an increasingly important role in applied mathematics, engineering analysis, plasma physics, and related scientific fields, the demand for flexible and structurally robust transform methods becomes more pronounced. The framework developed in this paper provides a solid analytical foundation for addressing such challenges.

Several promising directions for future research emerge from this work. First, establishing inversion theorems under more general conditions and extending the transform to broader functional spaces would strengthen its theoretical foundations. Second, investigating the applicability of the proposed transform to nonlinear fractional systems and developing higher-dimensional extensions such as triple and multi-fold Mellin-type transforms would expand its

scope. Third, extending the double Mellin transform to fractional stochastic PDEs would address real-world systems influenced by noise, while incorporating quantitative validation through error bounds and benchmark comparisons would strengthen practical interpretability. Fourth, extending the transform to control and optimization problems would broaden its applicability to decision-making systems. Additionally, integrating the proposed transform with numerical and hybrid analytical–computational techniques may further enhance its effectiveness in modeling complex real-world phenomena.
Overall, the general double Mellin integral transform introduced in this work represents a substantial contribution to the theory of integral transforms, offering both theoretical enrichment and practical analytical capabilities for contemporary research in differential equations and applied mathematical sciences

8. Author Contributions

Conceptualization, Z.W and M.A.Z.; methodology, Z.W, M.A.Z. and M.H.A.; software, Z.W and M.A.Z.; validation, M.H.A.; formal analysis, M.H.A.; writing—original draft preparation, Z.W and M.A.Z.; writing—review and editing, Z.W, M.A.Z. and M.H.A.; supervision,M.H.A. All authors have read and agreed to the published version of the manuscript.

9. Data Availability Statement

Not applicable.

10. Aknowledgement

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11. Ethical considerations

This article has no associated data.

12. Funding, Conflict of interest

The authors have no financial or proprietary interests in any material discussed in this article. The authors declare no conflict of interest.

References

[1] Akrami, M. H., Poya, A., & Zirak, M. A. (2024). New general single, double and triple conformable integral transforms: Definitions, properties and applications. Partial Differential Equations in Applied Mathematics, 12, 100991. https://doi.org/10.1016/j.padiff.2024.100991

[2] Andrews, G. E., Askey, R., Roy, R., Roy, R., & Askey, R. (1999). Special functions (Vol. 71, pp. xvi+-664). Cambridge: Cambridge university press. https://doi.org/10.1016/j.padiff.2024.100991

[3] Aziz, T. (2022). Generalized Mellin transform and its applications in fractional calculus. Computational & Applied Mathematics, 41(3), 1. https://doi.org/10.1007/s40314-022-01802-9

[4] Belmor, S., Ravichandran, C., & Jarad, F. (2020). Nonlinear generalized fractional differential equations with generalized fractional integral conditions. Journal of Taibah University for Science, 14(1), 114-123. https://doi.org/10.1080/16583655.2019.1709265

[5] Butzer, P. L., & Jansche, S. (1997). A direct approach to the Mellin transform. Journal of Fourier analysis and applications, 3(4), 325-376. https://doi.org/10.1007/BF02649101

[6] Debnath, L., & Bhatta, D. (2016). Integral transforms and their applications. Chapman and Hall/CRC. https://doi.org/10.1201/9781420010916

[7] Elzaki, T. M. (2011). The new integral transform Elzaki transform. Global Journal of pure and applied mathematics, 7(1), 57-64.

[8] Erden, S., & Uyanik, N. (2025). Fractional inequalities involving double integrals of Riemann-Liouville for higher-order partial differential functions. Thermal Science, 29(4 Part B), 3013-3022. https://doi.org/10.2298/tsci2504013e

[9] Erdoğan, E., Kocabaş, S., & Dernek, N. (2019). Some results on the generalized Mellin transforms and applications. Konuralp Journal of Mathematics, 7(1), 175-181. https://izlik.org/JA73AR45LW

[10] Fitouhi, A., Bettaibi, N., & Brahim, K. (2006). The Mellin transform in quantum calculus. Constructive Approximation, 23(3), 305-323. https://doi.org/10.1007/s00365-005-0597-6

[11] Goufo, E. F. D., Ravichandran, C., & Birajdar, G. A. (2021). Self-similarity techniques for chaotic attractors with many scrolls using step series switching. Mathematical Modelling and Analysis, 26(4), 591-611. https://doi.org/10.3846/mma.2021.13126

[12] Jafari, H. (2021). A new general integral transform for solving integral equations. Journal of Advanced Research, 32, 133-138. https://doi.org/10.1016/j.jare.2020.08.016

[13] Jia, H., Nie, Y., & Zhao, Y. (2025). General Conformable Fractional Double Laplace-Sumudu Transform and its Application. J. Appl. Anal. Comput, 15, 9-20. https://doi.org/10.11948/20220344

[14] Jumarie, G. (2008). Fourier's transform of fractional order via Mittag-leffler function and modified Riemann-Liouville derivative. Journal of applied mathematics and informatics, 26(5-6), 1101-1121. https://doi.org/10.1016/j.aml.2009.05.011

[15] Khan Zafar, H., & Khan Waqar, A. (2008). N-Transform-Properties and Applications, NUST J. Engg. Sci, 1(1), 127-133. https://doi.org/10.24949/njes.v1i1.69

[16] Kilbas, A. A., Luchko, Y. F., Martinez, H., & Trujillo, J. J. (2010). Fractional Fourier transform in the framework of fractional calculus operators. Integral Transforms and Special Functions, 21(10), 779-795. https://doi.org/10.1080/10652461003676099

[17] Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). Theory and applications of fractional differential equations (Vol. 204). elsevier. https://doi.org/10.1016/S0304-0208(06)80001-0

[18] Kumar, V. (2020). On a generalized fractional Fourier transform. Palestine Journal of Mathematics, 9(2), 903-907. https://doi.org/10.2478/mjpaa-2021-0030

[19] Levit, B. Y. (2025). Minimax linear estimation on the half-line via Mellin transform. Theory of Probability & Its Applications, 70(2), 185-199. https://doi.org/10.1137/S0040585X97T992331

[20] Mahor, T. C., Mishra, R., & Jain, R. (2020). Fractionalization of Fourier sine and Fourier cosine transforms and their applications. International journal of scientific and technology research 9, 4.

[21] Romero, L., Cerutti, R., & Luque, L. (2011). A new Fractional Fourier Transform and convolutions products. International Journal of Pure and Applied Mathematics, 66(4), 397-408.

[22] Salehi, Y., & Schiesser, W. E. (2022). Numerical integration of space fractional partial differential equations: vol 2-applications from classical integer PDEs. Springer Nature. https://doi.org/10.1007/978-3-031-08961-5

[23] Upadhyaya, L. M. (2019). Introducing the Upadhyaya Integral Transform Bulletin of Pure & Applied Sciences-Mathematics, 38E(1) 471–510. https://doi.org/10.5958/2320-3226.2019.00051.1

[24] Vasileva, T., & Vasileva, O. (2009). Application Mellin transforms to the Black-Scholes equations. Mathematical Physics and Computer Modeling, (12), 55-63.

[25] Veeresha, P., Baskonus, H. M., & Gao, W. (2021). Strong interacting internal waves in rotating ocean: novel fractional approach. Axioms, 10(2), 123. https://doi.org/10.3390/axioms10020123

[26] Veeresha, P., Ilhan, E., & Baskonus, H. M. (2021). Fractional approach for analysis of the model describing wind-influenced projectile motion. Physica Scripta, 96(7), 075209. https://doi.org/10.1088/1402-4896/abf868

[27] Watugala, G. K. (1993). Sumudu transform: a new integral transform to solve differential equations and control engineering problems. Integrated Education, 24(1), 35-43. https://doi.org/10.1080/0020739930240105

[28] Yang, J., Sarkar, T. K., & Antonik, P. (2007). Applying the Fourier–modified Mellin transform (FMMT) to Doppler-distorted waveforms. Digital Signal Processing, 17(6), 1030-1039. 17(6), 1030-1039. https://doi.org/10.1016/j.dsp.2007.01.003

[29] Yao, S. W., Ilhan, E., Veeresha, P., & Baskonus, H. M. (2021). A powerful iterative approach for quintic complex Ginzburg–Landau equation within the frame of fractional operator. Fractals, 29(05), 2140023. https://doi.org/10.1142/S0218348X21400235

[30] Zolotarev, V. M. (1957). Mellin-Stieltjes transforms in probability theory. Theory of Probability & Its Applications, 2(4), 433-460. https://doi.org/10.1137/1102031

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