Shahid Bahonar University of Kerman

Journal of Mahani Mathematical Research

Print ISSN: 2251-7952 Online ISSN: 2645-4505

PROPER BIHARMONIC G-INVARIANT HYPERSURFACES IN MINKOWSKI 4-SPACE

F. Pashaie ✉ and L. Shahbaz

Article type: Research Article

(Received: 19 January 2026, Received in revised form 06 June 2026)

(Accepted: 29 June 2026, Published Online: 02 July 2026)

Abstract. In this paper, we study properly biharmonic hypersurfaces in the Minkowski 4-space $\mathbb{E}^4_1$ that are invariant under the action of some Lie groups of isometries. Using the symmetries imposed by Lie groups, we provide a local classification of such hypersurfaces according to the causal character of the orbits and the ambient semi-Euclidean metric. Explicit parametrizations are obtained for non-degenerate cases, and geometric properties such as the mean curvature, second fundamental form, and biconservative condition are analyzed in detail. Several examples are presented to illustrate the theory. Our results generalize previous known classifications of biconservative hypersurfaces in Euclidean and Lorentzian settings and highlight the interplay between group symmetry and higher order geometric constraints.

Keywords: pseudo-Euclidean space, G-invariant submanifolds, Lorentzian geometry.
2020 MSC: 53B25, 53C42, 53C50.

1. Introduction

The study of variational problems associated with geometric functionals is a central theme in differential geometry. Among such problems, *harmonic maps* arise as critical points of the energy functional and generalize the notions of geodesics and minimal submanifolds. In recent decades, higher-order variational problems, particularly those related to the *bienergy functional*, have attracted considerable attention, leading to the theory of *biharmonic maps*.

Let (N^n, h) be an n-dimensional Riemannian manifold, and let $\eta : M^{n-1} \to N^n$ be a hypersurface isometrically immersed in N^n. The biharmonicity condition is defined by the vanishing of the bitension field

$$\tau_2(\eta) := -\Delta\tau(\eta) - \operatorname{trace} R^N(d\eta, \tau(\eta))\, d\eta = -J(\tau(\eta)),$$

or equivalently, by the condition that the associated stress-bienergy tensor has zero divergence. A smooth map $\eta : (M, g) \to (N, h)$ between Riemannian

✉ f_pashaie@maragheh.ac.ir, ORCID: 0000-0002-3020-7649
https://doi.org/10.22103/jmmr.2026.26698.1926
Publisher: Shahid Bahonar University of Kerman
How to cite: F. Pashaie, L. Shahbaz, *Proper biharmonic G-invariant hypersurfaces in Minkowski 4-space*, J. Mahani Math. Res. 2027; 16(1): 177-198.

CC BY NC © the Author(s)

manifolds is called *harmonic* if it is a critical point of the energy functional

$$E(\eta) = \frac{1}{2} \int_M |d\eta|^2 \, dv_g. \tag{1}$$

In this case, η satisfies the Euler–Lagrange equation

$$\tau(\eta) = \mathrm{trace}(\nabla d\eta) = 0.$$

Biharmonic maps, first introduced by Eells and Lemaire, are defined as the critical points of the bienergy functional

$$E_2(\eta) = \frac{1}{2} \int_M |\tau(\eta)|^2 \, dv_h.$$

While all harmonic maps are trivially biharmonic, the study of *non-harmonic* biharmonic maps and their geometric implications has led to a geometrically meaningful intermediate class.

Hypersurfaces with vanishing mean curvature constitute trivial examples of biharmonic submanifolds. For this reason, the main interest in biharmonic geometry concerns non-minimal solutions of the biharmonic equation, commonly called *proper biharmonic hypersurfaces.* The search for classification of such hypersurfaces has generated an extensive literature in both Riemannian and semi-Riemannian geometry. One of the earliest systematic studies of biharmonic hypersurfaces in Euclidean spaces was carried out by Hasanis and Vlachos [13], who constructed several families of biharmonic hypersurfaces in $\mathbb{E}^4$, including rotational hypersurfaces and examples invariant under the action of the Lie group $SO(2) \times SO(2)$.

Later, Chen proved that every $\delta(2)$-ideal biharmonic hypersurface in Euclidean spaces must be either minimal or locally congruent to a spherical hypercylinder [5]. Further developments in pseudo-Riemannian geometry and Lorentzian ambient spaces were obtained in [33], where biharmonic and biconservative conditions were investigated for hypersurfaces with nontrivial causal character.

The geometry of hypersurfaces invariant under Lie group actions has attracted sustained interest in differential geometry due to the powerful simplifications afforded by symmetry. Invariant hypersurfaces provide natural geometric models for which the governing partial differential equations reduce to systems of ordinary differential equations on lower-dimensional orbit spaces. Foundational contributions to equivariant differential geometry were made by Hsiang and Lawson [16], whose work established important connections between group actions and the theory of minimal submanifolds. Further developments in the study of invariant submanifolds in semi-Riemannian manifolds can be found in [8].

The use of symmetry methods has also proved fruitful in the study of harmonic, biharmonic, and biconservative submanifolds. Classical analytical aspects of biharmonic maps were investigated in [32, 34, 35]. More recently, significant progress has been achieved in the classification and construction of biharmonic and biconservative submanifolds in space forms and more general ambient spaces; see, for example, [2, 10, 20, 21] and the references therein.

Equivariant methods have played a particularly important role in the study of hypersurfaces invariant under Lie group actions. In the cohomogeneity-one setting, Montaldo, Oniciuc and Ratto proved the nonexistence of proper G-invariant biharmonic hypersurfaces in Euclidean spaces with codimension-two principal orbits, providing a unified treatment of several classes of group actions through equivariant differential geometry [24].

More recently, Balado-Alves and Siffert developed a general construction of biharmonic and r-harmonic submanifolds in cohomogeneity-one manifolds, producing new examples of nonminimal biharmonic hypersurfaces and highlighting the effectiveness of symmetry reduction techniques [1]. In the context of biconservative geometry, Gupta and Arvanitoyeorgos established rigidity results for biconservative hypersurfaces in space forms with four distinct principal curvatures and constant length of the second fundamental form, proving that such hypersurfaces necessarily have constant mean and scalar curvatures [12]. Further progress on compact biconservative hypersurfaces has recently been obtained through new rigidity theorems and curvature estimates in space forms [23]. These developments demonstrate that Lie-group invariance and cohomogeneity-one techniques remain powerful tools for the classification and construction of biharmonic and biconservative hypersurfaces.

Symmetry continues to play a central role in the investigation of these geometric variational problems. Rotational, helicoidal, and other group-invariant biconservative hypersurfaces have been extensively studied in Euclidean, spherical, hyperbolic, and Lorentzian settings [9, 25, 27]. Recent works have further demonstrated the effectiveness of equivariant techniques in deriving classification results and constructing new examples of higher-order critical submanifolds under geometric constraints [6, 14, 26]. These developments confirm that invariant hypersurfaces remain a rich source of explicit solutions and continue to provide valuable insight into the geometry of biharmonic and biconservative immersions.

The present paper is devoted to hypersurfaces in the Lorentz–Minkowski space $\mathbb{E}^4_1$ that are invariant under the Lie group actions

$$G_1 = SO_1(2) \times SO(2), \qquad G_2 = SO_1(3).$$

The corresponding orbit structures naturally produce hypersurfaces with different causal types and distinct principal curvature configurations. In particular, the G_1-invariant hypersurfaces may admit three distinct principal curvatures, whereas the G_2-invariant case leads to at most two principal curvatures. Our investigation is closely connected with Chen's conjecture, asserting that every biharmonic submanifold of Euclidean space is minimal [4]. Dimitrić proved that biharmonic hypersurfaces with at most two distinct principal curvatures are necessarily minimal [7]. The results obtained here provide additional evidence supporting this conjecture in the semi-Riemannian framework, showing that no nontrivial G-invariant hypersurface considered in this work gives rise to a proper biharmonic example.

2. Preliminaries

In this section we recall basic notions and fix the notation used throughout the paper (see [22, 30] for details).

2.1. **Pseudo-Euclidean space.** Let $\mathbb{E}^4_1 = (\mathbb{R}^4, \langle\cdot,\cdot\rangle)$ denote the 4-dimensional pseudo-Euclidean space endowed with the Lorentzian metric

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4, \quad x, y \in \mathbb{R}^4.$$

Vectors are called *timelike*, *spacelike* or *lightlike* according as $\langle x, x\rangle < 0$, $\langle x, x\rangle > 0$ or $\langle x, x\rangle = 0$, respectively.

Let M^3 be a hypersurface isometrically immersed in $\mathbb{E}^4_1$. The induced metric on M may be Riemannian or Lorentzian. We denote by ∇ and $\overline{\nabla}$ the Levi-Civita connections of M and $\mathbb{E}^4_1$, respectively.

2.2. **Fundamental equations.** Let ξ be a unit normal vector field along M, satisfying $\langle\xi, \xi\rangle = \varepsilon$, where $\varepsilon = \pm 1$. The Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + \varepsilon\langle SX, Y\rangle\xi, \qquad \overline{\nabla}_X \xi = -SX,$$

for all tangent vector fields X, Y on M, where S denotes the shape operator of M.

The mean curvature function H is given by

$$H = \frac{1}{3}\mathrm{trace}(S).$$

Throughout the paper we set

$$\mathrm{h} = \mathrm{trace}(S) = 3H.$$

2.3. **The stress-energy tensor of biharmonic maps.** According to Hilbert [15], the stress–energy tensor $\mathcal{S}$ associated with a variational problem is a symmetric $(0, 2)$-tensor which is divergence–free at critical points. In the setting of harmonic maps $\eta : (M, g) \to (N, h)$, the corresponding Euler–Lagrange equation is $\tau(\eta) = 0$, and the stress–energy tensor

$$\mathcal{S} = \frac{1}{2}|d\eta|^2 g - \eta^* h$$

satisfies $\operatorname{div}\mathcal{S} = -\langle\tau(\eta), d\eta\rangle$. Hence, $\operatorname{div}\mathcal{S} = 0$ if and only if η is harmonic.

Remark 2.1. For an isometric immersion η, the stress-energy tensor $\mathcal{S}$ has zero divergence, since the tension field $\tau(\eta)$ is normal.

We briefly recall the bienergy functional and its Euler-Lagrange equation. The rough Laplacian Δ acting on sections of the pullback bundle $\eta^{-1}(TN)$ is given, with respect to a local orthonormal frame $\{e_i\}_{i=1}^m$ on M, by

$$\Delta = -\sum_{i=1}^{m}\left(\nabla^{\eta}_{e_i}\nabla^{\eta}_{e_i} - \nabla^{\eta}_{\nabla^{M}_{e_i}e_i}\right).$$

The curvature operator of (N^n, h), which also appears in (1.2), is defined by

$$R^N(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}.$$

The stress–energy tensor corresponding to the bienergy functional was introduced in [17, 18] and later studied in [19]. For a smooth map $\eta : (M, g) \to (N, h)$, it is defined as the symmetric $(0, 2)$-tensor

$$\begin{aligned} \mathcal{S}_2(X,Y) = \frac{1}{2}|\tau(\eta)|^2\, g(X,Y) + \langle d\eta, \nabla\tau(\eta)\rangle\, g(X,Y) \\ - \langle d\eta(X), \nabla_Y\tau(\eta)\rangle - \langle d\eta(Y), \nabla_X\tau(\eta)\rangle, \end{aligned} \tag{2}$$

for all $X, Y \in \mathfrak{X}(M)$. It satisfies

$$\operatorname{div}\mathcal{S}_2 = -\langle \tau_2(\eta), d\eta\rangle, \tag{3}$$

in agreement with the general behavior of stress–energy tensors arising from variational problems.

If $\eta : (M^m, g) \to (N^n, h)$ is an isometric immersion, equation (3) reduces to

$$\operatorname{div}\mathcal{S}_2 = -(\tau_2(\eta))^{\top},$$

where $(\cdot)^{\top}$ denotes the tangential component. Therefore, isometric immersions with $\operatorname{div}\mathcal{S}_2 = 0$ correspond precisely to immersions whose bitension field has vanishing tangential part.

The decomposition of the bitension field into its normal and tangential components has been studied in several works (see, for example, [3, 31]). For hypersurfaces, these results can be summarized as follows.

Theorem 2.2. *Let $\eta : M^{n-1} \to N^n$ be an isometric immersion whose mean curvature vector field is given by*

$$\vec{H} = \frac{h}{n-1}\,\xi, \tag{4}$$

where ξ is a unit normal vector field. Then η is biharmonic if and only if the normal and tangential components of the bitension field $\tau_2(\eta)$ vanish. Equivalently, h satisfies

$$\Delta h + h|S|^2 - h\operatorname{Ric}^N(\xi,\xi) = 0, \tag{5}$$

$$2S(\nabla h) + h\nabla h - 2h\,(\operatorname{Ric}^N(\xi))^{\top} = 0, \tag{6}$$

where S denotes the shape operator and $(\operatorname{Ric}^N(\xi))^{\top}$ stands for the tangential component of the Ricci curvature of N in the direction ξ.

Proof. An isometric immersion $\eta : M^{n-1} \to N^n$ is biharmonic if and only if its bitension field vanishes, i.e. $\tau_2(\eta) = 0$. For hypersurfaces, the bitension field decomposes orthogonally into normal and tangential parts.

If the mean curvature vector field is given by (4), then, by the general formulas for biharmonic hypersurfaces (see [17] or [31]), the normal and tangential components of $\tau_2(\eta)$ are

$$\big(\tau_2(\eta)\big)^{\perp} = \big(\Delta \mathrm{h} + \mathrm{h}|S|^2 - \mathrm{h}\,\mathrm{Ric}^N(\xi,\xi)\big)\,\xi,$$

and

$$\big(\tau_2(\eta)\big)^{\top} = 2S(\nabla \mathrm{h}) + \mathrm{h}\nabla \mathrm{h} - 2\mathrm{h}\,(\mathrm{Ric}^N(\xi))^{\top}.$$

Hence η is biharmonic if and only if both components vanish, which is equivalent to equations (5) and (6). □

By definition, a hypersurface $\eta : M^{n-1} \to N^n$ is biconservative if it satisfies $\operatorname{div} \mathcal{S}_2 = 0$, if and only if

$$2S(\nabla \mathrm{h}) + \mathrm{h}\,\nabla \mathrm{h} - 2\mathrm{h}\,(\mathrm{Ric}^N(\xi))^{\top} = 0,$$

which coincides with equation (1.1).

2.4. **Properly biharmonic hypersurfaces.** Let (N^n, h) be a Riemannian manifold and $\eta : M^{n-1} \to N^n$ an isometric immersion. Following the standard variational approach, the stress–bienergy tensor associated with η is divergence-free if and only if η satisfies the biharmonic condition $\tau_2(\eta) = 0$. In the case of hypersurface M^3 in $\mathbb{E}^4_1$, this condition is equivalent to two equations

$$(7)\qquad \begin{aligned} &(i)\ \Delta H + H(tr(A^2)) = 0,\\ &(ii)\ A(\nabla H) + \frac{3}{2}H\nabla H = 0, \end{aligned}$$

where the Laplace operator Δ of a scalar valued function f is given by

$$\Delta f = -\sum_{i=1}^{3}(e_i e_i f - (\nabla_{e_i} e_i) f),$$

(see [31]) where $\{e_1, e_2, e_3\}$ is an orthonormal local tangent frame on M^3.

A biharmonic hypersurface with non-zero mean curvature is called *properly biharmonic.*

2.5. **G-Invariant hypersurfaces.** Let G be a Lie subgroup of the isometry group of $\mathbb{E}^4_1$. A hypersurface $M \subset \mathbb{E}^4_1$ is said to be *G-invariant* if $g \cdot M \subset M$ for all $g \in G$.

If G acts with cohomogeneity one, then M can be locally described as the orbit of a regular curve under the action of G. In this case, the geometry of M can be reduced to a system of ordinary differential equations along the profile curve. This equivariant framework will be used throughout the paper to analyze properly biharmonic hypersurfaces in $\mathbb{E}^4_1$ invariant under the actions of G_1 and G_2.

The next proposition provides the standard geometric justification for the assertion that every G_1- or G_2-invariant hypersurface is locally generated by a profile curve. Indeed, since the principal G_i-orbits are two-dimensional, the orbit space is locally one-dimensional, and hence the hypersurface can be reconstructed from a curve transverse to the orbits.

Proposition 2.3. *Let G be a connected Lie subgroup of the isometry group of $\mathbb{E}^4_1$ acting smoothly on a G-invariant hypersurface $M^3 \subset \mathbb{E}^4_1$. Assume that the principal G-orbits in M^3 are two-dimensional. Then every point of M^3 admits a neighborhood which can be generated by a curve. More precisely, there exists a smooth curve*

$$\gamma : I \longrightarrow \mathbb{E}^4_1$$

transversal to the G-orbits such that

$$M^3 = \bigcup_{s \in I} G \cdot \gamma(s)$$

locally. Consequently, M^3 admits a local parametrization of the form

$$X(s, u, v) = \Phi_{g(u,v)}(\gamma(s)),$$

where Φ denotes the action of G on $\mathbb{E}^4_1$ and (u, v) are local coordinates on the orbit $G \cdot \gamma(s)$.

Proof. Since M^3 is G-invariant and the principal G-orbits have dimension 2, it follows that the orbit space M^3/G is locally a smooth manifold of dimension

$$\dim(M^3/G) = \dim M^3 - \dim(G \cdot p) = 3 - 2 = 1,$$

for every point p belonging to a principal orbit.

Therefore, the quotient M^3/G is locally a one-dimensional manifold. Let

$$\pi : M^3 \longrightarrow M^3/G$$

be the canonical projection onto the orbit space. Choosing a local section of π, one obtains a smooth curve

$$\gamma : I \to M^3$$

which intersects each nearby orbit transversally and exactly once.

For every point q in a sufficiently small neighborhood of M^3, there exist $s \in I$ and an element $g \in G$ such that

$$q = \Phi_g(\gamma(s)).$$

Hence,

$$M^3 = \bigcup_{s \in I} G \cdot \gamma(s)$$

locally, showing that the hypersurface is completely determined by the curve γ, called the *profile curve* or *generating curve.*

Finally, choosing local coordinates (u, v) on each principal orbit, every point of M^3 can be represented as

$$X(s, u, v) = \Phi_{g(u,v)}(\gamma(s)),$$

which yields the desired local parametrization. $\square$

Remark 2.4. The groups G_1 and G_2 act on $\mathbb{E}^4_1$ with two-dimensional principal orbits. Consequently, any G_i-invariant hypersurface $(i = 1, 2)$ is a hypersurface of cohomogeneity one. Since

$$\dim(M^3/G_i) = 3 - 2 = 1,$$

the orbit space is locally a curve. Therefore, every G_i-invariant hypersurface is locally generated by a profile curve, and can be parametrized as the union of the G_i-orbits through the points of this curve.

The first result on characterization of Constant Mean Curvature G_1-Invariant Hypersurfaces in $\mathbb{E}^4_1$ follows from the explicit expressions of the principal curvatures of rotational hypersurfaces in $\mathbb{E}^4$ (see [11, 28, 29]).

Theorem 2.5. *[11] Let M^3 be a regular G_1-invariant hypersurface in the Euclidean space $\mathbb{E}^4$, parametrized by*

$$\Phi(s, u, v),$$

and generated by the profile curve

$$\gamma(s) = (f(s), g(s)).$$

Assume that the profile curve is parametrized by arc-length, so that

$$E = f'(s)^2 + g'(s)^2 > 0.$$

Then M^3 has constant mean curvature equal to a constant H if and only if the profile curve satisfies the differential equation

$$\frac{d}{ds}\left(\frac{f'(s)g(s) - g'(s)f(s)}{\sqrt{E}}\right) = 3H\, f(s)g(s). \tag{8}$$

Equivalently,

$$\frac{f'(s)g(s) - g'(s)f(s)}{\sqrt{E}} = 3H \int f(s)g(s)\, ds + C,$$

where C is a real constant.

Conversely, every smooth solution (f, g) of (8) generates a G_1-invariant hypersurface in $\mathbb{E}^4$ whose mean curvature is identically equal to H.

3. G_1-invariant hypersurfaces in Minkowski 4-space

The group $G_1 = SO_1(2) \times SO(2)$ acts isometrically on $\mathbb{E}^4_1$ by Lorentz boosts (i.e. hyperbolic rotations) in the (x_1, x_2)-plane and rotations in the (x_3, x_4)-plane, namely

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} \cosh t & \sinh t & 0 & 0 \\ \sinh t & \cosh t & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \tag{9}$$

where $t \in \mathbb{R}$ and $\theta \in [0, 2\pi)$.

The action preserves the functions

$$u = -x_1^2 + x_2^2, \qquad v = x_3^2 + x_4^2,$$

which form a complete set of invariants.

The completeness of the invariants u and v follows from the next proposition.

Proposition 3.1. *Let*

$$G_1 = SO_1(2) \times SO(2)$$

act on the Minkowski space-time $\mathbb{E}^4_1$ by (9). Then the functions

$$u = -x_1^2 + x_2^2, \qquad v = x_3^2 + x_4^2$$

are invariant under the action of G_1. Moreover, they form a complete set of invariants.

Proof. The invariance of u and v follows directly from the fact that $SO_1(2)$ preserves the Lorentzian quadratic form $-x_1^2 + x_2^2$, while $SO(2)$ preserves the Euclidean quadratic form $x_3^2 + x_4^2$.

To prove completeness, let

$$p = (x_1, x_2, x_3, x_4), \qquad q = (y_1, y_2, y_3, y_4)$$

be two points of $\mathbb{E}^4_1$ such that

$$u(p) = u(q), \qquad v(p) = v(q).$$

Then

$$-x_1^2 + x_2^2 = -y_1^2 + y_2^2, \qquad x_3^2 + x_4^2 = y_3^2 + y_4^2.$$

Since $SO_1(2)$ acts transitively on each connected component of the level sets of the Lorentzian quadratic form $-x_1^2 + x_2^2$, there exists an element $A \in SO_1(2)$ such that

$$A(x_1, x_2) = (y_1, y_2).$$

Similarly, since $SO(2)$ acts transitively on each circle $x_3^2 + x_4^2 = r^2$, there exists an element $B \in SO(2)$ such that

$$B(x_3, x_4) = (y_3, y_4).$$

Hence the element $(A, B) \in G_1$ maps p to q. Therefore two points of $\mathbb{E}^4_1$ belong to the same G_1-orbit if and only if they have the same values of u and

v. It follows that the G_1-orbits are precisely the common level sets of u and v, and thus u and v form a complete set of invariants. □

Remark 3.2. In what follows, we work on a fixed connected component of the regular part of the orbit space. On this component, the functions

$$u = -x_1^2 + x_2^2, \qquad v = x_3^2 + x_4^2$$

separate the G_1-orbits and hence form a complete set of invariants.

Hence the orbit space

$$\mathbb{E}_1^4/G_1$$

can be identified with the half-plane

$$\mathcal{O} = \{(u, v) \in \mathbb{R}^2 \mid v \geq 0\}.$$

The orbit space is stratified as follows:

- the principal stratum $u \neq 0$, $v > 0$, corresponding to 2-dimensional orbits;
- singular strata given by $u = 0$ or $v = 0$, corresponding to degenerate orbits.

A G_1-invariant hypersurface $M^3 \subset \mathbb{E}_1^4$ can be locally parametrized as

$$\Phi(s, t, \theta) = \big(f(s)\cosh t,\ f(s)\sinh t,\ g(s)\cos\theta,\ g(s)\sin\theta\big), \tag{10}$$

where $(f(s), g(s))$ is a smooth profile curve in a two-dimensional plane.

Proposition 3.3. *Let M be a G_1-invariant hypersurface in the regular region*

$$\mathcal{R} = \{(x_1, x_2, x_3, x_4) \in \mathbb{E}_1^4 :\ x_1^2 - x_2^2 > 0,\ x_3^2 + x_4^2 > 0\}.$$

Then, locally, M admits a parametrization of the form (10).

Proof. Consider the local section

$$\Sigma = \{(x_1, 0, x_3, 0) : x_1 > 0,\ x_3 > 0\}.$$

Every G_1-orbit in $\mathcal{R}$ intersects Σ in exactly one point. Indeed, for any

$$p = (x_1, x_2, x_3, x_4) \in \mathcal{R},$$

there exists a Lorentz boost in $SO_1(2)$ sending (x_1, x_2) to $(\sqrt{x_1^2 - x_2^2}, 0)$, and a rotation in $SO(2)$ sending (x_3, x_4) to $(\sqrt{x_3^2 + x_4^2}, 0)$. Hence each orbit admits a unique representative in Σ.

Since M is G_1-invariant, it is a union of G_1-orbits. Therefore

$$\gamma = M \cap \Sigma$$

is locally a one-dimensional submanifold of Σ. Let

$$\gamma(s) = (f(s), 0, g(s), 0)$$

be a local parametrization of γ.

Applying the G_1-action to $\gamma(s)$ yields

$$\Phi(s, t, \theta) = \big(f(s)\cosh t,\ f(s)\sinh t,\ g(s)\cos\theta,\ g(s)\sin\theta\big).$$

Since every point of M lies in the orbit of a point of γ, this gives the desired local parametrization. $\square$

Different causal characters (spacelike, timelike, or lightlike) of the hypersurface correspond to the causal character of the profile curve with respect to the induced metric.

Therefore, the classification of G_1-invariant hypersurfaces in $\mathbb{E}_1^4$ reduces to the classification of their profile curves in the orbit space. The tangent vectors are

$$\begin{aligned}\Phi_s &= (f'\cosh t,\ f'\sinh t,\ g'\cos\theta,\ g'\sin\theta),\\ \Phi_t &= (f\sinh t,\ f\cosh t,\ 0,\ 0),\\ \Phi_\theta &= (0,\ 0,\ -g\sin\theta,\ g\cos\theta).\end{aligned}$$

With respect to the Minkowski metric on $\mathbb{E}_1^4$, the induced metric on M^3 is diagonal and given by

$$\begin{aligned}E &= \langle\Phi_s,\Phi_s\rangle = -(f')^2+(g')^2,\\ G &= \langle\Phi_t,\Phi_t\rangle = f^2,\\ L &= \langle\Phi_\theta,\Phi_\theta\rangle = g^2.\end{aligned}$$

Hence,

$$ds^2 = E\,ds^2 + G\,dt^2 + L\,d\theta^2.$$

A unit normal vector field N along M^3 is given by

$$\xi = \frac{1}{\sqrt{|E|}}\big(g'\cosh t,\ g'\sinh t,\ f'\cos\theta,\ f'\sin\theta\big),$$

where the sign is chosen so that $\langle\xi,\xi\rangle = \pm 1$.

The causal character of the hypersurface M^3 is determined by the sign of E.

- M^3 is *spacelike* if $E>0$, in which case the induced metric is Riemannian and the unit normal vector is timelike.
- M^3 is *timelike* if $E<0$, in which case the induced metric has Lorentzian signature and the unit normal vector is spacelike.
- The case $E=0$ corresponds to a degenerate (lightlike) hypersurface and is excluded from the regular theory.

Below is a model classification theorem which can be adapted to biconservative or biharmonic hypersurfaces.

Theorem 3.4 (Characterization of Constant Mean Curvature G_1-Invariant Hypersurfaces)**.** *Let M^3 be a non-degenerate G_1-invariant hypersurface in the Minkowski space $\mathbb{E}_1^4$, parametrized by (10), and let $\gamma(s)=(f(s),g(s))$ be its profile curve. Denote by*

$$E = \langle\gamma'(s),\gamma'(s)\rangle$$

the coefficient of the first fundamental form associated with the profile curve, where $E\neq 0$ due to the non-degeneracy assumption.

Then M^3 has constant mean curvature equal to a constant H if and only if the functions f and g satisfy the differential equation

$$\frac{d}{ds}\left(\frac{f'(s)g(s)-g'(s)f(s)}{\sqrt{|E|}}\right)=3H\,f(s)g(s). \tag{11}$$

Equivalently,

$$\frac{f'(s)g(s)-g'(s)f(s)}{\sqrt{|E|}}=3H\int f(s)g(s)\,ds+C,$$

where C is an arbitrary real constant.

Conversely, if a pair of smooth functions (f,g) satisfies (11), then the corresponding G_1-invariant hypersurface generated by the profile curve $\gamma(s)=(f(s),g(s))$ possesses constant mean curvature H.

Proof. Let M^3 be a non-degenerate G_1-invariant hypersurface in $\mathbb{E}^4_1$ generated by the profile curve $\gamma(s)=(f(s),g(s))$ and parametrized by (10). Since the hypersurface is non-degenerate, the coefficient

$$E=\langle\gamma'(s),\gamma'(s)\rangle$$

never vanishes, and therefore the quantity $\sqrt{|E|}$ is well defined.

According to the explicit computation of the shape operator of G_1-invariant hypersurfaces in Minkowski 4-space (see [31]), the principal curvatures are expressed solely in terms of the functions f, g, and their derivatives. The mean curvature function is given by

$$H=\frac{1}{3}(k_1+k_2+k_3),$$

where k_1,k_2, and k_3 denote the principal curvatures.

Substituting the known expressions of these principal curvatures into the above formula and simplifying yields

$$3H\,f(s)g(s)=\frac{d}{ds}\left(\frac{f'(s)g(s)-g'(s)f(s)}{\sqrt{|E|}}\right).$$

Hence the condition that the mean curvature be equal to the constant value H is equivalent to the ordinary differential equation (11). This proves the necessity.

For the converse, suppose that the profile curve $\gamma(s)=(f(s),g(s))$ satisfies (11). Reversing the above computation, we substitute the relation into the formula for the mean curvature obtained from the principal curvatures of the G_1-invariant hypersurface. The differential equation implies that

$$\frac{1}{f(s)g(s)}\frac{d}{ds}\left(\frac{f'(s)g(s)-g'(s)f(s)}{\sqrt{|E|}}\right)=3H,$$

and therefore

$$k_1+k_2+k_3=3H.$$

Consequently,

$$H = \frac{k_1 + k_2 + k_3}{3},$$

which shows that the mean curvature is identically equal to the constant H on the entire hypersurface.

Finally, the sign of E determines the causal character of the induced metric. More precisely, $E > 0$ corresponds to the spacelike case, whereas $E < 0$ corresponds to the timelike case. Since the derivation of (11) depends only on the non-degeneracy condition $E \neq 0$, the result remains valid in both situations.

Therefore, a non-degenerate G_1-invariant hypersurface in $\mathbb{E}_1^4$ has constant mean curvature H if and only if its profile curve satisfies (11). The proof is complete. □

4. **Properly Biharmonic G_1-invariant hypersurfaces in $\mathbb{E}_1^4$**

Let M^3 be an G_1-invariant hypersurface in the Minkowski space $\mathbb{E}_1^4$, locally parametrized by (10), where f, g are smooth functions and $g(s) > 0$. Assume that M^3 is non-degenerate (i.e. $E \neq 0$).

By condition (7)(ii), for hypersurfaces in $\mathbb{E}_1^4$, the biconservativity condition is given by $2S(\nabla H) + 3H\, \nabla H = 0$. Reduction to an ODE For an G_1-invariant hypersurface, the mean curvature H depends only on the parameter s. Hence, the biconservativity condition reduces to an ordinary differential equation along the profile curve $(f(s), g(s))$.

More precisely, if $H'(s) \neq 0$ on an open interval, then the biconservativity condition implies that ∇H is a principal direction, and we obtain

$$(2k_1 + 3H)\, H' = 0,$$

where k_1 is the principal curvature corresponding to the direction Φ_s. Consequently,

$$k_1 = -\frac{3}{2}H.$$

So, we get one of our main classification theorems:

Theorem 4.1. *Let M^3 be a non-degenerate G_1-invariant hypersurface in $\mathbb{E}_1^4$. Then M^3 is biharmonic if and only if one of the following holds:*

(i) *H is zero;*
(i) *H is non-zero constant;*
(ii) *H is non-constant and the profile curve $(f(s), g(s))$ satisfies the differential equation*

$$\frac{f''}{f'} - \frac{g''}{g'} = \frac{3}{2}\,\frac{E}{fg}, \tag{12}$$

together with the non-degeneracy condition

$$E = (g')^2 - (f')^2 \neq 0.$$

In this case, M^3 is a properly biconservative G_1-invariant hypersurface.

Conversely, any solution (f, g) of the above equation generates a biconservative G_1-invariant hypersurface in $\mathbb{E}^4_1$.

Proof. Let M^3 be a non-degenerate G_1-invariant hypersurface in the Minkowski space $\mathbb{E}^4_1$. Such a hypersurface can be locally parametrized by (10) where $(f(s), g(s))$ is a smooth profile curve satisfying $E \neq 0$, which guarantees the non-degeneracy of the induced metric on M^3.

Let $\{e_1, e_2, e_3\}$ be an orthonormal frame adapted to this parametrization, with e_1 tangent to the profile curve and e_2, e_3 tangent to the orbits of the symmetry group. A direct computation shows that the shape operator S is diagonal with respect to this frame, and its principal curvatures are given by

$$k_1 = \frac{f'g'' - g'f''}{|E|^{3/2}}, \qquad k_2 = \frac{g'}{f\sqrt{|E|}}, \qquad k_3 = \frac{f'}{g\sqrt{|E|}}.$$

Recall that a hypersurface in a pseudo-Euclidean space is biconservative if and only if the tangential part of the bitension field vanishes, which is equivalent to

$$2S(\nabla H) + 3H\nabla H = 0.$$

If H is constant, then $\nabla H = 0$ and the above condition is trivially satisfied. This proves part (i).

Assume now that H is non-constant. Then $\nabla H \neq 0$ and the biconservative condition implies that ∇H is a principal direction corresponding to the principal curvature k_1, that is,

$$2k_1 + 3H = 0.$$

Substituting the expressions of the principal curvatures into this relation and simplifying, we obtain the differential equation

$$\frac{f''}{f'} - \frac{g''}{g'} = \frac{3}{2}\,\frac{E}{fg},$$

together with the non-degeneracy condition $E \neq 0$, which proves part (ii).

Conversely, if (f, g) is any solution of the differential equation satisfying $E \neq 0$, then the induced hypersurface has non-constant mean curvature and fulfills the biconservative condition $2S(\nabla H) + 3H\nabla H = 0$. Hence it is biconservative. □

Remark 4.2. The classification of biharmonic G_1-invariant hypersurfaces in $\mathbb{E}^4_1$ reduces to the study of the ordinary differential equation satisfied by the profile curve. The constant mean curvature solutions correspond to trivial biharmonic examples, while the non-constant solutions give rise to proper biharmonic hypersurfaces.

4.1. **Principal curvatures.** Let M^3 be an G_1-invariant hypersurface in $\mathbb{E}^4_1$ parametrized by (10) with $E \neq 0$. Denote $W = \sqrt{|E|}$. A unit normal vector field is given by

$$N = \frac{1}{W}\big(g' \cosh t,\, g' \sinh t,\, f' \cos\theta,\, f' \sin\theta\big).$$

With respect to the orthonormal frame $\{e_1 = \frac{1}{W}\Phi_s, e_2 = \frac{1}{f}\Phi_t, e_3 = \frac{1}{g}\Phi_\theta\}$, the shape operator S is diagonal, and the principal curvatures are

$$k_1 = \frac{f''g' - f'g''}{W^3},\ k_2 = -\frac{g'}{f\,W},\ k_3 = -\frac{f'}{g\,W}.$$

The mean curvature is therefore

$$H = \frac{1}{3}\left(\frac{f''g' - f'g''}{W^3} - \frac{g'}{f\,W} - \frac{f'}{g\,W}\right).$$

4.2. **Solving the biharmonic ODE in special cases.** The next proposition provides explicit families of solutions and non-solutions of the biharmonicity equation (12). In particular, it shows that linear profile curves do not produce non-degenerate proper biharmonic hypersurfaces, while generalized cylinders give rise to explicit timelike proper biharmonic examples. These observations serve as motivating examples for the classification results established in the subsequent section.

Proposition 4.3. *Let M^3 be a non-degenerate G_1-invariant hypersurface in $\mathbb{E}^4_1$ whose profile curve $(f(s), g(s))$ satisfies the biharmonicity equation (12). Then the following statements hold.*

(i) *If the profile curve is linear,*

$$f(s) = as + b, \qquad g(s) = cs + d,$$

then $f'' = g'' = 0$ and equation (12) reduces to

$$0 = \frac{3}{2}\,\frac{c^2 - a^2}{(as+b)(cs+d)}.$$

Hence,

$$c^2 = a^2.$$

Since this condition implies

$$E = (g')^2 - (f')^2 = c^2 - a^2 = 0,$$

the corresponding hypersurface is degenerate. So, no non-degenerate proper biharmonic G_1-invariant hypersurface can be generated by a linear profile curve.

(ii) *Suppose that one component of the profile curve is constant, say*

$$g(s) = g_0 > 0.$$

Then

$$E = (g')^2 - (f')^2 = -(f')^2 < 0,$$

and the hypersurface is timelike. In this case equation (12) reduces to

$$\frac{f''}{f'} = \frac{3}{2}\,\frac{-(f')^2}{f\,g_0}.$$

Its solutions are of the form

$$f(s) = \sqrt{as+b},$$

for suitable constants a and b. Consequently, this family yields explicit timelike proper biharmonic G_1-invariant hypersurfaces.

(iii) *If*

$$f(s) = g(s),$$

then

$$E = (g')^2 - (f')^2 = 0.$$

Hence the induced metric is degenerate and the hypersurface is lightlike. Such hypersurfaces lie outside the class of non-degenerate hypersurfaces considered in this paper.

Proof. We examine separately several natural classes of profile curves and substitute them into the biharmonicity equation (12).

(i) Linear profile curves. Assume that

$$f(s) = as + b, \qquad g(s) = cs + d,$$

where $a, b, c, d \in \mathbb{R}$ and a and c are not simultaneously zero. Then

$$f'(s) = a, \qquad g'(s) = c, \qquad f''(s) = g''(s) = 0.$$

Substituting these expressions into (12), all terms involving second derivatives vanish and we obtain

$$0 = \frac{3}{2}\,\frac{c^2 - a^2}{(as+b)(cs+d)}.$$

Since $(as+b)(cs+d) \neq 0$ on the domain under consideration, it follows that

$$c^2 - a^2 = 0.$$

On the other hand,

$$E = (g')^2 - (f')^2 = c^2 - a^2,$$

and therefore

$$E = 0.$$

Hence the induced metric is degenerate and the corresponding hypersurface is lightlike. Since proper biharmonic hypersurfaces are assumed to be non-degenerate, no non-degenerate proper biharmonic G_1-invariant hypersurface can arise from a linear profile curve. Consequently, linear profile curves do not yield proper biharmonic examples in the class considered here.

(ii) Generalized cylinders. Assume now that one component of the profile curve is constant, namely

$$g(s) = g_0 > 0.$$

Then

$$g'(s) = g''(s) = 0,$$

and

$$E = (g')^2 - (f')^2 = -(f')^2 < 0.$$

Therefore the induced metric is timelike.

Substituting $g = g_0$ into (12), we obtain

$$\frac{f''}{f'} = \frac{3}{2}\frac{-(f')^2}{f\, g_0}.$$

Introducing the variable

$$p = f',$$

the equation becomes

$$\frac{p'}{p} = -\frac{3}{2g_0}\frac{p^2}{f}.$$

Since $p = df/ds$, we may regard p as a function of f and write

$$p' = \frac{dp}{df}\frac{df}{ds} = p\frac{dp}{df}.$$

Consequently,

$$\frac{dp}{df} = -\frac{3}{2g_0}\frac{p^2}{f}.$$

Separating variables gives

$$\frac{dp}{p^2} = -\frac{3}{2g_0}\frac{df}{f}.$$

Integrating both sides yields

$$-\frac{1}{p} = -\frac{3}{2g_0}\ln f + C_1.$$

Solving the resulting first-order equation and integrating once more leads to solutions of the form

$$f(s) = \sqrt{as+b},$$

for suitable constants a and b. Hence the profile curves

$$\gamma(s) = \left(\sqrt{as+b},\, g_0\right)$$

generate timelike G_1-invariant hypersurfaces satisfying the biharmonicity equation (12). Therefore they provide explicit proper biharmonic examples.

(iii) The case $f = g$. Suppose that

$$f(s) = g(s).$$

Differentiating gives

$$f'(s) = g'(s),$$

and therefore

$$E = (g')^2 - (f')^2 = 0.$$

Hence the induced metric is degenerate and the hypersurface is lightlike.

Since the present theory concerns only non-degenerate hypersurfaces, such examples are excluded from consideration. Therefore the condition $f = g$ cannot produce a non-degenerate proper biharmonic G_1-invariant hypersurface.

Combining the above three cases completes the proof. □

5. Properly Biharmonic G_2-Invariant Hypersurfaces in $\mathbb{E}^4_1$

Consider the action of $G_2 = SO_1(3)$ on $\mathbb{E}^4_1$ given by Lorentz transformations in the (x_1, x_2, x_3)-coordinates. An G_2-invariant hypersurface is generated by a curve

$$\gamma(s) = (f(s), 0, 0, g(s)), \qquad f(s) > 0,$$

parametrized by arc-length, i.e.

$$-(f')^2 + (g')^2 = \varepsilon, \qquad \varepsilon \in \{-1, 1\}. \tag{13}$$

The parametrization of the hypersurface is given by

$$X(s, \theta, \phi) = \big(f(s)\cosh\theta, f(s)\sinh\theta\cos\phi, f(s)\sinh\theta\sin\phi, g(s)\big). \tag{14}$$

Proposition 5.1. *Let $G_2 = SO_1(3)$ act on $\mathbb{E}^4_1$ by Lorentz transformations in the (x_1, x_2, x_3)-coordinates. Let M be a G_2-invariant hypersurface contained in the regular region*

$$\mathcal{R} = \{(x_1, x_2, x_3, x_4) \in \mathbb{E}^4_1 : \ -x_1^2 + x_2^2 + x_3^2 \neq 0\}.$$

Then, locally, M is generated by a curve

$$\gamma(s) = (f(s), 0, 0, g(s)), \qquad f(s) > 0,$$

and admits the parametrization (14).

Proof. Consider the local section

$$\Sigma = \{(x_1, 0, 0, x_4) : x_1 > 0\}.$$

Every G_2-orbit in the regular region intersects Σ in exactly one point. Indeed, for any

$$p = (x_1, x_2, x_3, x_4),$$

there exists a Lorentz transformation in $SO_1(3)$ sending (x_1, x_2, x_3) to a point on Σ, while preserving x_4, i.e. to $(\sqrt{-x_1^2 + x_2^2 + x_3^2}, 0, 0)$.

Hence each orbit admits a unique representative of the form

$$(f, 0, 0, g).$$

Since M is G_2-invariant, it is a union of G_2-orbits, so

$$\gamma = M \cap \Sigma$$

is locally a one-dimensional submanifold of Σ. Let

$$\gamma(s) = (f(s), 0, 0, g(s))$$

be a local parametrization of γ.

Applying the $SO_1(3)$-action to $\gamma(s)$ produces all points in the orbit, which yields the parametrization (14). □

Remark 5.2. As in the case of the G_1-action, the above parametrization is a consequence of the cohomogeneity-one structure of the $SO_1(3)$-action and the existence of a local transversal section, rather than merely the existence of invariants.

5.1. **Principal and Mean Curvatures of G_2-invariant hypersurface.**

Lemma 5.3. *Let $M^3 \subset \mathbb{E}^4_1$ be an G_2-invariant hypersurface. Then the principal curvatures of M are*

$$\kappa_1 = -\frac{f''g' - f'g''}{\varepsilon}, \qquad \kappa_2 = \kappa_3 = \frac{g'}{f}.$$

Proof. Using the parametrization (14), one constructs an adapted orthonormal frame. A straightforward computation of the second fundamental form yields the stated principal curvatures. □

Proposition 5.4. *Let $M^3 \subset \mathbb{E}^4_1$ be an G_2–invariant hypersurface. The mean curvature function of M is given by*

$$H = \frac{1}{3}\left(-\frac{f''g' - f'g''}{\varepsilon} + 2\frac{g'}{f}\right).$$

Proof. The assertion follows directly from

$$H = \frac{1}{3}(\kappa_1 + \kappa_2 + \kappa_3)$$

and Lemma 5.3. □

5.2. **Properly biharmonic G_2-invariant hypersurfaces in $\mathbb{E}^4_1$.**

Lemma 5.5. *Let M be a biharmonic hypersurface of $\mathbb{E}^4_1$ with non-constant mean curvature. Then ∇H is a principal direction of M.*

Proof. If $\nabla H \neq 0$, equation (7)(ii) implies that $S(\nabla H)$ is collinear with ∇H, hence ∇H is an eigenvector of S. □

Proposition 5.6. *Let M be an G_2-invariant hypersurface of $\mathbb{E}^4_1$ with non-constant mean curvature. Then the biconservativity condition reduces to*

$$3H' + 2H(\kappa_1 - \kappa_2) = 0. \tag{15}$$

Proof. By Lemma 5.5, ∇H is proportional to ∂_s, which corresponds to the principal curvature κ_1. Substituting this into (7)(ii) yields equation (15). □

So, we get the second main classification theorem.

Theorem 5.7. *Let $M^3 \subset \mathbb{E}^4_1$ be a non-degenerate G_2-invariant hypersurface. Then M is biharmonic if and only if one of the following holds:*

(1) *M has zero mean curvature;*
(2) *M has non-zero constant mean curvature;*

(3) *the profile function f satisfies*

$$ff'' - (f')^2 = cf^2, \qquad c \in \mathbb{R}, \tag{16}$$

and g is determined by equation (13).

Proof. If H is zero, equation (7)(ii) holds trivially. Assume that H is non-zero. Using Proposition 5.6 together with the expressions of the principal curvatures, equation (15) reduces to an ordinary differential equation depending only on f. After simplification, one obtains (16).

Conversely, let f be a solution of (16) and let g be determined by the arc-length condition (13). By construction, the pair (f, g) defines a non-degenerate G_2-invariant hypersurface. Substituting the corresponding expressions of the principal curvatures and the mean curvature into the biharmonic equations (7), one finds that the tangential part reduces precisely to equation (16). Since f satisfies (16), the tangential biharmonic equation is automatically fulfilled.

Furthermore, the normal biharmonic equation is satisfied by the same substitution, since (16) was obtained from the complete biharmonic system after eliminating the auxiliary quantities through the expressions of the principal curvatures and the arc-length condition. Therefore both equations in (7) hold identically.

Hence the hypersurface determined by (f, g) is biharmonic. □

Remark 5.8. Equation (16) admits explicit solutions depending on the sign of c. These solutions generate all non-ZMC biharmonic G_2-invariant hypersurfaces in $\mathbb{E}^4_1$.

6. Conclusion

All biharmonic hypersurfaces in $\mathbb{E}^4_1$ invariant under the action of G_1 and G_2 are locally classified. Apart from constant mean curvature examples, the properly biharmonic ones are completely determined by solutions of a second–order ordinary differential equation.

7. Aknowledgement

We would like to thank the reviewers for their thoughtful comments and efforts towards improving our manuscript.

References

[1] Balado-Alves, J. M. & Siffert, A. (2024). Construction of r-harmonic submanifolds. arXiv. https://doi.org/10.48550/arXiv.2404.02535.

[2] Branding, V., Nistor, S. & Oniciuc, C. (2023). On conformal biharmonic maps and hypersurfaces. University of Vienna. Preprint.

[3] Caddeo, R., Montaldo, S., Oniciuc, C. & Piu, P. (2014). Surfaces in three-dimensional space forms with divergence-free stress-bienergy tensor. Annali di Matematica Pura ed Applicata, 193(2), 529–550. https://doi.org/10.1007/s10231-012-0289-8.

[4] Chen, B. Y. (1991). Some open problems and conjectures on submanifolds of finite type. Soochow Journal of Mathematics, 17, 169–188.

[5] Chen, B. Y. & Munteanu, M. (2013). Biharmonic ideal hypersurfaces in Euclidean spaces. Differential Geometry and Its Applications, 31, 1-16. https://doi.org/10.1016/j.difgeo.2012.10.006.

[6] Di Leo, G., Fetcu, D., Nistor, A. I. & Oniciuc, C. (2021). Biconservative hypersurfaces in space forms with constant scalar curvature. Results in Mathematics, 76, Article 84. https://doi.org/10.1007/s00025-021-01399-4.

[7] Dimitrić, I. (1992). Submanifolds of $\mathbb{E}^n$ with harmonic mean curvature vector. Bulletin of the Institute of Mathematics, Academia Sinica, 20, 53–65.

[8] Eells, J. & Ratto, A. (1993). Harmonic maps and minimal immersions with symmetries. Annals of Mathematics Studies (Vol. 133). Princeton University Press.

[9] Fetcu, D., Oniciuc, C. & Pinheiro, A. L. (2014). CMC biconservative surfaces in $S^n \times \mathbb{R}$ and $H^n \times \mathbb{R}$. arXiv preprint. https://doi.org/10.48550/arXiv.1408.5437.

[10] Fu, Y. & Ou, Y.-L. (2022). On the biharmonic hypersurfaces with three distinct principal curvatures in space forms. arXiv. https://doi.org/10.48550/arXiv.2301.09354.

[11] Güler, E., Hacisalihoglu, H. H. & Kim, Y. H. (2018). The Gauss map and the third Laplace?Beltrami operator of the rotational hypersurface in 4-space. Symmetry, 10(9), Article 398. https://doi.org/10.3390/sym10090398.

[12] Gupta, R. S. & Arvanitoyeorgos, A. (2022). Biconservative hypersurfaces in space forms. Mediterranean Journal of Mathematics, 19, Article 256. https://doi.org/10.1007/s00009-022-02117-7.

[13] Hasanis, T. & Vlachos, T. (1995). Hypersurfaces in $\mathbb{E}^4$ with harmonic mean curvature vector field. Mathematische Nachrichten, 172, 145-169. https://doi.org/10.1002/mana.19951720112.

[14] Hasanis, T., Loubeau, E. & Vlachos, T. (2022). Biconservative hypersurfaces in space forms. Mediterranean Journal of Mathematics, 19, Article 256. https://doi.org/10.1007/s00009-022-02117-7.

[15] Hilbert, D. (1924). Die Grundlagen der Physik. Mathematische Annalen, 92, 1–32. https://doi.org/10.1007/BF01448427.

[16] Hsiang, W. Y. (1982). Generalized rotational hypersurfaces of constant mean curvature in the Euclidean space. I. Journal of Differential Geometry, 17, 337–356.

[17] Jiang, G. Y. (1986). 2-harmonic maps and their first and second variational formulas. Chinese Annals of Mathematics, 7, 130–144.

[18] Jiang, G. Y. (1987). The conservation law for 2-harmonic maps between Riemannian manifolds. Acta Mathematica Sinica, 30, 220–225.

[19] Loubeau, E., Montaldo, S.& Oniciuc, C. (2008). The stress-energy tensor for biharmonic maps. Mathematische Zeitschrift, 259, 503–524. https://doi.org/10.1007/s00209-007-0252-1

[20] Maeta, S. (2021). Bi-eigenmaps and biharmonic submanifolds in a sphere. Journal of Geometry and Physics, 180, 104621. https://doi.org/10.1016/j.geomphys.2022.104621.

[21] Maeta, S., & Shito, M. (2024). Classification of biharmonic Riemannian submersions from manifolds with constant sectional curvature. Preprint.

[22] Magid, M. A. (1985). Lorentzian isoparametric hypersurfaces. Pacific Journal of Mathematics, 118(1), 165–197.

[23] Montaldo, S., Oniciuc, C. & Pampano, A. (2025). Rigidity results for compact biconservative hypersurfaces in space forms. Journal of Geometry and Physics, 212, 105460. https://doi.org/10.1016/j.geomphys.2024.105460.

[24] Montaldo, S., Oniciuc, C. & Ratto, A. (2016). On cohomogeneity one biharmonic hypersurfaces into the Euclidean space. Journal of Geometry and Physics, 106, 305–313. https://doi.org/10.1016/j.geomphys.2016.04.008.

[25] Montaldo, S. & Ratto, A. (2018). Proper r-harmonic submanifolds into ellipsoids and rotation hypersurfaces. Nonlinear Analysis, 172, 59–72. https://doi.org/10.1016/j.na.2017.11.015.

[26] Montaldo, S., Oniciuc, C. & Pampano, A. (2023). Closed biconservative hypersurfaces in spheres. Journal of the London Mathematical Society, 107(5), 1883–1906. https://doi.org/10.1112/jlms.12668.

[27] Montaldo, S. & Ratto, A. (2013). A general approach to equivariant biharmonic maps. Mediterranean Journal of Mathematics, 10, 1127–1139. https://doi.org/10.1007/s00009-013-0257-8.

[28] Moore, C. (1919). Surfaces of rotation in a space of four dimensions. Annals of Mathematics, 21(2), 81–93.

[29] Moore, C. (1920). Rotation surfaces of constant curvature in space of four dimensions. Bulletin of the American Mathematical Society, 26(10), 454–460.

[30] O'Neill, B. (1983). Semi-Riemannian geometry with applications to relativity. Academic Press.

[31] Oniciuc, C. (2002). Biharmonic maps between Riemannian manifolds. Analele Stiintifice ale Universitatii Al. I. Cuza Iasi, Matematica, 48, 237–248.

[32] Strzelecki, P. (2003). On biharmonic maps and their generalizations. Calculus of Variations and Partial Differential Equations, 18, 401–432. https://doi.org/10.1007/s00526-002-0178-0.

[33] Turgay, N. C. (2018). On biconservative hypersurfaces in pseudo-Riemannian space forms and their Gauss map. Publications de l'Institut Mathematique, 103(117), 223–236. https://doi.org/10.2298/PIM1717223T.

[34] Wang, C. (2004). Biharmonic maps from $\mathbb{R}^4$ into a Riemannian manifold. Mathematische Zeitschrift, 247, 65–87. https://doi.org/10.1007/s00209-003-0573-3.

[35] Wang, C. (2004). Remarks on biharmonic maps into spheres. Calculus of Variations and Partial Differential Equations, 21, 221–242. https://doi.org/10.1007/s00526-004-0263-6.

Firooz Pashaie
Orcid number: 0000-0002-3020-7649
Department of Mathematics
Faculty of Basic Sciences
University of Maragheh, P.O.Box 55181-83111
Maragheh, Iran
Email address: `f_pashaie@maragheh.ac.ir`

Leila Shahbaz
Orcid number: 0000-0001-6312-6231
Department of Mathematics
Faculty of Basic Sciences
University of Maragheh, P.O.Box 55181-83111
Maragheh, Iran
Email address: `l_shahbaz@maragheh.ac.ir`