Shahid Bahonar University of Kerman
Journal of Mahani Mathematical Research
Print ISSN: 2251-7952 Online ISSN: 2645-4505

DUALITY ON A CERTAIN CLASS OF ANALYTIC FUNCTIONS RELATED TO PARABOLIC DOMAIN

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Article type: Research Article
(Received: 10 May 2026, Received in revised form 15 June 2026)
(Accepted: 02 July 2026, Published Online: 02 July 2026)

Abstract. In the present paper a certain subclass of analytic functions related to parabolic domains is defined. We characterize duality of the class

$$M(\alpha)=\left\{f\in\mathcal{A}:\mathrm{Re}\{\alpha\frac{f(z)}{z}+(1-\alpha)f'(z)\}>\left|\alpha\frac{f(z)}{z}+(1-\alpha)f'(z)-1\right|,\ z\in\mathbb{D}\right\},$$

which gives some interesting results such as neighborhood, necessary and sufficient condition for normalized analytic function f in open unit disk $\mathbb{D}$ to be in $M(\alpha)$ in terms of convolution.

Keywords: Starlike convex functions, Neighborhoods, Stability, Parabolic domain, Convolution
2020 MSC: 30C45, 30C50.

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions f in open unit disk $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ with the Maclaurin series of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n. \tag{1}$$

The subclass of $\mathcal{A}$ consisting of all univalent functions is denoted by $\mathcal{S}$. The subclasses starlike and convex functions of $\mathcal{S}$ are denoted by $\mathcal{S}^*$ and $\mathcal{K}$ respectively. It is known that

$$f\in\mathcal{S}^*(f\in\mathcal{K})\iff \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0\left(\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0\right)$$

for $z\in\mathbb{D}$. Goodman [7] introduced the classes uniformly starlike and uniformly convex functions as subclasses of starlike and convex functions. A starlike function (or convex function) is said to be uniformly starlike (or uniformly convex) if the image of every circular arc γ contained in $\mathbb{D}$, with center at ζ also in $\mathbb{D}$ is starlike (or convex) with respect to $f(\zeta)$. The class of uniformly starlike functions is represented by UST and the class of uniformly convex functions is represented by UCV. The class of parabolic starlike functions is represented by S_p. Rønning [16] and Ma-Minda [11, 12] independently gave

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https://doi.org/10.22103/jmmr.2026.27156.1966

Publisher: Shahid Bahonar University of Kerman
How to cite: M. Mokhtari Heydari, S. Shams, A. Ebadian, *Duality on a certain class of analytic functions related to parabolic domain*, J. Mahani Math. Res. 2027; 16(1): 199-213.

the characterization for the classes S_p and UCV as follows. A function $f \in \mathcal{A}$ is said to be in the class S_p if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \left|\frac{zf'(z)}{f(z)} - 1\right|, \; z \in \mathbb{D}.$$

A function $f \in \mathcal{A}$ is said to be in the class UCV if and only if

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right|, \; z \in \mathbb{D}.$$

Also, it is clear that

$$f \in UCV \iff zf'(z) \in S_p.$$

In the study of geometric function theory, the concept of conic domains specifically those related to classes of analytic functions represents a sophisticated intersection of classical complex analysis and modern operator theory. By restricting the image of a function $f(z)$ to a specific geometric region in the complex plane, one can classify functions based on their growth, distorsion and boundary behavior. For $0 \leqslant k < \infty$ a function $f \in \mathcal{A}$ is said to be k-uniformly convex in $\mathbb{D}$, if the image of every circular arc γ containd in $\mathbb{D}$, with center ζ where $|\zeta| \leqslant k$, is convex. The class of k-uniformly convex functions is denoted by $k - UCV$. Kanas and Wiśniowska [9] showed that for $f \in \mathcal{A}$ and $0 \leqslant k < \infty$

$$f \in k - UCV \iff \mathrm{Re}\left\{1 + (z - \zeta)\frac{f''(z)}{f'(z)}\right\} \geqslant 0, z \in \mathbb{D}, \; |\zeta| \leqslant k,$$

also for $f \in \mathcal{A}$ and $0 \leqslant k < \infty$

$$f \in k - UCV \iff \mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > k\left|\frac{zf''(z)}{f'(z)}\right|, \; z \in \mathbb{D}.$$

Consider the domain

$$\Omega_k = \{w = u + iv : \; u^2 > k^2(u-1)^2 + k^2v^2\}.$$

For fixed k, Ω_k represents the conic region bounded successively by the imaginary axis ($k = 0$), the right branch of a hyperbola ($0 < k < 1$), a parabola ($k = 1$) and an ellipse ($k > 1$). This domain was studied by Kanas [8, 9, 10]. The function p_k, with $p_k(0) = 1, \; p'_k(0) > 0$ plays the role of extremal and is given by

$$p_k(z) = \begin{cases} \frac{1+z}{1-z}, & k = 0, \\ 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & k = 1, \\ 1 + \frac{2}{1-k^2}\sinh^2[(\frac{2}{\pi}\arccos k)\arctan h\sqrt{z}], & 0 < k < 1, \\ 1 + \frac{1}{k^2-1}\sin[\frac{\pi}{2R(t)}\int_0^{\frac{u(z)}{\sqrt{t}}}\frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}dx] + \frac{1}{k^2-1}, & k > 1, \end{cases}$$

where $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tz}}, \; t \in (0,1), \; z \in \mathbb{D}$ and t is chosen such that $k = \cosh(\frac{\pi R'(t)}{4R(t)})$, with $R(t)$ is Legendre's complete elliptic integral of the first kind and $R(t)$ is the complementary integral of $R(t)$ (see [8, 9]). By definition of the class $k-UCV$, it is clear that

$$f \in k-UCV \iff 1+\frac{zf''(z)}{f'(z)} \in \Omega_k.$$

The class of k-starlike functions is denoted by $k-ST$ and is defined as

$$k-ST=\{f \in \mathcal{S} : f(z)=zg'(z),\ g \in k-UCV\}.$$

Kanas and Wiśniowska [8] showed that for $f \in \mathcal{A}$ and $0 \leqslant k < \infty$,

$$f \in k-ST \iff \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > k\left|\frac{zf'(z)}{f(z)}-1\right|.$$

The generalization of uniformly starlike and convex functions recently were studied by many authors. We refer readers to ([1, 3, 13, 20, 23]).
We recall that the convolution (or Hadamard product) of two functions

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n \text{ and } g(z)=z+\sum_{n=2}^{\infty} b_n z^n$$

is given by

$$(f*g)(z)=z+\sum_{n=2}^{\infty} a_n b_n z^n=(g*f)(z) \quad (z \in \mathbb{D}).$$

Next, here the earlier investigations about neighborhoods, duality and stability by Goodman [7], Ruscheweyh [17], Silverman [22] and Ebadian, et al. [5] (See also [2, 6, 8, 14, 18, 19, 21]) are presented. The T-δ neighborhood of a function $f \in \mathcal{A}$ of the form (1) is defined by:

$$TN_\delta(f)=\left\{g(z)=z+\sum_{n=2}^{\infty} b_n z^n \in \mathcal{A} : \sum_{n=2}^{\infty} |a_n-b_n| T_n \leqslant \delta\right\},$$

where $T=\{T_n\}_{n=1}^{\infty}$ is a sequence of real numbers. The duality technique in the context of univalent functions often refers to the relationships between various classes of functions and their properties. This can involve dual classes, applications of duality. In recent years, there has been significant research into the application of duality techniques to study various subclasses of analytic functions. For our study, we need some definitions due to Ruscheweyh [18].

Definition 1.1. For every $V \subseteq \mathcal{A}$, the dual set of V is denoted by V^* and is defined by

$$V^*=\left\{g \in \mathcal{A} : \frac{(g*f)(z)}{z} \neq 0, \quad z \in \mathbb{D}, f \in V\right\}.$$

Definition 1.2. Let $A, B, C \subseteq \mathcal{A}$ and $A*B \subseteq C$, then the Hadamard product on pair (A, B) is C-invariant if there exists $\delta > 0$ such that $TN_\delta(A)*TN_\delta(B) \subseteq C$.

Definition 1.3. Let $A, B, C \subseteq \mathcal{A}$ and $A * B \subseteq C$. Suppose that Hadamard product on pair (A, B) is C-invariant then

$$\delta(A * B, C) := \sup\{\delta > 0 : TN_\delta(A) * TN_\delta(B) \subseteq C\}$$

is called the invariant radius of Hadamard product.

The integral convolution for $f, g \in \mathcal{A}$ is defined by

$$(f \otimes g)(z) = z + \sum_{n=2}^{\infty} \frac{a_n b_n}{n} z^n \quad (z \in \mathbb{D}).$$

Definition 1.4. Let $A, B, C \subseteq \mathcal{A}$ and $A \otimes B \subseteq C$, then the integral convolution on pair (A, B) is C-invariant if there exists $\delta > 0$ such that $TN_\delta(A) \otimes TN_\delta(B) \subseteq C$.

Definition 1.5. Let $A, B, C \subseteq \mathcal{A}$ and $A \otimes B \subseteq C$. Suppose that integral convolution on pair (A, B) is C-invariant then

$$\delta(A \otimes B, C) := \sup\{\delta > 0 : TN_\delta(A) \otimes TN_\delta(B) \subseteq C\}$$

is called the invariant radius of integral convolution.

Several authors have published papers regarding the properties of analytic functions, focusing on conic domains in complex plane (See [1, 3, 8, 9, 10, 13, 20, 23]). These research efforts have motivated us to introduce the new subclass of analytic functions characterized by the Maclaurin series (1), as presented in the following definition.

Definition 1.6. For $0 \leqslant \alpha \leqslant 1$, we define

$$M(\alpha) = \left\{ f \in \mathcal{A} : \mathrm{Re}\left\{ \alpha \frac{f(z)}{z} + (1-\alpha) f'(z) \right\} > \left| \alpha \frac{f(z)}{z} + (1-\alpha) f'(z) - 1 \right|,\ z \in \mathbb{D} \right\},$$

that is, $\alpha \frac{f(z)}{z} + (1-\alpha) f'(z) \in \Omega$ where Ω is a domain surrounded by parabola, $u - \frac{1}{2} = \frac{1}{2} v^2$.

Note that, for $\alpha = 0$ we get $\mathrm{Re}\{f'(z)\} > |f'(z) - 1|$ which implies that $\mathrm{Re}\{f'(z)\} > \frac{1}{2}$, so f is univalent. Also for $\alpha = 1$ we have $\mathrm{Re}\{\frac{f(z)}{z}\} > |\frac{f(z)}{z} - 1|$ thus $\mathrm{Re}\{\frac{f(z)}{z}\} > \frac{1}{2}$.

Example 1.7. *Let*

$$p_1(z) = 1 + \frac{2}{\pi^2} \left(\log \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2 = 1 + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{2k+1} \right) z^n,$$

where all branches are principle. According to [6, 8, 9] we have

$$f \in M(\alpha) \iff \alpha \frac{f(z)}{z} + (1-\alpha) f'(z) \prec p_1(z),$$

where the symbol $\prec$ shows the notation of subordination. Putting

$$\Psi(z) := \alpha \frac{f(z)}{z} + (1-\alpha) f'(z),$$

we get $\Psi(z) \prec p_1(z)$. *Solving this first order linear differential equation we have*

$$f(z) = \frac{z^{\frac{\alpha}{\alpha-1}}}{1-\alpha}\int_0^z u^{\frac{\alpha}{1-\alpha}}\Psi(u)\,du = \int_0^z u^{\frac{\alpha}{1-\alpha}}p_1(w(u))\,du,$$

where $w(z)$ *is the Schwarz function. In particular if we set* $w(z) = z$ *then we find a sample member of* $M(\alpha)$.

The aim of this paper is to investigate the problem of invariance for the class $M(\alpha)$, determining necessary and sufficient conditions for f to be in $M(\alpha)$. Also for $f \in M(\alpha)$ we show that $f * g$ and $f \otimes g$ are in $M(\alpha)$ where $g \in \mathcal{K}$.

2. Neighborhoods of $M(\alpha)$

In the first theorem we obtain a criterion for f to be in $M(\alpha)$ using Hadamard product.

Theorem 2.1. *For* $0 \leqslant \alpha \leqslant 1$, $f \in M(\alpha)$ *if and only if*

$$\frac{f(z) * \varphi(z)}{z} \neq 0, \quad z \in \mathbb{D} - \{0\},$$

where

$$\varphi(z) = \frac{1}{1-w(t)}\left[\frac{\alpha z}{1-z} + \frac{(1-\alpha)z}{(1-z)^2} - zw(t)\right], \tag{2}$$

with

$$w(t) = \frac{1}{2} + \frac{1}{2}t^2 + it, \quad -\infty < t < \infty.$$

Proof. It is clear that $f \in M(\alpha)$ if and only if $\alpha\frac{f(z)}{z} + (1-\alpha)f'(z) \in G$ where $G = \{u + iv : u - \frac{1}{2} > \frac{1}{2}v^2\}$. Therefore $f \in M(\alpha)$ if and only if $\alpha\frac{f(z)}{z} + (1-\alpha)f'(z) \neq w(t)$ with $w(t) = \frac{1}{2} + \frac{1}{2}t^2 + it, t \in \mathbb{R}$. Finally, making use of the following properties of convolution

$$f(z) = f(z) * \frac{z}{1-z},$$

$$zf'(z) = f(z) * \frac{z}{(1-z)^2},$$

we conclude that $f \in M(\alpha)$ if and only if $\frac{f(z)*\varphi(z)}{z} \neq 0$, $z \in D - \{0\}$ where $\varphi(z) = \frac{1}{1-w(t)}\left[\frac{\alpha z}{1-z} + \frac{(1-\alpha)z}{(1-z)^2} - zw(t)\right]$. The proof is complete. ☐

Remark 2.2. For the function φ given by (2) we can write

$$\varphi(z) = z + \sum_{n=2}^{\infty} e_n z^n = z + \sum_{n=2}^{\infty}\frac{\alpha + (1-\alpha)n}{1-w(t)}z^n.$$

Since the function $g(t) = \left(\frac{1}{2} - \frac{1}{2}t^2\right)^2 + t^2$ attains its minimum at $t = 0$, $g(t) \geqslant g(0) = \frac{1}{4}$, so we get

$$|e_n|^2 = \frac{[\alpha + (1-\alpha)n]^2}{\left(\frac{1}{2} - \frac{1}{2}t^2\right)^2 + t^2} \leqslant 4[\alpha + (1-\alpha)n]^2,$$

which implies that $|e_n| \leqslant 2[\alpha + (1-\alpha)n]$.

Corollary 2.3. *Let* $0 \leqslant \alpha \leqslant 1$ *and*

$$D = \left\{\varphi \in \mathcal{A} : \varphi(z) = \frac{1}{1-w(t)}\left[\frac{\alpha z}{1-z} + \frac{(1-\alpha)z}{(1-z)^2} - zw(t)\right], w(t) = \frac{1}{2} + \frac{1}{2}t^2 + it,\ t \in \mathbb{R}\right\},$$

then $D^* = M(\alpha)$.

Corollary 2.4. *Let* $0 \leqslant \alpha \leqslant 1$ *and* $g(z) = z + Az^n$, *then* $g(z) \in M(\alpha)$ *if and only if* $|A| \leqslant \frac{1}{2[\alpha+(1-\alpha)n]}$.

Proof. Suppose that $\varphi(z) = z + \sum_{n=2}^{\infty} e_n z^n$ is an arbitrary member of D, then

$$\left|\frac{(g * \varphi)(z)}{z}\right| = |1 + e_n A z^{n-1}| \geqslant 1 - |e_n||A||z|^{n-1} > 1 - |e_n||A| \geqslant 0,$$

so $\frac{(g*\varphi)(z)}{z} \neq 0$ and $g \in D^* = M(\alpha)$. Conversely let $g \in M(\alpha)$ and φ be of the form $\varphi(z) = z + \sum_{n=2}^{\infty} 2[\alpha + (1-\alpha)n]z^n$, then $\varphi \in D$ and we have

$$\frac{(g * \varphi)(z)}{z} = 1 + 2A[\alpha + (1-\alpha)n]z^{n-1}.$$

Suppose $|A| > \frac{1}{2[\alpha+(1-\alpha)n]}$, then $z_0 = \frac{-1}{2|A|[\alpha+(1-\alpha)n]} \in \mathbb{D}$ and $\frac{(g*\varphi)(z_0)}{z_0} = 0$. This contradicts Theorem 2.1. $\square$

Theorem 2.5 (Rogosinski's theorem [15]). *Let*

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

be subordinate to

$$H(z) = 1 + \sum_{n=1}^{\infty} d_n z^n.$$

If $H(z)$ *is univalent in* $\mathbb{D}$ *and* $H(\mathbb{D})$ *is convex, then* $|c_n| \leqslant |d_1|$.

Now we find upper bound for coefficients of $f(z)$ of the form (1) to be in $M(\alpha)$. Since

$$f \in M(\alpha) \iff \alpha\frac{f(z)}{z} + (1-\alpha)f'(z) \prec p_1(z),$$

where $p_1(z)$ is given in Example 1.7, by Theorem 2.5, we get

$$|\alpha + (1-\alpha)n||a_n| \leqslant \frac{8}{\pi^2},$$

which implies

$$|a_n| \leqslant \frac{8}{\pi^2(2-\alpha)}. \tag{3}$$

Theorem 2.6. *Let $f \in \mathcal{A}$ and for all complex numbers μ, with $|\mu| < \delta$, if $\frac{f(z)+\mu z}{1+\mu} \in M(\alpha)$ then $TN_\delta(f) \subseteq M(\alpha)$ where*

$$T = \{T_n\}_{n=2}^{\infty} = \{2[\alpha + (1-\alpha)n]\}_{n=2}^{\infty}.$$

Proof. Suppose $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in TN_\delta(f)$, then

$$\begin{aligned} \left|\frac{(g*\varphi)(z)}{z}\right| &\geqslant \left|\frac{(f*\varphi)(z)}{z}\right| - \left|\frac{((g-f)*\varphi)(z)}{z}\right| \\ &= \left|\frac{(f*\varphi)(z)}{z}\right| - \left|\sum_{n=2}^{\infty}(b_n-a_n)e_n z^{n-1}\right| \\ &\geqslant \left|\frac{(f*\varphi)(z)}{z}\right| - 2\sum_{n=2}^{\infty}(\alpha+(1-\alpha)n)|b_n-a_n|. \end{aligned} \tag{4}$$

Since $F_\mu(z) := \frac{f(z)+\mu z}{1+\mu} \in M(\alpha)$, $\frac{1}{z}\left[\frac{f(z)+\mu z}{1+\mu} * \varphi(z)\right] \neq 0, z \in \mathbb{D}$, which is equivalent to $\frac{f(z)*\varphi(z)}{z} \neq -\mu$, $z \in \mathbb{D}$. Consequently we obtain $\left|\frac{(f*\varphi)(z)}{z}\right| > \delta$. In view of (4), we have $\left|\frac{(g*\varphi)(z)}{z}\right| > 0$, and the proof is complete. □

Remark 2.7. Let $|\mu| < \delta$. If $F_\mu \in M(\alpha)$ then for every $\varphi \in D$ we have $\left|\frac{(f*\varphi)(z)}{z}\right| \geqslant \delta$, $z \in \mathbb{D}$.

In order to answer the question $M(\alpha) * \mathcal{K} \subset M(\alpha)$? we need the following result due to Ruscheweyh [19].

Theorem 2.8 ([19])**.** *Let $f \in \mathcal{K}$, $g \in \mathcal{S}^*$ then for every analytic function h in $\mathbb{D}$, $\frac{(f*hg)(\mathbb{D})}{(f*g)(\mathbb{D})} \subseteq \overline{\text{Co}}\ h(D)$, where $\overline{\text{Co}}\ h(D)$ is the closed convex hull of $h(\mathbb{D})$.*

Theorem 2.9. *Let $f \in M(\alpha)$ and $g \in \mathcal{K}$ then $f * g \in M(\alpha)$.*

Proof. It is enough to show that

$$\alpha\frac{(f*g)(z)}{z} + (1-\alpha)(f*g)'(z) \in G,$$

So by the properties of convolution and applying Theorem 2.8, we obtain

$$\begin{aligned} \frac{\alpha(f*g)(z)}{z} + (1-\alpha)(f*g)'(z) =& \frac{g(z)*[\alpha f(z) + (1-\alpha)zf'(z)]}{z} \\ =& \frac{g(z)*[\alpha\frac{f(z)}{z} + (1-\alpha)f'(z)]z}{g(z)*z} \\ \subseteq& \overline{\text{Co}}\ \left[\alpha\frac{f(z)}{z} + (1-\alpha)f'(z)\right](\mathbb{D}) \\ \subseteq& G. \end{aligned}$$

□

Remark 2.10. By Corollary 2.4, the function $f(z) = z+\frac{1}{n+1}z^n$ belongs to $M(\frac{1}{2})$ while convolution with starlike Koebe function $k(z) = \frac{z}{(1-z)^2}$ gives $(f*k)(z) = z + \frac{n}{n+1}z^n$ which is not in $M(\frac{1}{2})$. Therefore Theorem 2.9, does not hold for starlike functions.

Definition 2.11. For $f \in \mathcal{A}$ we say that $f \in N(\alpha)$ if $zf' \in M(\alpha)$.

Theorem 2.12. *If $f \in N(\alpha)$ then $F_\mu \in M(\alpha)$ for $|\mu| < \frac{1}{4}$.*

Proof. It is easy to see that

$$\begin{aligned} F_\mu(z) =& \frac{f(z)+\mu z}{1+\mu} \\ =& f(z) * \left[z + \sum_{n=2}^{\infty} \frac{z^n}{1+\mu}\right] \\ =& f(z) * \frac{z - \frac{\mu}{1+\mu}z^2}{1-z}. \end{aligned}$$

Set $g(z) = \frac{z-\frac{\mu}{1+\mu}z^2}{1-z}$, then differentiating logarithmically we get

$$\frac{zg'(z)}{g(z)} = 1 + \frac{1}{1-z} - \frac{1}{1-\frac{\mu}{1+\mu}z}.$$

Since

$$\mathrm{Re}\left\{\frac{1}{1-\frac{\mu}{1+\mu}z}\right\} \leqslant \left|\frac{1}{1-\frac{\mu}{1+\mu}z}\right| < \frac{1}{1-\frac{|\mu|}{1-|\mu|}},$$

we obtain

$$\mathrm{Re}\left\{\frac{zg'(z)}{g(z)}\right\} > 1 + \frac{1}{2} - \mathrm{Re}\left\{\frac{1}{1-\frac{\mu}{1+\mu}z}\right\}.$$

The last term of above inequality is positive if $|\mu| < \frac{1}{4}$, therefore for $|\mu| < \frac{1}{4}$ the function $g(z)$ is starlike. If ψ is of the form $\psi(z) = \int_0^z \frac{g(t)dt}{t}$, then $\psi \in \mathcal{K}$. Now considering the power series of $g(z)$ at the origin of the form

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n$$

we get $\psi(z) = g(z) * (-\log(1-z))$, making use of convolution properties we have $F_\mu(z) = f(z) * g(z) = zf'(z) * \psi(z)$. Since $zf' \in M(\alpha)$ and $\psi \in \mathcal{K}$ so from Theorem 2.9 we conclude that $F_\mu \in M(\alpha)$. □

Lemma 2.13. *If $f \in N(\alpha)$ and $\varphi \in D$, then $\left|\frac{(f*\varphi)(z)}{z}\right| \geqslant \frac{1}{4}$.*

Proof. Since $f \in N(\alpha)$, Theorem 2.12 gives $F_\mu \in M(\alpha)$ for $|\mu| < \frac{1}{4}$, then for every $\varphi \in D$, $\frac{(F_\mu * \varphi)(z)}{z} \neq 0$, equivalently $\frac{1}{1+\mu}\left[\frac{f(z)*\varphi(z)}{z} + \mu\right] \neq 0$, which gives $\left|\frac{f(z)*\varphi(z)}{z}\right| \geqslant \frac{1}{4}$. □

Theorem 2.14. *Let $f \in M(\alpha)$ and $g \in \mathcal{K}$ then $f \otimes g \in M(\alpha)$.*

Proof. Suppose that f and g are of the forms

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \ (z \in \mathbb{D}).$$

Then

$$(f \otimes g)(z) = f(z) * g(z) * (-\log(1-z)).$$

Since $g(z)$ and $-\log(1-z)$ are convex functions, Theorem 2.9 gives the desired result. □

Theorem 2.15. *For $0 \leqslant \delta \leqslant \frac{\sqrt{(\pi^2(2-\alpha)+8)^2+\pi^4(2-\alpha)}-(\pi^2(2-\alpha)+8)}{\pi^2}$ we have*

$$TN_\delta(M(\alpha)) \otimes TN_\delta(\mathcal{K}) \subseteq M(\alpha).$$

Proof. Let $f_0 \in M(\alpha)$, $g_0 \in \mathcal{K}$, $f \in TN_\delta(M(\alpha))$, $g \in TN_\delta(\mathcal{K})$ and $\varphi \in D$ be of the forms

$$f_0(z) = z+\sum_{n=2}^{\infty} a_n^0 z^n,\ g_0(z) = z+\sum_{n=2}^{\infty} b_n^0 z^n,\ f(z) = z+\sum_{n=2}^{\infty} a_n z^n,\ g(z) = z+\sum_{n=2}^{\infty} b_n z^n$$

and

$$\varphi(z) = z + \sum_{n=2}^{\infty} e_n z^n.$$

Then

$$|a_n^0| \leqslant \frac{8}{\pi^2(2-\alpha)}\ ,\ |b_n^0| \leqslant 1,\ |e_n| \leqslant 2(\alpha+(1-\alpha)n) = T_n, \sum_{n=2}^{\infty} |a_n - a_n^0| T_n \leqslant \delta,$$

and $\sum_{n=2}^{\infty} |b_n - b_n^0| T_n \leqslant \delta$. It is enough to show that

$$\frac{((f \otimes g) * \varphi)(z)}{z} \neq 0, z \in \mathbb{D} - \{0\}.$$

The following inequality is clear

$$\left|\frac{((f \otimes g) * \varphi)(z)}{z}\right| \geqslant \left|\frac{((f_0 \otimes g_0) * \varphi)(z)}{z}\right| - \left|\frac{([f_0 \otimes (g - g_0)] * \varphi)(z)}{z}\right| - \left|\frac{([(f - f_0) \otimes g_0] * \varphi)(z)}{z}\right| - \left|\frac{([(f - f_0) \otimes (g - g_0)] * \varphi)(z)}{z}\right|.$$

The right hand side of the above inequality has four terms. For the first term since $z(f_0 \otimes g_0)'(z) = (f_0 * g_0)(z) \in M(\alpha)$, $(f_0 \otimes g_0)(z) \in N(\alpha)$, so by Lemma 2.13 we obtain

$$\left|\frac{[(f_0 \otimes g_0) * \varphi](z)}{z}\right| \geqslant \frac{1}{4}.$$

For the second term since $f_0 \in M(\alpha)$, $|a_n^0| \leqslant \frac{8}{\pi^2(2-\alpha)}$, therefore

$$\begin{aligned}
\left|\frac{([f_0 \otimes (g-g_0)] * \varphi)(z)}{z}\right| &= \left|\sum_{n=2}^{\infty} \frac{a_n^0(b_n-b_n^0)e_n z^{n-1}}{n}\right| \\
&\leqslant \sum_{n=2}^{\infty} \frac{|a_n^0||(b_n-b_n^0)||e_n||z|^{n-1}}{n} \\
&< \sum_{n=2}^{\infty} \frac{|a_n^0||b_n-b_n^0||e_n|}{n} \\
&\leqslant \frac{4}{\pi^2(2-\alpha)} \sum_{n=2}^{\infty} |b_n-b_n^0| T_n \\
&\leqslant \frac{4\delta}{\pi^2(2-\alpha)}.
\end{aligned}$$

For the third term we get

$$\begin{aligned}
\left|\frac{([(f-f_0) \otimes g] * \varphi)(z)}{z}\right| &= \left|\sum_{n=2}^{\infty} \frac{(a_n-a_n^0)b_n^0 e_n z^{n-1}}{n}\right| \\
&\leqslant \sum_{n=2}^{\infty} \frac{|a_n-a_n^0)||b_n^0||e_n||z|^{n-1}}{n} \\
&< \sum_{n=2}^{\infty} \frac{|a_n-a_n^0||b_n^0||e_n|}{n} \\
&\leqslant \frac{1}{2} \sum_{n=2}^{\infty} |a_n-a_n^0| T_n \\
&\leqslant \frac{\delta}{2}.
\end{aligned}$$

Finally for the last term we obtain

$$\begin{aligned}\left|\frac{([(f-f_0)\otimes(g-g_0)]*\varphi)(z)}{z}\right| &= \left|\sum_{n=2}^{\infty}\frac{(a_n-a_n^0)(b_n-b_n^0)e_n z^{n-1}}{n}\right| \\ &< \sum_{n=2}^{\infty}\frac{|a_n-a_n^0||b_n-b_n^0||e_n|}{n} \\ &\leqslant \frac{1}{2}\sum_{n=2}^{\infty}|a_n-a_n^0||b_n-b_n^0|T_n \\ &\leqslant \frac{\delta}{4(2-\alpha)}\sum_{n=2}^{\infty}|b_n-b_n^0|T_n \\ &\leqslant \frac{\delta^2}{4(2-\alpha)}.\end{aligned}$$

So considering these inequalities we get

$$\left|\frac{((f\otimes g)*\varphi)(z)}{z}\right| \geqslant \frac{1}{4}-\frac{4\delta}{\pi^2(2-\alpha)}-\frac{\delta}{2}-\frac{\delta^2}{4(2-\alpha)}>0,$$

which gives the desired result. □

Corollary 2.16. *Let $f\in M(\alpha)$, then the Alexander operator given by $F(z)=\int_0^z \frac{f(t)}{t}\,dt \quad (z\in\mathbb{D})$ belongs to $M(\alpha)$.*

Proof. If

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n\in M(\alpha),$$

then

$$F(z)=z+\sum_{n=2}^{\infty}\frac{a_n}{n}z^n=f(z)*(-\log(1-z)).$$

Therefore by Theorem 2.9, $F(z)\in M(\alpha)$. □

Theorem 2.17. *For $0<\delta\leqslant\frac{\sqrt{64+4(2-\alpha)\pi^4}-8}{\pi^2}$, we have*

$$TN_\delta(I)\otimes TN_\delta(M(\alpha))\subseteq M(\alpha),$$

where I is the identity map on $\mathbb{D}$.

Proof. Let

$$f_0(z)=I(z)=z,\quad g_0(z)=z+\sum_{n=2}^{\infty}b_n^0 z^n\in M(\alpha),$$

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n\in TN_\delta(f_0),$$

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in TN_\delta(g_0),$$

then

$$\sum_{n=2}^{\infty} 2(\alpha+(1-\alpha)n)|a_n| \leqslant \delta, \quad \sum_{n=2}^{\infty} 2(\alpha+(1-\alpha)n)|b_n - b_n^0| \leqslant \delta \; and \; \sum_{n=2}^{\infty} T_n|a_n| \leqslant \delta.$$

Now for $\varphi(z) = z + \sum_{n=2}^{\infty} e_n z^n \in D$ it is enough to show that $\frac{((f\otimes g)*\varphi)(z)}{z} \neq 0$. It is easy to see that

$$\begin{aligned} \left|\frac{((f\otimes g)*\varphi)(z)}{z}\right| &\geqslant \left|\frac{((f_0\otimes g_0)*\varphi)(z)}{z}\right| - \left|\frac{([f_0\otimes (g-g_0)]*\varphi)(z)}{z}\right| \\ &- \left|\frac{([(f-f_0)\otimes g_0]*\varphi)(z)}{z}\right| - \left|\frac{([(f-f_0)\otimes (g-g_0)]*\varphi)(z)}{z}\right|. \end{aligned} \tag{5}$$

Since $((f_0\otimes g_0)*\varphi)(z) = z,\ ((f_0\otimes (g-g_0))*\varphi)(z) = 0$ and

$$\begin{aligned} \left|\frac{([(f-f_0)\otimes g_0]*\varphi)(z)}{z}\right| &= \left|\sum_{n=2}^{\infty} \frac{a_n b_n^0 e_n z^{n-1}}{n}\right| \leqslant \sum_{n=2}^{\infty} T_n|a_n|\frac{|b_n^0|}{n} \\ &\leqslant \frac{8}{(2-\alpha)\pi^2}\sum_{n=2}^{\infty}\frac{T_n|a_n|}{n} \leqslant \frac{4\delta}{(2-\alpha)\pi^2}. \end{aligned} \tag{6}$$

It is clear that $2(\alpha+(1-\alpha)n)|a_n| \leqslant \delta$, which gives, $|a_n| \leqslant \frac{\delta}{2(2-\alpha)}$ so we get

$$\begin{aligned} \left|\frac{([(f-f_0)\otimes (g-g_0)]*\varphi)(z)}{z}\right| &\leqslant \sum_{n=2}^{\infty}\frac{|a_n||b_n-b_n^0|T_n}{n} \\ &\leqslant \frac{1}{2}\sum_{n=2}^{\infty} T_n|b_n-b_n^0||a_n| \\ &\leqslant \frac{\delta}{4(2-\alpha)}\sum_{n=2}^{\infty} T_n|b_n-b_n^0| \\ &\leqslant \frac{\delta^2}{4(2-\alpha)}. \end{aligned} \tag{7}$$

Using (6) and (7) in (5) we obtain

$$\left|\frac{((f\otimes g)*\varphi)(z)}{z}\right| \geqslant 1 - 0 - \frac{4\delta}{(2-\alpha)\pi^2} - \frac{\delta^2}{4(2-\alpha)}.$$

The right side of the recent inequality is positive and the proof is complete.

Conclusion

In this paper a new class of analytic functions based on conic domains is defined. The concepts of duality, neighborhood and stability for this class are investigated. Cauchy problem is a way to understanding analytic functions and their properties in complex analysis. This problem arises in various contexts like the Cauchy-Riemann equations, integral formulas, and boundary value problems. The concept of generators extends these facts into dynamical systems and stochastic processes, which provide tools for understanding how complex functions evolve over time and how they relate to PDEs and probability distributions. Recently Bracci [4] et al worked on the relation between geometric function theory and semigroup generators as well as filtration on class defined by

$$\mathcal{R}(\alpha) = \left\{ f \in \mathcal{A} : \mathrm{Re}\{\alpha \frac{f(z)}{z} + (1-\alpha) f'(z)\} > 0 \right\}.$$

This leads to an interesting question: how can the properties of one-parameter semigroups, their generators, and filters within class $\mathcal{R}(\alpha)$ be established and analyzed for class $M(\alpha)$?

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