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REFINED-BASED FUNDAMENTAL PYTHAGOREAN FUZZY GRAPHS

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Abstract. This paper introduces the notion of refined Pythagorean fuzzy superhypergraphs (RPFS) as a mathematical model for clustering complex data under uncertainty. We define a fundamental relation β on an RPFS and prove that its quotient is a Pythagorean fuzzy graph (PFG). We also prove that every PFG can be embedded in an RPFS, and any RPFS is a quotientable PFG. A combinatorial formula is proven for the number of distinct RPFSs on any given set and for complete-based PFGs. This study presents the link-degree and superedge-degree points and investigates the relation of link-degree in fundamental PFG and superedge-degree in any related RPFS. Furthermore, it is analyzed that the direct product of two RPFSs is again an RPFS, and its quotient equals the direct product of their quotients. In the categorical property, it is established that the set of all RPFSs forms a commutative monoid under the direct product operation. Finally, we considered the relation between fuzzy data clustering and RPFS, and clustered, modeled, and combined some real-world problems via RPFS and the concept of its direct product.

Keywords: Refined RPFS, Fundamental (complete-based) PFG, link-degree, superedge-degree.
2020 MSC: 03B52, 54C70, 94A17.

1. Introduction

Superhypergraphs have emerged as a powerful hypernetwork, incorporating supervertices to represent clusters of entities and superedges to model intricate relationships between these clusters. Indeed, a superhypergraph is an extension of a graph such that if in a graph at most two elements can be connected at a time, in a superhypergraph several groups of elements can be connected in the form of general-general, partial-general, and partial-partial correspondences. Since mathematical structures and hyperstructures in an abstract and pure form cannot be applied in a world full of complex problems, mixing pure concepts with labels and value identifiers can be more useful in real problems. On the other hand, in the real world, not every value is exact, and its value can be presented in a vague, indefinite, uncertain, and approximate form, so

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fuzzy set theory is of great importance in imprecise and uncertain problems. Therefore, in modeling everyday problems, the integration and combination of mathematical modeling tools such as superhypergraphs and fuzzy concepts are of great importance and application. This paper aims to introduce the notion of refined Pythagorean fuzzy superhypergraphs as a mathematical model for real-world problems. Refined Pythagorean fuzzy superhypergraphs are retained as a tool to cluster and organize the primary data of any real-world problem based on its properties. A refined Pythagorean fuzzy superhypergraph consists of three components: Pythagorean fuzzy supervertices, Pythagorean fuzzy superedges, and Pythagorean fuzzy links, each with unique properties. The primary data is placed within Pythagorean fuzzy supervertices, which are related via Pythagorean fuzzy links. A connection exists between any two Pythagorean fuzzy supervertices in the form of a Pythagorean fuzzy superedge, formed based on the properties of the two supervertices. Consequently, when a connection needs to be established between groups of complex elements, Pythagorean fuzzy superhypergraphs become important for achieving suitable clustering. In this manner, they play a central role in the optimal clustering of complex data, making the analysis of the degree of Pythagorean fuzzy supervertices crucial. We characterize the degree of supervertices via superedges and links. Given that Pythagorean fuzzy graphs have established applications, we aim to create a connection between Pythagorean fuzzy superhypergraphs and Pythagorean fuzzy graphs. For this, we introduce a fundamental equivalence relation β, such that the quotient of any Pythagorean fuzzy superhypergraph under β is a Pythagorean fuzzy graph. The notions of quotientable Pythagorean fuzzy graphs and fundamental Pythagorean fuzzy graphs are defined. This research presents the concept of isomorphism in Pythagorean fuzzy superhypergraphs and proves that any Pythagorean fuzzy superhypergraph is a quotientable Pythagorean fuzzy graph, and any quotientable Pythagorean fuzzy graph is isomorphic to a Pythagorean fuzzy superhypergraph. This study establishes a functor between the category of Pythagorean fuzzy superhypergraphs and the category of Pythagorean fuzzy graphs. To extend the class of Pythagorean fuzzy superhypergraphs, we define their direct product and prove that the direct product of two Pythagorean fuzzy superhypergraphs is itself a Pythagorean fuzzy superhypergraph. The quotient of this direct product under the fundamental relation β is investigated, and we prove that the fundamental direct product of Pythagorean fuzzy graphs is isomorphic to the direct product of the related fundamental Pythagorean fuzzy graphs. Finally, we present the notions of link-degree and superedge-degree points and discuss the link between the degree measure in Pythagorean fuzzy superhypergraphs and that in Pythagorean fuzzy graphs to highlight the role of fuzzy analysis.

Notable applications by researchers such as Hamidi et al. include investigations into the spectra of superhypergraphs via flows [7] and decision-making frameworks based on valued fuzzy superhypergraphs [9]. Despite these advancements, the data underlying such networks often lacks precision, being marked by uncertainty, ambiguity, and conflicting information. Fuzzy set theory and its extensions, including Pythagorean fuzzy sets (PFS) introduced by Yager [17], have successfully addressed imprecision within graph theory. Today, some researchers have investigated the works on graphs, hypergraphs, and fuzzy subsets that refer to some of the related papers to our research. Related works include algorithms for computing Pythagorean fuzzy average edge connectivity of Pythagorean fuzzy graphs [1], interval intuitionistic neutrosophic sets with its applications to interval intuitionistic neutrosophic graphs and climatic analysis [11], product of interval-valued fuzzy graphs and degree [16], vertex connectivity of fuzzy graphs with applications to human trafficking [3], multi-attribute decision-making and interval-valued picture (S, T)-fuzzy graphs [4], application of superhypergraphs-based domination number in real world [8], domination in Pythagorean fuzzy graphs [5], multi-criteria group decision making based on graph neural networks in Pythagorean fuzzy environment [13], an algorithm to compute the strength of competing interactions in the Bering Sea based on Pythagorean fuzzy hypergraphs [14] and interval-valued picture fuzzy hypergraphs with application towards decision making [12].

The research gap identified from this literature is that no existing mathematical tool combines hierarchical clustering, group-level connections, internal cluster structure, and Pythagorean uncertainty. The present paper tries to fill these problems.

Motivation: To improve and optimize any multivariate and multicriteria problem in today's real world, we need to design and classify each problem with a group model and then use this model to optimize the problem using mathematical tools. Grouping and clustering problems require powerful mathematical tools that fully examine the part-to-part, part-to-whole, and whole-to-whole relationships of the problems. Real-world complex networks such as scientific collaborations, supply chains, engineering sciences, social hypernetworks, economic networks, cultural networks, commercial networks, and financial networks involve hierarchical clusters and uncertain information that traditional (hyper)graphs cannot model. Existing fuzzy (hyper)graphs handle only pairwise relationships, while crisp superhypergraphs cannot express degrees of membership and non-membership. Pythagorean fuzzy sets offer greater flexibility than intuitionistic fuzzy sets, but no framework currently combines hierarchical clustering, group-level connections, internal cluster structure, and Pythagorean uncertainty. Therefore, we introduce refined Pythagorean fuzzy superhypergraphs (RPFS) as a generalization of fuzzy (hyper)graphs to fill these gaps, enabling optimal clustering and risk-reliability analysis in complex uncertain networks.

2. Preliminaries

This section revisits key definitions and results that will be instrumental in the discussions that follow. Throughout the paper, X represents a non-empty set.

Definition 2.1. [6] Let X be a finite set. A hypergraph on X is a family $H = (X, \{E_i\}_{i=1}^{m})$, such that for all $1 \leq i \leq m, \emptyset \neq E_i \subseteq X$ and $\bigcup_{i=1}^{m} E_i = X$.

Definition 2.2. [2] Let $m, n \in \mathbb{N}$. Then an (m, n)-fuzzy subset on non-empty set X is a set $E = \{(x, \mu_E(x), \nu_E(x)) \mid x \in X\}$, where $\mu_E, \nu_E, X \to [0, 1]$ and for every $x \in X$ $x \in E, 0 \leq \mu_E^m(x) + \nu_E^n(x) \leq 1$.

Definition 2.3. [17] A Pythagorean fuzzy subset of $X(PFS)$, is an ordered pair $P = (\mu_P, \nu_P)$, that $\mu_P, \nu_P : X \to [0, 1]$ and for any $x \in X, \mu_P^2(x) + \nu_P^2(x) \leq 1$.

From now on, will denote any PFS, $P = (\mu_P, \nu_P)$ by $P \in PFS(X)$ for simplify.

Definition 2.4. [15] A PFS Q in $X \times X$ is said to be a Pythagorean fuzzy relation (PFR) in X, denoted by $Q = \{(xy, \mu_Q(xy), \nu_Q(xy)) \mid xy \in X \times X\}$. A Pythagorean fuzzy graph (PFG) on a non-empty set X is a pair $G = (P, Q)$, where P is a PFS on X and Q is a PFR on X such that $\mu_Q(xy) \leq \min\{\mu_P(x), \mu_P(y)\}$ and $\nu_Q(xy) \geq \max\{\nu_P(x), \nu_P(y)\}$.

Definition 2.5. [15] The degree and total degree of a vertex $x \in V$ in a PFG G is defined as $d_G(x) = (d_\mu(x), d_\nu(x))$ and $\text{td}_G(x) = (\text{td}_\mu(x), \text{td}_\nu(x))$, respectively, where $d_\mu(x) = \sum_{x,y\neq x\in V} \mu_Q(xy)$, $\quad d_\nu(x) = \sum_{x,y\neq x\in V} \nu_Q(xy)$, $\text{td}_\mu(x) = \sum_{x,y\neq x\in V} \mu_Q(xy) + \mu_P(x)$, $\quad \text{td}_\nu(x) = \sum_{x,y\neq x\in V} \nu_Q(xy) + \nu_P(x)$.

Definition 2.6. [7] Let $X \neq \emptyset$. Then

- (i) $H = (X, S = \{S_i\}_{i=1}^{k}, \Phi = \{\varphi_{i,j}\}_{i,j})$ is a quasi-superhypergraph (QSH), if $\Phi \neq \emptyset$ and $X = \bigcup_{i=1}^{n} S_i$, where $k \geq 2$, and $S_i \in P^*(X)$ is a "supervertex," and for every pair $i \neq j$, the mapping (link) $\varphi_{i,j} : S_i \to S_j$, is a "superedge."
- (ii) A QSH H becomes a superhypergraph (SH), if for any $S_i \in P^*(X)$, there exists at least one $S_j \in P^*(X)$ that S_i has a link to S_j. However, not all supervertices are required to be interconnected.

Definition 2.7. [10] Assume that $\mathcal{V} = \{P_i = (\mu_{P_i}, \nu_{P_i})\}_{i=1}^{v}, \mathcal{E} = \{Q_{ij} = (\mu_{Q_{i,j}}, \nu_{Q_{ij}})\}_{i,j\geq 1}^{e}, \mathcal{D} = \{R_i = (\mu_{R_i}, \nu_{R_i})\}_{i=1}^{d}$ be a finite family of non-trivial PFS of X. Then $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ is a Pythagorean fuzzy quasi superhypergraph ($PFQS$) on X, if $X = \bigcup_{i=1}^{v} P_i^*$.

(i) $\forall(x,y) \in R_i^*, \mu_{R_i}((x,y)) \leq \min\{\mu_{P_i}(x), \mu_{P_i}(y)\}$ and $\nu_{R_i}((x,y)) \geq \max\{\nu_{P_i}(x), \nu_{P_i}(y)\}$,

(ii) $\forall(x,y) \in Q_{ij}^*, \mu_{Q_{ij}}((x,y)) \leq \min\{\frac{\mu_{P_i}(x)}{w(P_i)}, \frac{\mu_{P_j}(y)}{w(P_j)}\}$ and $\nu_{Q_{ij}}((x,y)) \geq \max\{\frac{\nu_{P_i}(x)}{w(P_i)}, \frac{\nu_{P_j}(y)}{w(P_j)}\}$,

where $w(P_i) = \sum\limits_{x \in P_i^*} (\mu_{P_i} + \nu_{P_i})(x)$. We will call $\mathcal{V}$ by Pythagorean fuzzy supervertices, $\mathcal{E}$ by Pythagorean fuzzy superedges and $\mathcal{D}$ by Pythagorean fuzzy links. Will call a PFQS $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ on X as a Pythagorean fuzzy superhypergraph ($PFSh$) on X, if $|\mathcal{V}| \leq |\mathcal{E}|$ and for any $1 \leq i \leq d, |R_i^*| \geq 1$. If for any $1 \leq i \leq v, w(P_i) < 1$, call H is a *Soft* $PFSh$, if for any $1 \leq i \leq v, w(P_i) > 1$, call H is a *Rough* $PFSh$ and if for any $1 \leq i \leq v, w(P_i) = 1$, call H is a *Smooth* $PFSh$. A PFQS $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ is *Coverable*, if any member of $\mathcal{E}$ is an onto map.

3. On refined Pythagorean fuzzy superhypergraph

In this section, we introduce the notion of *refined* Pythagorean fuzzy superhypergraph on any non-empty set and investigate the relation between refined Pythagorean fuzzy superhypergraphs and Pythagorean fuzzy graphs. The notions of quotientable Pythagorean fuzzy graphs and fundamental Pythagorean fuzzy graphs are defined, and it is proved that any refined Pythagorean fuzzy superhypergraph is a quotientable Pythagorean fuzzy graph.

Let $P = (\mu_P, \nu_P)$ be a PFS of X, then define support of P, by $P^* = \{x \in X \mid \mu_P(x) > 0, \nu_P(x) < 1\}$.

Definition 3.1. Let $\mathcal{V} = \{P_i = (\mu_{P_i}, \nu_{P_i})\}_{i=1}^v, \mathcal{E} = \{Q_{ij} = (\mu_{Q_{ij}}, \nu_{Q_{ij}})\}_{i,j\geq 1}^e, \mathcal{D} = \{R_i = (\mu_{R_i}, \nu_{R_i})\}_{i=1}^d$ be finite families of non-trivial PFS of X. Then $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ is a refined Pythagorean fuzzy superhypergraph ($RPFS$) on X, if $X = \bigcup_{i=1}^v P_i^*$, for all $i \neq j, P_i \cap P_j = \emptyset$ and for any,

(i) $xy := (x,y) \in R_i^*$, $\mu_{R_i}(xy) \leq \min\{\mu_{P_i}(x), \mu_{P_i}(y)\}$ and $\nu_{R_i}(xy) \geq \max\{\nu_{P_i}(x), \nu_{P_i}(y)\}$,

(ii) $xy := (x,y) \in Q_{ij}^*$, $\mu_{Q_{ij}}(xy) \leq \min\{\mu_{P_i}(x), \mu_{P_j}(y)\}$ and $\nu_{Q_{ij}}(xy) \geq \max\{\nu_{P_i}(x), \nu_{P_j}(y)\}$.

For $1 \leq i \leq v$, if $|P_i^*| \geq 2$, then $|P_i^*| - 1 \leq |R_i^*| \leq \binom{|P_i^*|}{2}$. In addition, for any $1 \leq i \leq v$ must exist some $1 \leq j \leq v$, such that $Q_{ij}^* \subseteq P_i^* \times P_j^*$ or $Q_{ji}^* \subseteq P_j^* \times P_i^*$ and if $|\mathcal{V}| = 1$, say that $RPFS$ is singleton. We will call $\mathcal{V}$ by Pythagorean fuzzy supervertices, $\mathcal{E}$ by Pythagorean fuzzy superedges and $\mathcal{D}$ by Pythagorean fuzzy links and $H^* = (\mathcal{V}^*, \mathcal{E}^*, \mathcal{D}^*)$.

We better illustrate this definition in the following example.

Example 3.2. *Let* $X = \{x_i\}_{i=1}^{7}$. *Then* $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ *is a RPFS on* X *as Figure 1, where* $\mathcal{V} = \{P_1, P_2, P_3\}, \mathcal{E} = \{Q_{12}, Q_{31}\}$ *and* $\mathcal{D} = \{R_1, R_2, R_3\}$.

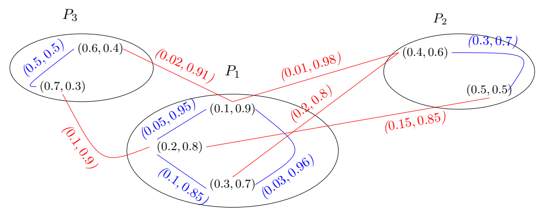

Figure 1. RPFS $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$

Indeed,

$$
\begin{aligned}
&P_1 = \{(x_1, 0.1, 0.9), (x_2, 0.2, 0.8), (x_3, 0.3, 0.7)\},\\
&P_2 = \{(x_4, 0.4, 0.6), (x_5, 0.5, 0.5)\},\\
&P_3 = \{(x_6, 0.6, 0.4), (x_7, 0.7, 0.3)\},\\
&Q_{12} = \{(x_1x_4, 0.01, 0.98), (x_2x_5, 0.15, 0.85),\\
&\qquad (x_3x_4, 0.2, 0.8)\},\\
&Q_{31} = \{(x_6x_1, 0.02, 0.91), (x_7x_2, 0.1, 0.9)\},\\
&R_1 = \{(x_1x_2, 0.05, 0.95), (x_2x_3, 0.1, 0.85),\\
&\qquad (x_1x_3, 0.03, 0.96)\},\\
&R_2 = \{(x_4x_5, 0.3, 0.7)\},\\
&R_3 = \{(x_6x_7, 0.5, 0.5)\}.
\end{aligned}
$$

Let X be a non-empty set and $\Phi = \{(x, \mu_{E_i}(x), \nu_{E_i}(x)) \mid x \in X\}_{i=1}^{m}$ be a family of (m, n)-fuzzy subsets of X such that $\bigcup_{i=1}^{m} supp(\mu_{E_i}, \nu_{E_i}) = X$. If $m = n = 1$, then Φ is called an intuitionistic fuzzy subset, and for $m = n = q \geq 1$, it is called a q-rung orthopair fuzzy subset, especially for $m = n = 2$, it is called a Pythagorean fuzzy subset. The general structure of intuitionistic fuzzy hypergraphs, and q-rung orthopair fuzzy hypergraphs are similar, and are different in degree of membership and the degree of non-membership of the element. So for instance, we compare intuitionistic fuzzy hypergraphs with $RPFS$. If $m = n = 1$, then $H = (X, \Phi)$ is called an intuitionistic fuzzy hypergraph on X, that $\bigcup_{i=1}^{m} supp(\mu_{E_i}, \nu_{E_i}) = X$. In any intuitionistic fuzzy hypergraph, we can analyze the intuitionistic fuzzy vertices belonging to an intuitionistic fuzzy hyperedge, but we can not investigate the intuitionistic fuzzy vertices that belong to an intuitionistic fuzzy hyperedge with another intuitionistic fuzzy hyperedge that does not belong to it, partially or generally. But in any $RPFS$, apart from that $m = n = 2$, we can analyze the partial and general relation between Pythagorean fuzzy supervertices, Pythagorean fuzzy superedges, and Pythagorean fuzzy links(this is established in the items $(i), (ii)$ of Definition 3.1). The capability allows modeling and solving real-world problems whose elements are completely dependent, both in part and in whole, with this tool. In addition, other algebraic and analytical properties, including dimension, degree, path complement, connectivity, and other properties, differ from each other. Based the definitions of $RPFS$ and PFG, have the following proposition.

Proposition 3.3. *Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be a $RPFS$ on X. Then for any $1 \leq i \leq |\mathcal{V}|, G_i = (P_i, R_i)$ is a PFG on R_i^*.*

For any given non-empty set X, consider, $\mathbb{RPFS}(X) = \{H \mid H \text{ is an RPFS on } X\}$, and $\mathbb{RPFS}^*(X) = \{H^* \mid H \in \mathbb{RPFS}(X)\}$, then have the following result.

Theorem 3.4. *Let $\emptyset \neq X$. Then*

(i)

$$|\mathbb{RPFS}^*(X)| = \sum_{e=v-1}^{2^{v(v-1)}} \left(\sum_{i,j\geq 1}^{e} \left[\left(\sum_{k=|P_i^*|-1}^{\binom{|P_i^*|}{2}} \binom{\binom{|P_i^*|}{2}}{k} \right) \cdot \left(\sum_{k=|P_j^*|-1}^{\binom{|P_j^*|}{2}} \binom{\binom{|P_j^*|}{2}}{k} \right) \cdot |P_i^*|^{|P_j^*|} \right] \right).$$

(ii) $|\mathbb{RPFS}(X)| = \mathfrak{c} = |([0,1]^3)|$.

Proof. (i) Let $\mathcal{V} = \{P_i = (\mu_{P_i}, \nu_{P_i})\}_{i=1}^{v}, \mathcal{E} = \{Q_{ij} = (\mu_{Q_{ij}}, \nu_{Q_{ij}})\}_{i,j\geq 1}^{e}, \mathcal{D} = \{R_i = (\mu_{R_i}, \nu_{R_i})\}_{i=1}^{d}$ and $H = (\mathcal{V}, \mathcal{E}, \mathcal{D}) \in \mathbb{RPFS}(X)$ be arbitrary. Then, for any $1 \leq i \leq d, R_i$ is a Pythagorean fuzzy link, and by definition, $|P_i^*| - 1 \leq |R_i^*| \leq \binom{|P_i^*|}{2}$. Hence, for any $1 \leq i \leq v$, get $|R_i^*| = \sum_{k=|P_i^*|-1}^{\binom{|P_i^*|}{2}} \binom{\binom{|P_i^*|}{2}}{k}$, because $|R_i^*|$ is the number of support of Pythagorean fuzzy graphs on P_i^*. In addition, for any $1 \leq i, j \leq v, |Q_{ij}| = |P_j^*|^{|P_i^*|}$ and $|Q_{ji}| = |P_i^*|^{|P_j^*|}$, because $|Q_{ij}|$ is the number of maps from Pythagorean fuzzy supervertex P_i to Pythagorean fuzzy supervertex P_j. Moreover, for any $1 \leq i \leq v$ there exists $1 \leq j \leq v$, such that $Q_{ij}^* \subseteq P_i^* \times P_j^*$. Hence, for any $1 \leq i, j \leq v$, the number of choosing the R_i^* is equal to $\sum_{k=|P_i^*|-1}^{\binom{|P_i^*|}{2}} \binom{\binom{|P_i^*|}{2}}{k}$, the number of choosing the R_j^* is equal to

$\displaystyle\sum_{k=|P_j^*|-1}^{\binom{|P_j^*|}{2}} \binom{\binom{|P_j^*|}{2}}{k}$ and so the number of choosing the R_i^* and R_j^* is equal to $\displaystyle(\sum_{k=|P_i^*|-1}^{\binom{|P_i^*|}{2}} \binom{\binom{|P_i^*|}{2}}{k})(\sum_{k=|P_j^*|-1}^{\binom{|P_j^*|}{2}} \binom{\binom{|P_j^*|}{2}}{k})$. In addition, the number of choosing of Q_{ij}^* from R_i^* to R_j^* is equal to $|P_j^*|^{|P_i^*|}$, so the number of cases from P_i to P_j is equal to $\alpha_{ij} = \displaystyle(\sum_{k=|P_i^*|-1}^{\binom{|P_i^*|}{2}} \binom{\binom{|P_i^*|}{2}}{k})(\sum_{k=|P_j^*|-1}^{\binom{|P_j^*|}{2}} \binom{\binom{|P_j^*|}{2}}{k})|P_j^*|^{|P_i^*|}$. Thus for any given $H \in \mathbb{RPFS}(X)$, have

$\displaystyle\sum_{i,j\geq 1}^{e} \alpha_{ij} = \sum_{i,j\geq 1}^{e}(\sum_{k=|P_i^*|-1}^{\binom{|P_i^*|}{2}} \binom{\binom{|P_i^*|}{2}}{k})(\sum_{k=|P_j^*|-1}^{\binom{|P_j^*|}{2}} \binom{\binom{|P_j^*|}{2}}{k})|P_j^*|^{|P_i^*|}$, cases for the choosing of H^*. In addition, for any $H \in \mathbb{RPFS}(X), v-1 \leq e \leq 2^{v(v-1)}$, thus

$$|\mathbb{RPFS}^*(X)| = \sum_{e=v-1}^{2^{v(v-1)}}\Bigg(\sum_{i,j\geq 1}^{e}\Bigg[\left(\sum_{k=|P_i^*|-1}^{\binom{|P_i^*|}{2}} \binom{\binom{|P_i^*|}{2}}{k}\right)\cdot\left(\sum_{k=|P_j^*|-1}^{\binom{|P_j^*|}{2}} \binom{\binom{|P_j^*|}{2}}{k}\right)\cdot|P_i^*|^{|P_j^*|}\Bigg]\Bigg). \tag{1}$$

(ii) Let $\mathcal{V} = \{P_i = (\mu_{P_i}, \nu_{P_i})\}_{i=1}^{v}, \mathcal{E} = \{Q_{ij} = (\mu_{Q_{ij}}, \nu_{Q_{ij}})\}_{i,j\geq 1}^{e}, \mathcal{D} = \{R_i = (\mu_{R_i}, \nu_{R_i})\}_{i=1}^{d}$ and $H = (\mathcal{V}, \mathcal{E}, \mathcal{D}) \in \mathbb{RPFS}(X)$ be arbitrary. Since $\mu_{P_i}, \nu_{P_i}, \mu_{Q_{ij}}, \nu_{Q_{ij}}, \mu_{R_i}, \nu_{R_i} \in FS(X)$, get $|\mathbb{RPFS}(X)| = \mathfrak{c} = |([0,1]^3)|$. $\square$

Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be a *RPFS* on X. We say that H is *quotientable PFG*, if there exists an equivalence relation R on X such that H/R can be a PFG. From now on, we will call R a fundamental relation and H/R is *fundamental PFG*.

Theorem 3.5. *Any RPFS is a quotientable PFG.*

Proof. Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be a *RPFS* on X. For any $x, y \in X$, define a relation β on X, by $x\beta y$ if and only if there exists a $P_i \in \mathcal{V}$ such that $\{x, y\} \subseteq P_i^*$. Then β is an equivalence relation on X and denote by $X/\beta = \{\beta(x) \mid x \in X\}$ and $X/\beta = (\mathcal{V}/\beta, \mathcal{E}/\beta)$, that $\mathcal{V}/\beta = \{\beta(x) \mid x \in P_i^*\}$ and $\mathcal{E}/\beta = \{\beta(xy) \mid xy \in P_i^* \times P_j^*\}$. For any $P_i \in \mathcal{V}$, define $(P_i/\beta)(\beta(x)) = (\bigwedge\limits_{y\beta x} \mu_{P_i}(y), \bigwedge\limits_{y\beta x} \nu_{P_i}(y))$ and

$(Q_{ij}/\beta)(\beta(x)\beta(x')) = (\bigwedge_{y\beta x, y'\beta x'} \mu_{Q_{ij}}(yy'), \bigwedge_{y\beta x, y'\beta x'} \nu_{Q_{ij}}(yy'))$. Now consider, $P/\beta = \{P_i/\beta \mid 1 \leq i \leq |P_i|\}$ and $Q/\beta = \{Q_{ij}/\beta \mid 1 \leq i \neq j \leq |P_i|\}$ and show that $H/\beta = (P/\beta, Q/\beta)$ is a PFG on X/β. For any $x \in X$, since $x\beta x$:

$$\mu^2_{P/\beta}(\beta(x)) + \nu^2_{P/\beta}(\beta(x)) = (\bigwedge_{y\beta x} \mu_{P_i}(y))^2 + (\bigwedge_{y\beta x} \nu_{P_i}(y)))^2 =$$

$$(\bigwedge_{y\beta x} \mu^2_{P_i}(y)) + (\bigwedge_{y\beta x} \nu^2_{P_i}(y))) \leq \mu^2_{P_i}(x) + \nu^2_{P_i}(x) \leq 1, and$$

$$\mu^2_{Q/\beta}(\beta(x)\beta(y)) + \nu^2_{Q/\beta}(\beta(x)\beta(y))$$

$$= (\bigwedge_{x'\beta x, y'\beta y} \mu_{Q_{ij}}(x'y'))^2 + (\bigwedge_{x'\beta x, y'\beta y} \nu_{Q_{ij}}(x'y'))^2 =$$

$$\bigwedge_{x'\beta x, y'\beta y} \mu^2_{Q_{ij}}(x'y') + \bigwedge_{x'\beta x, y'\beta y} \nu^2_{Q_{ij}}(x'y') \leq \mu^2_{Q_{ij}}(xy) + \nu^2_{Q_{ij}}(xy) \leq 1.$$

Now, for any $\beta(x), \beta(y) \in X/\beta$,

$$\mu_{Q/\beta}(\beta(x)\beta(y)) = \bigwedge_{x'\beta x, y'\beta y} \mu_{Q_{ij}}(x'y') \leq \bigwedge_{x'\beta x, y'\beta y} \mu_{P_i}(x') \wedge \mu_{P_j}(y')$$

$$= \bigwedge_{x'\beta x} \mu_{P_i}(x') \wedge \bigwedge_{y'\beta y} \mu_{P_j}(y') =$$

$$\mu_{P/\beta}(\beta(x)) \wedge \mu_{P/\beta}(\beta(x)) \ and$$

$$\nu_{Q/\beta}(\beta(x)\beta(y)) = \bigwedge_{x'\beta x, y'\beta y} \nu_{Q_{ij}}(x'y') \geq \bigwedge_{x'\beta x, y'\beta y} \nu_{P_i}(x') \vee \mu_{P_j}(y')$$

$$= \bigwedge_{x'\beta x} \nu_{P_i}(x') \vee \bigwedge_{y'\beta y} \mu_{P_j}(y') =$$

$$\nu_{P/\beta}(\beta(x)) \vee \nu_{P/\beta}(\beta(x)).$$

Thus $H/\beta = (P/\beta, Q/\beta)$ is a PFG on X/β. □

Example 3.6. *Consider the RPFS, H in Figure 1. Then by Theorem 3.5, H/β is a PFG as Figure 2.*

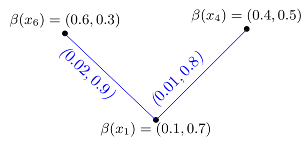

FIGURE 2. Fundamental PFG H/β.

Let $\emptyset \neq X$. Consider $\mathbb{PFG}(X) = \{H \mid H$ is a PFG on $X\}$ and a relation $\Phi \subseteq \mathbb{PFG} \times \mathbb{RPFS}$. Then have the following results.

Theorem 3.7. *Let $\emptyset \neq X$. Then for any $H \in \mathbb{PFG}(X)$, there exists at least $H' \in \mathbb{RPFS}(X)$ that $H \Phi H'$.*

Proof. Let $G = (P, Q)$ be a PFG on $\varnothing \neq X$. Then, $P = \{(x, \mu_P(x), \nu_P(x)) \mid x \in P^*\}$ and $Q = \{(xy, \mu_Q(xy), \nu_Q(xy)) \mid xy := (x, y) \in Q^* \times Q^*\}$, that $P, Q \in PFS(X)$ and $\mu_Q(xy) \leq \min\{\mu_P(x), \mu_P(y)\}$ and $\nu_Q(xy) \geq \max\{\nu_P(x), \nu_P(y)\}$, that $xy \in E \subseteq X \times X$, represents an edge between vertices x and y. Since X is finite set, then there exist a family of non-empty subsets $\{X_i\}_{i=1}^n$ of X such that $X = \bigcup\limits_{i=1}^{n} X_i$ and for any $i \neq j, X_i \cap X_j = \emptyset$. Now, consider $\mathcal{V} = \{P_i = (\mu_{P_i}, \nu_{P_i})\}_{i=1}^n$, that $P_i = \{(x_i, \mu_{P_i}(x_i), \nu_{P_i}(x_i)) \mid x_i \in X_i\}$ and $\mathcal{E} = \{Q_{ij} = (\mu_{Q_{ij}}, \nu_{Q_{ij}})\}_{i,j\geq 1}^e$, that $e \leq n-1$ and $Q_{ij} = \{(x_i x_j, \mu_{Q_{ij}}(x_i x_j), \nu_{Q_{ij}}(x_i x_j)) \mid x_i \in X_i, x_j \in X_j, |X_i| \geq |X_j|\}$ and for any $xy := (x, y) \in P_i^* \times P_j^*, \mu_{Q_{ij}}(xy) = \mu_Q(xy)$ and $\nu_{Q_{ij}}(xy) = \nu_Q(xy)$. Also, set $\mathcal{D} = \{R_i = (\mu_{R_i}, \nu_{R_i})\}_{i=1}^d$, that $d \leq n$ and for any $xy \in P_i^* \times P_i^*, \mu_{R_i}(xy) = \mu_Q(xy)$ and $\nu_{R_i}(xy) = \nu_Q(xy)$. Now, $X = \bigcup\limits_{i=1}^{v} P_i^* = \bigcup\limits_{i=1}^{n} X_i = X$, for all $i \neq j, P_i \cap P_j = X_i \cap X_j = \emptyset$. In addition,

(i) for all $xy := (x, y) \in P_i^* \times P_i^*$,
$\mu_{R_i}(xy) = \mu_Q(xy) \leq \min\{\mu_{P_i}(x), \mu_{P_i}(y)\}$ and
$\nu_{R_i}(xy) = \nu_{R_i}(xy) \geq \max\{\nu_{P_i}(x), \nu_{P_i}(y)\}$,

(ii) for all $xy := (x, y) \in P_i^* \times P_j^*$,
$\mu_{Q_{ij}}(xy) = \mu_Q(xy) \leq \min\{\mu_{P_i}(x), \mu_{P_j}(y)\}$ and
$\nu_{Q_{ij}}(xy) = \nu_Q(xy) \geq \max\{\nu_{P_i}(x), \nu_{P_j}(y)\}$.

$H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ is a $RPFS$ on X. □

By the following example, one can see that $\Phi : \mathbb{PFG} \to \mathbb{RPFS}$ is not a map, necessarily.

Example 3.8. *Consider the PFG $G = (P, Q)$ on $X = \{a, b, c, d, e, f\}$ as Figure 3.*

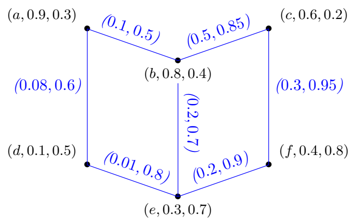

FIGURE 3. PFG $G = (P, Q)$.

Based Theorem 3.7, obtain the $RPFSs$ $H_1 = (\mathcal{V}_1, \mathcal{E}_1, \mathcal{D}_1)$ and $H_2 = (\mathcal{V}_2, \mathcal{E}_2, \mathcal{D}_2)$, as Figures 4, 5.

Corollary 3.9. *Let $\emptyset \neq X$. Then any PFG on X can be a $RPFS$ on X.*

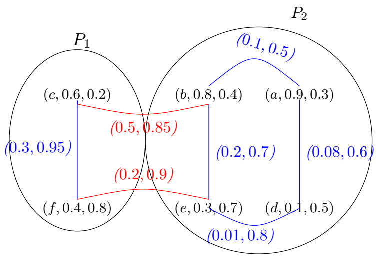

FIGURE 4. RPFS $H_1 = (\mathcal{V}_1, \mathcal{E}_1, \mathcal{D}_1)$.

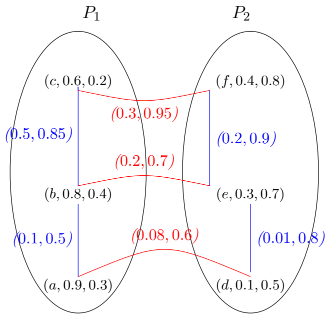

FIGURE 5. RPFS $H_2 = (\mathcal{V}_2, \mathcal{E}_2, \mathcal{D}_2)$.

Let $G = (P, Q)$ be a PFG. Will call G is complete-based, if $G^* = (P^*, Q^*)$ is a complete graph. Consider the $RPFS$, $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$, which $\mathcal{V} = \{P_i = (\mu_{P_i}, \nu_{P_i})\}_{i=1}^{v}, \mathcal{E} = \{Q_{i,j} = (\mu_{Q_{i,j}}, \nu_{Q_{ij}})\}_{i,j\geq 1}^{e}, \mathcal{D} = \{R_i = (\mu_{R_i}, \nu_{R_i})\}_{i=1}^{d}$.

In what follow, for any $m \in \mathbb{N}$, B_m means the Bell number for $m \in \mathbb{N}$.

Theorem 3.10. *Let $G = (P, Q)$ be a complete-based. Then $1 \leq |Ran(\Phi)| \leq \beta$, where $P^* = \bigcup_{i=1}^{m} P_i^*$,*

$$\beta = (B_{|P^*|})\Big(\sum_{e=m-1}^{2^{m(m-1)}} \Big(\sum_{i,j\geq 1}^{e} \Big[\sum_{k=|P_i^*|-1}^{\binom{|P_i^*|}{2}} \binom{\binom{|P_i^*|}{2}}{k}\Big)\Big(\sum_{k=|P_j^*|-1}^{\binom{|P_j^*|}{2}} \binom{\binom{|P_j^*|}{2}}{k}\Big)|P_i^*|^{|P_j^*|}\Big]\Big)\Big)\Big),$$

and $n_1, n_2, \ldots, n_m$ are the block sizes of P^* and α_i is the number of blocks of size i.

Proof. Since $G = (P, Q)$ is a PFG, get $P, Q \in PFS(X)$ and $\mu_Q(xy) \leq \min\{\mu_P(x), \mu_P(y)\}$ and $\nu_Q(xy) \geq \max\{\nu_P(x), \nu_P(y)\}$. If $xy \in Q^*$, then $\mu_Q(xy) > 0$ and $\nu_Q(xy) < 1$, follow that $\min\{\mu_P(x), \mu_P(y)\} > 0$ and $\max\{\nu_P(x), \nu_P(y)\} < 1$. Hence $xy \in P^* \times P^*$, and so $Q^* \subseteq P^* \times P^*$. Because, $G = (P, Q)$ is a complete-based, get $G^* = (P^*, Q^*)$ is a complete graph. Then for given PFG, $G = (P, Q)$, using Theorem 3.7, there exists at least $H \in \mathbb{RPFS}(X)$ that $G\Phi H$. Suppose that $X = \{X_i\}_{i=1}^{m}$ is an arbitrary partition of P^* that for any $1 \leq i \leq m, |X_i| = n_i$. Then for any $1 \leq i \leq m$, the number of cases for choosing X is equal to $\dfrac{\binom{|P^*|}{n_1}\binom{|P^*|-n_1}{n_2}\cdots\binom{|P^*|-n_1-\cdots-n_{m-1}}{n_m}}{\alpha_1!\,\alpha_2!\cdots\alpha_{|P^*|}!}$, where any α_i is a block with i-order. Assume $\mathcal{P}(P^*) = \{X \mid X \text{ is a partation of } P^*\}$, then $|\mathcal{P}(P^*)| = \sum_{\sum_{i=1}^{m} n_i = |P^*|} \frac{\binom{|P^*|}{n_1}\binom{|P^*|-n_1}{n_2}\cdots\binom{|P^*|-n_1-\cdots-n_{m-1}}{n_m}}{\alpha_1!\,\alpha_2!\,\cdots\,\alpha_{|P^*|}!}$. Now, consider the $RPFS$, $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ that $H = \Phi(G)$ and $\mathcal{V}^* = \{P_i^*\}_{i=1}^m$. Then by Theorem 3.4, $|\mathbb{RPFS}^*(X)| =$

$$(\sum_{e=m-1}^{2^{m(m-1)}} \Bigg(\sum_{i,j\geq 1}^{e} \Bigg[\sum_{k=|P_i^*|-1}^{\binom{|P_i^*|}{2}} \binom{\binom{|P_i^*|}{2}}{k}\Bigg)(\sum_{k=|P_j^*|-1}^{\binom{|P_j^*|}{2}} \binom{\binom{|P_j^*|}{2}}{k}\Bigg)|P_i^*|^{|P_j^*|}\Bigg]\Bigg))),$$ and so

$\beta =$

$$(B_{|P^*|})(\sum_{e=m-1}^{2^{m(m-1)}} \Bigg(\sum_{i,j\geq 1}^{e} \Bigg[\sum_{k=|P_i^*|-1}^{\binom{|P_i^*|}{2}} \binom{\binom{|P_i^*|}{2}}{k}\Bigg)(\sum_{k=|P_j^*|-1}^{\binom{|P_j^*|}{2}} \binom{\binom{|P_j^*|}{2}}{k}\Bigg)|P_i^*|^{|P_j^*|}\Bigg]\Bigg))),$$

where $n_1, n_2, \ldots, n_m$ are the block sizes of P^*, α_i are the number of blocks of size i and $B_{|P^*|} = (\sum_{\sum_{i=1}^{m} n_i = |P^*|} \frac{\binom{|P^*|}{n_1}\binom{|P^*|-n_1}{n_2}\cdots\binom{|P^*|-n_1-\cdots-n_{m-1}}{n_m}}{\alpha_1!\,\alpha_2!\,\cdots\,\alpha_{|P^*|}!})$. $\square$

In general, we leave this as an open problem for interested researchers to solve it. Let $G = (P, Q)$ be an arbitrary PFG. Then $1 \leq |Ran(\Phi)| \leq ?$

Definition 3.11. Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be a $RPFS$ on X and $x \in X$. Define the

(i) link-degree of $x \in P_i^*$, by $deg_{\mathcal{V}}(x) = (deg(x, \mathcal{V}, \mu_{P_i}), deg(x, \mathcal{V}, \nu_{P_i})$ that $deg(x, \mathcal{V}, \mu_{P_i}) = \sum_{x\neq y\in P_i^*} \mu_{R_i}(xy)$ and $deg(x, \mathcal{V}, \mu_{P_i}) = \sum_{x\neq y\in P_i^*} \nu_{R_i}(xy)$,

(ii) superedge-degree of $x \in P_i^*$, by $deg_{\mathcal{E}}(x) = (deg(x, \mathcal{E}, \mu_{P_i}), deg(x, \mathcal{E}, \nu_{P_i})$ that $deg(x, \mathcal{E}, \mu_{P_i}) = \sum_{x\neq y\in P_j^*} \mu_{Q_{ij}}(xy)$ and

$deg(x, \mathcal{E}, \nu_{P_i}) = \sum_{x\neq y\in P_j^*} \nu_{Q_{ij}}(xy)$,

(iii) the degree of H by, $deg(H) = \sum_{x\in X}(deg_{\mathcal{V}}(x) + deg_{\mathcal{E}}(x))$.

Based on the example below, we emphasize that the degree of each vertex does not have to be smaller than 1.

Example 3.12. *For the RPFS, H in Figure 1, can see that* $deg_{\mathcal{V}}(x_1) = (0.08, 1.91)$ *and* $deg_{\mathcal{E}}(x_1) = (0.03, 1.89)$.

Definition 3.13. Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ and $H' = (\mathcal{V}', \mathcal{E}', \mathcal{D}')$ be two $RPFS$ on X and X', respectively. Then a map $\varphi : X \to X'$, is called a homomorphism, if

(i) for any $x \in P_i^*, \mu_{P_i}(x) \leq \mu_{P_i'}(\varphi(x))$ and $\nu_{P_i}(x) \geq \nu_{P_i'}(\varphi(x))$,

(ii) for any $xy := (x, y) \in R_i^*, \mu_{R_i}(x) \leq \mu_{R_i'}(\varphi(x))$ and $\nu_{R_i}(x) \geq \nu_{R_i'}(\varphi(x))$,

(iii) for any $xy := (x, y) \in {Q_{ij}}^*, \mu_{Q_{ij}}(x) \leq \mu_{Q_{ij}'}(\varphi(x))$ and $\nu_{Q_{ij}}(x) \geq \nu_{Q_{ij}'}(\varphi(x))$.

Also H, H' are isomorphic (denote by $H \cong H'$), if the items $(i), (ii), (iii)$, be equality and $\varphi : X \to X'$ be a bijection.

Based the definition have the following proposition.

Proposition 3.14. *Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ and $H' = (\mathcal{V}', \mathcal{E}', \mathcal{D}')$ be two isomorphic RPFS on X and X', respectively. If $H \cong H'$, then $deg(H) = deg(H')$.*

Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ and $H' = (\mathcal{V}', \mathcal{E}', \mathcal{D}')$ be two $RPFS$ on X and X', respectively. Assume β_H and $\beta'_{H'}$ be the defined fundamental relations (based Theorem 3.5), on H, H', respectively, so have the following results.

Theorem 3.15. *Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ and $H' = (\mathcal{V}', \mathcal{E}', \mathcal{D}')$ be two isomorphic RPFS on X and X', respectively. If $H \cong H'$, then $H/\beta_H \cong H'/\beta_{H'}$.*

Proof. Since $H \cong H'$, there exists a bijection $\varphi : X \to X'$. Now, define a map $\overline{\varphi} : X/\beta_H \to X'/\beta_{H'}$ by $\overline{\varphi}(\beta_H(x)) = \beta_{H'}(\varphi(x))$, that $x \in X$. Assume $\beta_H(x) = \beta_H(y)$, there exists a $P_i \in \mathcal{V}$ such that $\{x, y\} \subseteq P_i$. Hence, there exists a $P'_i \in \mathcal{V}'$ such that $\{\varphi(x), \varphi(y)\} \subseteq P'_i$, and so $\beta_{H'}(\varphi(x)) = \beta_{H'}(\varphi(y))$. It follows that $\overline{\varphi}$ is well-defined. If $\beta_{H'}(\varphi(x)) = \beta_{H'}(\varphi(y))$, then there exists a $P'_i \in \mathcal{V}'$ such that $\{\varphi(x), \varphi(y)\} \subseteq P'_i$. Since φ is a bijection, $\{x, y\} = \{\varphi^{-1}(\varphi(x)), \varphi^{-1}(\varphi(y))\} \subseteq \varphi^{-1}(P'_i) \in \mathcal{V}$. Thus $\beta_H(x) = \beta_H(y)$, and so $\overline{\varphi}$ is one to one. Clearly, $\overline{\varphi}$, hence, $\overline{\varphi}$ is a bijection. Let $\beta(x) \in P^*_i/\beta$. Since for any $x, y \in P^*_i$, $(\mu_{P_i}(x), \nu_{P_i}(x)) = (\mu_{P'_i}(\varphi(x)), \nu_{P'_i}\varphi(x)))$, get

$$
\begin{aligned}
&(\mu_{\frac{P_i}{\beta_H}}(\beta_H(x)), \nu_{\frac{P_i}{\beta_H}}(\beta_H(x))) = (\bigwedge_{y\beta x} \mu_{P_i}(y), \bigwedge_{y\beta x} \nu_{P_i}(y))) = \\
&(\bigwedge_{\varphi(y)\beta\varphi(x)} \mu_{P'_i}(\varphi(y)), \bigwedge_{\varphi(y)\beta\varphi(x)} \nu_{P'_i}(\varphi(y)))) \\
&= (\mu_{P'_i/\beta_{H'}}(\beta_{H'}(\varphi(\beta(x))), \nu_{P'_i/\beta_{H'}}(\beta_{H'}(\varphi(\beta(x))) = \\
&(\mu_{P'_i/\beta_{H'}}(\overline{\varphi}(\beta_H(x))), \nu_{P'_i/\beta_{H'}}(\overline{\varphi}(\beta_H(x))).
\end{aligned}
$$

Let $\beta(x)\beta(z) := (\beta(x), \beta(z)) \in P_i^*/\beta \times P_j^*/\beta$. Since for any $xy := (x, y) \in P_i^* \times P_j^*, (\mu_{Q_{ij}}(xy), \nu_{Q_{ij}}(xy)) = (\mu_{Q'_{ij}}(\varphi(x)\varphi(y)), \nu_{Q'_{ij}}(\varphi(x)\varphi(y)))$, get

$$\begin{aligned}
&\Big(\mu_{Q_{ij}/\beta_H}(\beta_H(x)\beta_H(z)), \nu_{Q_{ij}/\beta_H}(\beta_H(x)\beta_H(z))\Big)\\
&= \Big(\bigwedge_{y\beta x, w\beta z} \mu_{Q_{ij}}(yw), \bigwedge_{y\beta x, w\beta z} \nu_{Q_{ij}}(yw)\Big)\\
&= \Big(\bigwedge_{\varphi(y)\beta\varphi(x), \varphi(w)\beta\varphi(z)} \mu_{Q'_{ij}}(\varphi(y)\varphi(w)),\\
&\quad \bigwedge_{\varphi(y)\beta\varphi(x), \varphi(w)\beta\varphi(z)} \nu_{Q'_{ij}}(\varphi(y)\varphi(w))\Big)\\
&= \Big(\mu_{Q'_{ij}/\beta_{H'}}\big(\beta_{H'}(\varphi(\beta(x)))\beta_{H'}(\varphi(\beta(z)))\big),\\
&\quad \nu_{Q'_{ij}/\beta_{H'}}\big(\beta_{H'}(\varphi(\beta(x)))\beta_{H'}(\varphi(\beta(z)))\big)\Big)\\
&= \Big(\mu_{Q'_{ij}/\beta_{H'}}\big(\overline{\varphi}(\beta_H(x))\overline{\varphi}(\beta_H(z))\big), \nu_{Q'_{ij}/\beta_{H'}}\big(\overline{\varphi}(\beta_H(x))\overline{\varphi}(\beta_H(z))\big)\Big).
\end{aligned}$$

□

Theorem 3.16. *Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be a RPFS on X and $x \in X$. Then for all $y \in \beta(x)$ have, $deg_{\mathcal{V}/\beta}(\beta(x)) \le deg_{\mathcal{E}}(y)$.*

Proof. Let $x \in X$. Then define
$deg_{\mathcal{V}/\beta}(\beta(x)) = (\sum_{\beta(x)\neq\beta(y)\in\mathcal{V}/\beta} \mu_{Q/\beta}(\beta(x)\beta(y)), \sum_{\beta(x)\neq\beta(y)\in\mathcal{V}/\beta} \nu_{Q/\beta}(\beta(x)\beta(y)))$.
Assume that $\beta(x) \in \mathcal{V}/\beta$. Because $x\beta x$, and $\beta(x) \neq \beta(y)$, implies that $x \neq y$, get

$$\begin{aligned}
&\sum_{\beta(x)\neq\beta(y)\in\mathcal{V}/\beta} \mu_{Q/\beta}(\beta(x)\beta(y)) = \sum_{\beta(x')\neq\beta(y')\in\mathcal{V}/\beta}\ \bigwedge_{x'\beta x, y'\beta y} \mu_{Q_{ij}}(x'y')\\
&\le \sum_{\beta(x)\neq\beta(y)\in\mathcal{V}/\beta} \mu_{Q_{ij}}(xy) \le \sum_{x\neq y\in X} \mu_{Q_{ij}}(xy).
\end{aligned}$$

In similar to,

$$\begin{aligned}
&\sum_{\beta(x)\neq\beta(y)\in\mathcal{V}/\beta} \nu_{Q/\beta}(\beta(x)\beta(y)) = \sum_{\beta(x')\neq\beta(y')\in\mathcal{V}/\beta}\ \bigwedge_{x'\beta x, y'\beta y} \nu_{Q_{ij}}(x'y')\\
&\le \sum_{x\neq y\in X} \nu_{Q_{ij}}(xy).
\end{aligned}$$

Hence,

$$\begin{aligned}&\Big(\sum_{\beta(x)\neq\beta(y)\in\mathcal{V}/\beta}\mu_{Q/\beta}(\beta(x)\beta(y)),\sum_{\beta(x)\neq\beta(y)\in\mathcal{V}/\beta}\nu_{Q/\beta}(\beta(x)\beta(y))\Big)\\&\leq\Big(\sum_{x\neq y\in X}\mu_{Q_{ij}}(xy),\sum_{x\neq y\in X}\nu_{Q_{ij}}(xy)\Big),\end{aligned}$$

and so $deg_{\mathcal{V}/\beta}(\beta(x))\leq deg_{\mathcal{E}}(x)$. □

3.1. **Direct product in category of $RPFS$.** In this subsection, we introduce the notion of direct product on the category of refined Pythagorean fuzzy superhypergraphs and investigate the properties of link-degree and superedge-degree in direct product on refined Pythagorean fuzzy superhypergraphs and related fundamental Pythagorean fuzzy graphs.

Definition 3.17. Let $H=(\mathcal{V},\mathcal{E},\mathcal{D})$ and $H'=(\mathcal{V}',\mathcal{E}',\mathcal{D}')$ be two $RPFS$ on X and X', respectively. Define the direct product $H\times H'=(\mathcal{V}\times\mathcal{V}',\mathcal{E}\times\mathcal{E}',\mathcal{D}\times\mathcal{D}')$, by $(\mu_P\times\mu_{P'})(x,x')=\mu_P(x)\wedge\mu_{P'}(x')$, $(\nu_P\times\nu_{P'})(x,x')=\nu_P(x)\vee\nu_{P'}(x')$, $(\mu_R\times\mu_{R'})(xx',yy')=\mu_R(xy)\wedge\mu_{R'}(x'y')$, $(\nu_R\times\nu_{R'})(xx',yy')=\nu_R(xy)\vee\nu_{R'}(x'y')$, $(\mu_Q\times\mu_{Q'})(xx',yy')=\mu_Q(xy)\wedge\mu_{Q'}(x'y')$ and $(\nu_Q\times\nu_{Q'})(xx',yy')=\nu_Q(xy)\vee\nu_{Q'}(x'y')$.

Based the definition, have the following result.

Theorem 3.18. *Let $H=(\mathcal{V},\mathcal{E},\mathcal{D}),H'=(\mathcal{V}',\mathcal{E}',\mathcal{D}')$ and $H''=(\mathcal{V}'',\mathcal{E}'',\mathcal{D})''$ be $RPFS$ on X,X' and X'', respectively. Then*

(i) *$H\times H'=(\mathcal{V}\times\mathcal{V}',\mathcal{E}\times\mathcal{E}',\mathcal{D}\times\mathcal{D}')$ is an $RPFS$ on $X\times X'$.*
(ii) *$H\times H'\cong H'\times H$.*
(iii) *$(H\times H')\times H''\cong H\times(H'\times H'')$.*
(iv) *For any $1\leq i\leq|\mathcal{V}|,1\leq i'\leq|\mathcal{V}'|,G_i\times G'_i=(P_i\times P'_{i'},R_i\times R'_{i'})$ is a PFG on $P_i^*\times {P'}_{i'}^{*}$.*

Example 3.19. *Consider the RPFS's, H,H' in Figures 6, 7.*

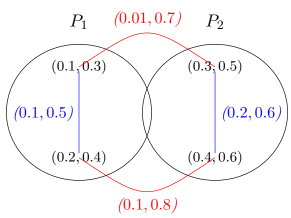

Figure 6. H

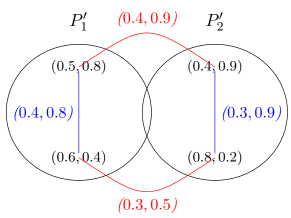

Figure 7. H'

Indeed,

$P_1 = \{(x_1, 0.1, 0.3), (x_2, 0.2, 0.4)\}, P_2 = \{(x_3, 0.3, 0.5), (x_4, 0.4, 0.6)\},$
$P_1' = \{(x_5, 0.5, 0.8), (x_6, 0.6, 0.4)\}, Q_{12} = \{(x_1x_3, 0.01, 0.7), (x_2x_4, 0.1, 0.8)\},$
$Q_{12}' = \{(x_5x_7, 0.4, 0.9), (x_6x_8, 0.3, 0.5)\}, R_1 = \{(x_1x_2, 0.1, 0.5)\},$
$R_2 = \{(x_3x_4, 0.2, 0.6)\}, R_3 = \{(x_5x_6, 0.4, 0.8)\}, R_4 = \{(x_7x_8, 0.5, 0.9)\}.$

Based definition, obtain the RPFS $H \times H'$ as Figure 8.

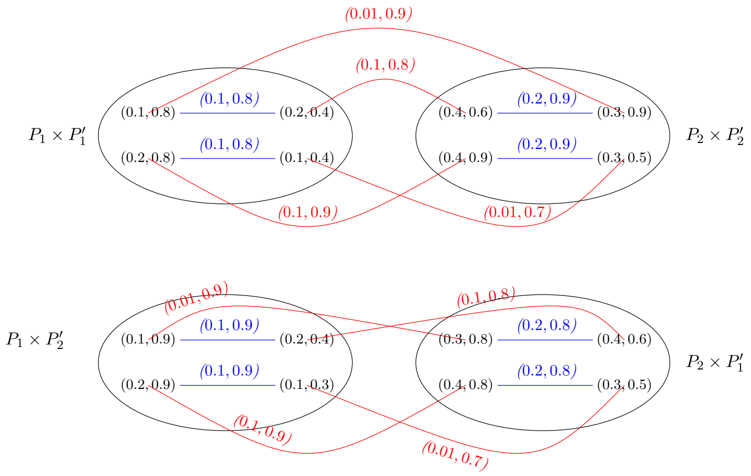

FIGURE 8. RPFS, $H \times H'$.

Indeed,

$P_1 \times P_1' = \{(x_1x_5, 0.1, 0.8), (x_2x_6, 0.2, 0.4), (x_2x_5, 0.2, 0.8), (x_1x_6, 0.1, 0.4)\},$
$P_2 \times P_2' = \{(x_3x_7, 0.3, 0.9), (x_4x_8, 0.4, 0.6), (x_4x_7, 0.4, 0.9), (x_3x_8, 0.3, 0.5)\}$
$P_1 \times P_2' = \{(x_1x_7, 0.1, 0.9), (x_2x_8, 0.2, 0.4), (x_2x_7, 0.2, 0.9), (x_1x_8, 0.1, 0.3)\},$
$P_2 \times P_1' = \{(x_3x_5, 0.3, 0.8), (x_4x_6, 0.4, 0.6), (x_4x_5, 0.4, 0.8), (x_3x_6, 0.3, 0.5)\}$
$R_1 \times R_1' = \{((x_1x_5, x_2x_6), 0.1, 0.8), ((x_2x_5, x_1x_6), 0.1, 0.8)\},$
$R_2 \times R_2' = \{((x_3x_7, x_4x_8), 0.2, 0.9), ((x_4x_7, x_3x_8), 0.2, 0.9)\},$
$R_1 \times R_2' = \{((x_1x_7, x_2x_8)0.1, 0.9), ((x_2x_7, x_1x_8)0.1, 0.9)\},$
$R_2 \times R_1' = \{((x_3x_5, x_4x_6)0.2, 0.8), ((x_4x_5, x_3x_6)0.2, 0.8)\},$
$Q_{1,2} \times Q_{1,2}' = \{((x_1x_5, x_3x_7), 0.01, 0.9), ((x_2x_6, x_4x_8), 0.1, 0.8),$
$((x_2x_5, x_4x_7), 0.1, 0.9), ((x_1x_6, x_3x_8), 0.01, 0.7), \},$
$Q_{1,2} \times Q_{2,1}' = \{((x_1x_7, x_3x_5), 0.01, 0.9), ((x_2x_8, x_4x_6), 0.1, 0.8),$
$((x_2x_7, x_4x_5), 0.1, 0.9), ((x_1x_8, x_3x_6), 0.01, 0.7), \}.$

Let $X = \{x_0\}$. For the singleton $RPFS$, $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$, define the $RPFS$, $\triangle = (\mathcal{V}_\triangle, \mathcal{E}_\triangle, \mathcal{D}_\triangle)$, by $P_\triangle = \{(x_0, 1, 0)\}$ and the following theorem is helpful in the category of Pythagorean fuzzy superhypergraphs(Objects are Pythagorean fuzzy superhypergraphs and Morphisms are homomorphisms).

Theorem 3.20. *Let $X \neq \emptyset$. Then*

(*i*) $(\mathbb{RPFS}(X), \times, \triangle)$ *is a commutative monoid.*

(*ii*) *For any $H \in \mathbb{RPFS}(X)$, there exists a unique homomorphism $\Theta : H \to \triangle$.*

Proof. (*i*) Let $H, H', H'' \in \mathbb{RPFS}(X)$. Based Theorem 3.18, $H \times H' \in \mathbb{RPFS}(X), H \times H' \cong H' \times H$ and $(H \times H') \times H'' \cong H \times (H' \times H'')$, so $(\mathbb{RPFS}(X), \times)$ is a commutative monoid. Clearly $\mathcal{V}^* \times \{x_0\} \cong \mathcal{V}^*, \mathcal{D}^* \times \{x_0x_0\} \cong \mathcal{D}^*$ and $\mathcal{E}^* \times \{(x_0, x_0)\} \cong \mathcal{E}^*$, so $H^* \times \triangle \cong H^*$. In addition, for any $(x, x_0) \in P_i^* \times P_\triangle^*, (xx', x_0x_0) \in R_i^* \times R_\triangle^*, (xx', x_0x_0) \in {Q_{ij}}^* \times Q_\triangle^*$,

$$\begin{aligned}
\mu_{P_i \times P_\triangle}((x, x_0)) &= \mu_{P_i}(x) \wedge 1 = \mu_{P_i}(x),\\
\nu_{P_i \times P_\triangle}((x, x_0)) &= \nu_{P_i}(x) \wedge 0 = \nu_{P_i}(x),\\
\mu_{R_i \times R_\triangle}((xx', x_0x_0)) &= \mu_{R_i}(xx') \wedge 1 = \mu_{R_i}(xx'),\\
\nu_{R_i \times R_\triangle}((xx', x_0x_0)) &= \nu_{R_i}(xx') \wedge 0 = \nu_{R_i}(xx'),\\
\mu_{Q_{ij} \times Q_\triangle}((x, x_0)) &= \mu_{Q_{ij}}(x) \wedge 1 = \mu_{Q_{ij}}(x),\\
\nu_{Q_{ij} \times Q_\triangle}((x, x_0)) &= \nu_{Q_{ij}}(x) \wedge 0 = \nu_{Q_{ij}}(x).
\end{aligned}$$

Therefore, $H \times \triangle \cong H$.

(*ii*) Define $\Theta : H \to \triangle$ by $\Theta(x) = x_0$, where $x \in X$. Clearly, Θ is a homomorphism. Assume $\Lambda : H \to \triangle$ is a homomorphism. Since for any $x \in P_i^*, \Lambda(x) = x_0$, for any $xx' \in R_i^*, \Lambda(xx') = x_0x_0$ and for any $xx' \in {Q_{ij}}^*, \Lambda(xx') = x_0x_0$, get $\Lambda = \Theta$ and so Θ is unique. □

Definition 3.21. Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ and $H' = (\mathcal{V}', \mathcal{E}', \mathcal{D}')$ be two $RPFS$ on X and X', respectively. then for any $(x, x') \in X \times X'$, define

$$, deg(xx', \mathcal{V} \times \mathcal{V}', \mu_{P_i} \times \mu_{P_i'}) = \sum_{x \neq x', y \neq y' \in X} (\mu_{R_i} \times \mu_{R_i})(xx', yy'),$$

$$deg(xx', \mathcal{V} \times \mathcal{V}', \nu_{P_i} \times_{P_i'}) = \sum_{x \neq x', y \neq y' \in X} (\nu_{R_i} \times \nu_{R_i})(xx', yy'),$$

$$deg(xx', \mathcal{E} \times \mathcal{E}', \mu_{P_i} \times \mu_{P_i'}) = \sum_{x \neq x', y \neq y' \in X} (\mu_{Q_{ij}} \times \mu_{Q_{ij}'})(xx', yy'),$$

$$deg(xx', \mathcal{E} \times \mathcal{E}', \nu_{P_i} \times_{P_i'}) = \sum_{x \neq x', y \neq y' \in X} (\nu_{Q_{ij}} \times \nu_{Q_{ij}'})(xx', yy').$$

In what follow, investigate the link-degree and superedge-degree in the product of $RPFS$'s based Definition 3.21.

Theorem 3.22. *Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ and $H' = (\mathcal{V}', \mathcal{E}', \mathcal{D}')$ be two $RPFS$ on X and X', respectively. Then for any $x \in P_i^*, x' \in {P_i'}^*$:*

(i) $deg(xx', \mathcal{V} \times \mathcal{V}', \mu_{P_i} \times \mu_{P'_i}) \leq \bigwedge\{deg(x, \mathcal{V}, \mu_{P_i}), deg(x', \mathcal{V}', \mu_{P'_i})\}$.

(ii) $deg(xx', \mathcal{V} \times \mathcal{V}', \nu_{P_i} \times_{P'_i}) \geq \bigvee\{deg(x, \mathcal{V}, \nu_{P_i}), deg(x', \mathcal{V}', \nu_{P'_i})\}$.

(iii) $deg(xx', \mathcal{E} \times \mathcal{E}', \mu_{P_i} \times \mu_{P'_i}) \leq \bigwedge\{deg(x, \mathcal{E}, \mu_{P_i}), deg(x', \mathcal{E}', \mu_{P'_i})\}$.

(iv) $deg(xx', \mathcal{E} \times \mathcal{E}', \nu_{P_i} \times_{P'_i}) \geq \bigvee\{deg(x, \mathcal{E}, \nu_{P_i}), deg(x', \mathcal{E}', \nu_{P'_i})\}$.

Proof. (i) Let $x \in P_i^*, x' \in P_i'^*$. Then

$$\begin{aligned}
\deg(xx', \mathcal{V} \times \mathcal{V}', \mu_{P_i} \times \mu_{P'_i}) &= \sum_{x \neq x', y \neq y' \in X} (\mu_{R_i} \times \mu_{R_i})(xx', yy') \\
&= \sum_{x \neq y, x' \neq y' \in X} (\mu_R(xy) \wedge \mu_{R'}(x'y')) \\
&\leq \left(\sum_{x \neq y \in X} \mu_R(xy) \right) \wedge \left(\sum_{x' \neq y' \in X} \mu_{R'}(x'y') \right) \\
&\leq \sum_{x \neq y \in X} \mu_R(xy) = \deg(x, \mathcal{V}, \mu_{P_i}).
\end{aligned}$$

In similar to $deg(xx', \mathcal{V} \times \mathcal{V}', \mu_{P_i} \times \mu_{P'_i}) \leq (\sum_{x' \neq y' \in X} \mu_R(x'y')) = deg(x', \mathcal{V}', \mu_{P'_i})$, so $deg(xx', \mathcal{V} \times \mathcal{V}', \mu_{P_i} \times \mu_{P'_i}) \leq \bigwedge\{deg(x, \mathcal{V}, \mu_{P_i}), deg(x', \mathcal{V}', \mu_{P'_i})\}$.

(ii) It is similar to (i).

We proved (v) and (iv) is similar to (i). Let $xx' \in P_i^* \times P_j'^*, yy' \in P_i{}^* \times P_j'^*$. Then

$$\begin{aligned}
deg(xx', \mathcal{E} \times \mathcal{E}', \nu_{P_i} \times_{P'_i}) &= \sum_{x \neq x', y \neq y' \in X} (\nu_{Q_{ij}} \times \nu_{Q'_{i,j}})(xx'yy') \\
= \sum_{x \neq x', y \neq y' \in X} (\nu_{Q_{ij}})(xy) \vee (\nu_{Q'_{i,j}}(x'y')) \\
\geq \sum_{x \neq y \in X} (\nu_{Q_{ij}})(xy) = deg(x, \mathcal{E}, \nu_{P_i}).
\end{aligned}$$

In similar to, one can see that $deg(xx', \mathcal{E} \times \mathcal{E}', \nu_{P_i} \times_{P'_i}) \geq deg(x', \mathcal{E}', \nu_{P'_i})$. Therefore, $deg(xx', \mathcal{E} \times \mathcal{E}', \nu_{P_i} \times_{P'_i}) \geq \max\{deg(x, \mathcal{E}, \nu_{P_i}), deg(x', \mathcal{E}', \nu_{P'_i})\}$. □

Let $x \in P_i^*, x' \in P_i'^*$. Consider, $\alpha := deg(xx', \mathcal{V} \times \mathcal{V}', \mu_{P_i} \times \mu_{P'_i}), \alpha' := deg(x, \mathcal{V}, \mu_{P_i}), \beta := deg(xx', \mathcal{V} \times \mathcal{V}', \nu_{P_i} \times_{P'_i})$ and $\beta' := deg(x, \mathcal{V}, \nu_{P_i})$. Define a binary relation R on $[0, 1]^2$, by $(\alpha, \beta)R(\alpha', \beta')$, that $\alpha \leq \alpha'$ and $\beta \geq \beta'$. Then by Theorem 3.22, have the following result.

Corollary 3.23. *Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ and $H' = (\mathcal{V}', \mathcal{E}', \mathcal{D}')$ be two RPFS on X and X', respectively. Then*

(i) *$([0, 1]^2, R)$ is a poset, where $R = \{((deg_{\mathcal{V} \times \mathcal{V}'}(xx'), deg_{\mathcal{V}}(x)) \mid x \in P_i^*, x' \in P_i'^*\}$.*

(*ii*) $([0,1]^2, S)$ *is a poset, where* $S = \{((deg_{\mathcal{E}\times\mathcal{E}'}(xx'), deg_{\mathcal{E}}(x)) \mid x \in P_i^*, x' \in P_i'^*\}$.

Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ and $H' = (\mathcal{V}', \mathcal{E}', \mathcal{D}')$ be two $RPFS$ on X and X', respectively. Assume $\beta_H, \beta_{H'}$ and $\beta_{H\times H'}$ be the defined fundamental relations (based Theorem 3.5), on $H, H', H \times H'$, respectively, so in what follow, investigate the relation of link-degree and superedge-degree and their quotient via the fundamental relation β.

Based Theorem 3.16, for the degree of vertices in fundamental Pythagorean fuzzy graphs have the following theorem.

Theorem 3.24. *Let* $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ *and* $H' = (\mathcal{V}', \mathcal{E}', \mathcal{D}')$ *be two* $RPFS$ *on* X *and* X'*, respectively. Then for any* $x \in P_i^*, x' \in P_i'^*$:

(*i*) $deg_{\mathcal{V}\times\mathcal{V}'/\beta_{H\times H'}}(\beta_{H\times H'}(xx'), \mathcal{V}\times\mathcal{V}'/\beta_{H\times H'}, \mu_{P_i}\times\mu_{P_i'}/\beta_{H\times H'}) \leq deg_{\mathcal{V}/\beta_H}(\beta_H(x), \mathcal{V}/\beta_H, \mu_{P_i}/\beta_H)$.
(*ii*) $deg_{\mathcal{V}\times\mathcal{V}'/\beta_{H\times H'}}(\beta_{H\times H'}(xx'), \mathcal{V}\times\mathcal{V}'/\beta_{H\times H'}, \nu_{P_i}\times\nu_{P_i'}/\beta_{H\times H'}) \geq deg_{\mathcal{V}/\beta_H}(\beta_H(x), \mathcal{V}/\beta_H, \nu_{P_i}/\beta_H)$.

In the following theorem, we show that the fundamental direct product of Pythagorean fuzzy graphs is isomorphic to the direct product of related fundamental Pythagorean fuzzy graphs.

Theorem 3.25. *Let* $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ *and* $H' = (\mathcal{V}', \mathcal{E}', \mathcal{D}')$ *be two* $RPFS$ *on* X *and* X'*, respectively. Then* $H \times H'/\beta_{H\times H'} = H/\beta_H \times H'/\beta_{H'}$.

Proof. Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ and $H' = (\mathcal{V}', \mathcal{E}', \mathcal{D}')$ be two $RPFS$'s on X and X', respectively. Define $\phi : H \times H'/\beta_{H\times H'} \to H/\beta_H \times H'/\beta_{H'}$, by $\phi(\beta_{H\times H'}(xx')) = (\beta_H(x), \beta_{H'}(x'))$, where $xx' := (x, x') \in X \times X'$. Assume $\beta_{H\times H'}(xx'), \beta_{H\times H'}(yy') \in H \times H'/\beta_{H\times H'}$ and $\beta_{H\times H'}(xx') = \beta_{H\times H'}(yy')$. Then there exists P_i, P_j' that $\{xx', yy'\} \subseteq P_i^* \times P'^*_j$. It follows that $\{x, y\} \subseteq P_i^*$ and $\{x', y'\} \subseteq P'^*_j$ and so $(\beta_H(x), \beta_H(x')) = (\beta_H(y), \beta_H(y'))$. Hence, $\phi(\beta_{H\times H'}(xx')) = \phi(\beta_{H\times H'}(xx'))$ and then ϕ is a well-defined map. Suppose that $\phi(\beta_{H\times H'}(xx')) = \phi(\beta_{H\times H'}(yy'))$. Then $(\beta_H(x), \beta_{H'}(x')) = (\beta_H(y), \beta_{H'}(y'))$. Hence there exist P_i, P_j' that $\{x, y\} \times \{x', y'\} \subseteq P_i \times P_j'$ and so $\{xx', yy'\} \subseteq P_i \times P_j'$. It concludes that ϕ is a one to one map. Let $xx' = (x, x') \in P_i^* \times P'^*_i$. Then,

$$
\begin{aligned}
&\Big(\mu_{P_i\times P_i'/\beta_{H\times H'}}(\beta_{H\times H'}(xx')),\\
&\quad \nu_{P_i\times P_i'/\beta_{H\times H'}}(\beta_{H\times H'}(xx'))\Big)\\
&=\Big(\bigwedge_{xx'\beta\, yy'}(\mu_{P_i}\times\mu_{P_i'})(yy'),\\
&\quad \bigwedge_{xx'\beta\, yy'}(\nu_{P_i}\times\nu_{P_i'})(yy')\Big)\\
&=\Big(\bigwedge_{x\beta y,\ x'\beta y'}(\mu_{P_i}(y)\wedge\mu_{P_i'}(y')),\\
&\quad \bigwedge_{x\beta y,\ x'\beta y'}(\nu_{P_i}(y)\wedge\nu_{P_i'}(y'))\Big)\\
&=\Big((\bigwedge_{x\beta y}\mu_{P_i}(y))\wedge(\bigwedge_{x'\beta y'}\mu_{P_i'}(y')),\\
&\quad (\bigwedge_{x\beta y}\nu_{P_i}(y))\vee(\bigwedge_{x'\beta y'}\nu_{P_i'}(y'))\Big)\\
&=\Big(\mu_{P_i/\beta_H}(\beta_H(x))\wedge\mu_{P_i'/\beta_{H'}}(\beta_{H'}(x')),\\
&\quad \nu_{P_i/\beta_H}(\beta_H(x))\vee\nu_{P_i'/\beta_{H'}}(\beta_{H'}(x'))\Big)\\
&=\Big((\mu_{P_i/\beta_H}\times\mu_{P_i'/\beta_{H'}})(\beta_H(x),\beta_{H'}(x')),\\
&\quad (\nu_{P_i/\beta_H}\times\nu_{P_i'/\beta_{H'}})(\beta_H(x),\beta_{H'}(x'))\Big)\\
&=\Big((\mu_{P_i/\beta_H}\times\mu_{P_i'/\beta_{H'}})(\phi(\beta_{H\times H'}(xx'))),\\
&\quad (\nu_{P_i/\beta_H}\times\nu_{P_i'/\beta_{H'}})(\phi(\beta_{H\times H'}(xx')))\Big).
\end{aligned}
$$

Assume $(xx' = (x, x'), yy' = (y, y')) \in Q^*_{ij} \times {Q'}^*_{ij}$. Then,

$$
\begin{aligned}
&\Big(\mu_{Q_{ij}\times Q'_{ij}/\beta_{H\times H'}}\big(\beta_{H\times H'}(xx')\beta_{H\times H'}(yy')\big),\\
&\quad \nu_{Q_{ij}\times Q'_{ij}/\beta_{H\times H'}}\big(\beta_{H\times H'}(xx')\beta_{H\times H'}(yy')\big)\Big)\\
&= \Big(\bigwedge_{ww'\beta xx', zz'\beta yy'} (\mu_{Q_{ij}} \times \mu_{Q'_{ij}})(ww', zz'),\\
&\quad \bigwedge_{ww'\beta xx', zz'\beta yy'} (\nu_{Q_{ij}} \times \nu_{Q'_{ij}})(ww', zz')\Big)\\
&= \Big(\bigwedge_{ww'\beta xx', zz'\beta yy'} \big(\mu_{Q_{ij}}(ww') \wedge \mu_{Q'_{ij}}(zz')\big),\\
&\quad \bigwedge_{ww'\beta xx', zz'\beta yy'} \big(\nu_{Q_{ij}}(ww') \wedge \nu_{Q'_{ij}}(zz')\big)\Big)\\
&= \Big(\big(\bigwedge_{ww'\beta xx'} \mu_{Q_{ij}}(ww')\big) \wedge \big(\bigwedge_{zz'\beta yy'} \mu_{Q'_{ij}}(zz')\big),\\
&\quad \big(\bigwedge_{ww'\beta xx'} \nu_{Q_{ij}}(ww')\big) \wedge \big(\bigwedge_{zz'\beta yy'} \nu_{Q'_{ij}}(zz')\big)\Big)\\
&= \Big(\mu_{Q_{ij}/\beta_H}(\beta_H(xx')) \wedge \mu_{Q'_{ij}/\beta_{H'}}(\beta_{H'}(yy')),\\
&\quad \nu_{Q_{ij}/\beta_H}(\beta_H(xx')) \wedge \nu_{Q'_{ij}/\beta_{H'}}(\beta_{H'}(yy'))\Big)\\
&= \Big((\mu_{Q_{ij}/\beta_H} \times \mu_{Q'_{ij}/\beta_{H'}})(\beta_H(xx'), \beta_{H'}(yy')),\\
&\quad (\nu_{Q_{ij}/\beta_H} \times \nu_{Q'_{ij}/\beta_{H'}})(\beta_H(xx'), \beta_{H'}(yy'))\Big)\\
&= \Big((\mu_{Q_{ij}/\beta_H} \times \mu_{Q_{ij}/\beta_{H\times H'}})\big(\phi(\beta_{H\times H'}(xx')\beta_{H\times H'}(yy'))\big),\\
&\quad (\nu_{Q_{ij}/\beta_H} \times \nu_{Q_{ij}/\beta_{H\times H'}})\big(\phi(\beta_{H\times H'}(xx')\beta_{H\times H'}(yy'))\big)\Big).
\end{aligned}
$$

Since ϕ is on to, get $H \times H'/\beta_{H\times H'} = H/\beta_H \times H'/\beta_{H'}$. □

Example 3.26. *Consider the RPFS's, $H, H', H \times H'$ in Figures 6, 7 and 8. Based Theorem 3.5, obtain the fundamental PFG's $H'/\beta_{H'}, H/\beta_H, H \times H'/\beta_{H\times H'}$, $H/\beta_H \times H'/\beta_{H'}$, respectively, as Figures 9, 10,11 and 12.*

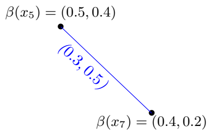

FIGURE 9. H'/β'_H.

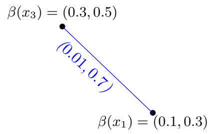

FIGURE 10. H/β_H.

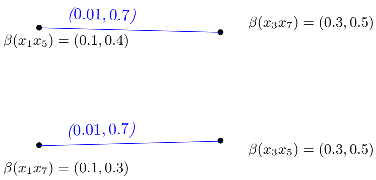

FIGURE 11. $H \times H'/\beta_{H\times H'}$.

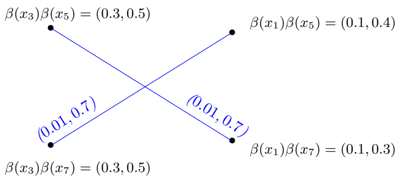

FIGURE 12. $H/\beta_H \times H'/\beta_{H'}$.

By Theorem 3.25, $H \times H'/\beta_{H\times H'} = H/\beta_H \times H'/\beta_{H'}$.

3.2. **Application of $RPFS$ in real world problems.** In this section, state the role of $RPFS$ in real-world problems. Any $RPFS$ is a universal model for complex and uncertain systems, such as networks and decision-making processes. Indeed, we can modelify any complex uncertain system as an $RPFS$, where in this real problem, there are some groups of objects that are related to this system in both whole and detail. In addition, the **direct product** of two related $RPFS$ establishes an optimal aggregation that balances reliability and risk, solving interconnected problems in a mathematically sound and conservative way.

Scientific Network: Let X, Y be two universities with two departments P_1, P_2, P'_1, P'_2 that are independent and want to establish a secure and reliable communication channel to exchange scientific and research achievements. Each university has its own internal network modeled as an $RPFS$'s H_1, H_2, respectively, as the Tables 1,2. In this network, x_is and y_is are the faculty representatives in the related departments, and x_ix_j is a connection between of x_i and x_j and y_iy_j is a connection between of y_i and y_j. For instance, in department P_1, for people x_1, degree of knowledge, precision, fairness, and scientific integrity is 90% and degree of misunderstanding, dependency, conflict of interest, information disclosure, ambiguity in responsibility is 20%, that we denote by $(x_1, 0.9, 0.2)$ in the Table 1.

Table 1. University X (H_1)

Label	**(scientific reliability, collaboration risk)**
P_1	$(x_1, 0.9, 0.2)$, $(x_2, 0.8, 0.3)$
P_2	$(x_3, 0.7, 0.4)$, $(x_4, 0.8, 0.3)$
R_1	$(x_1x_2, 0.75, 0.35)$
R_2	$(x_3x_4, 0.65, 0.45)$
Q_{12}	$(x_1x_3, 0.70, 0.40)$, $(x_1x_4, 0.75, 0.35)$, $(x_2x_3, 0.70, 0.40)$, $(x_2x_4, 0.75, 0.35)$

Table 2. University Y (H_2)

Label	**(scientific reliability, collaboration risk)**
P'_1	$(y_1, 0.85, 0.25)$, $(y_2, 0.75, 0.35)$
P'_2	$(y_3, 0.90, 0.15)$, $(y_4, 0.80, 0.30)$
R'_1	$(y_1y_2, 0.70, 0.40)$
R'_2	$(y_3y_4, 0.75, 0.35)$
Q'_{12}	$(y_1y_3, 0.80, 0.30)$, $(y_1y_4, 0.75, 0.35)$, $(y_2y_3, 0.70, 0.40)$, $(y_2y_4, 0.70, 0.40)$

Even if each university individually has good internal values, the inter-university connection requires simultaneous superedges from both universities. Now, we want to by aggregation of two university and find the maximal scientific reliability and minimum collaboration risk. So compute the Table 3.

Table 3. Aggregation of two universities

Label	**(Element, μ, ν) Pairs**
$P_1 \times P'_1$	$(x_1y_1, 0.85, 0.25)$, $(x_1y_2, 0.75, 0.35)$, $(x_2y_1, 0.80, 0.30)$, $(x_2y_2, 0.75, 0.35)$
$P_2 \times P'_2$	$(x_3y_3, 0.70, 0.40)$, $(x_3y_4, 0.70, 0.40)$, $(x_4y_3, 0.80, 0.30)$, $(x_4y_4, 0.75, 0.35)$
$R_1 \times R'_1$	$((x_1y_1)(x_2y_2), 0.75, 0.35)$
$R_2 \times R'_2$	$((x_3y_3)(x_4y_4), 0.65, 0.45)$
$Q_{(11)(22)}$	$((x_1y_1)(x_3y_3), 0.70, 0.40)$

It follows that for $(R_1 \times R'_1)= (x_1y_1)(x_2y_2), 0.75, 0.35)$ is the optimal with maximal scientific reliability (0.75) and minimum collaboration risk (0.35).

Fuzzy data clustering: Fuzzy data clustering is a method of grouping data points where a single point can belong to multiple clusters simultaneously, with varying degrees of membership. In $RPFS$, data are in a set X, each Pythagorean fuzzy supervertex P_i is denoted by a cluster, where $\mu_{P_i}(x)$ and $\nu_{P_i}(x)$ denote the membership and non-membership of a point x in that cluster. Within-cluster similarities are captured by internal links R_i, while between-cluster relationships are represented by superedges Q_{ij}.

Let X be a university. The head of the university wants to cluster 7 masters based on teaching effectiveness, research satisfaction, communication with a colleague, and colleague satisfaction. The data are in Table 4, with (P_i, teaching quality, student satisfaction), (R_i, communication with a colleague, colleague satisfaction), and (Q_{ij}, colleague satisfaction, student satisfaction), where teaching quality, student satisfaction, communication with a colleague, colleague satisfaction, and student satisfaction are dependent variables. For instance, in $(M_1, 0.9, 0.1) \in P_1$ means that the for master M_1, degree of teaching quality is 90% and degree of student satisfaction is 10% and degree of communication M_1 and M_2 is 70% and degree of satisfaction of M_1 and M_2 is 20%. In addition, degree of satisfaction M_1 in P_1 with M_4 in P_2 is 20% and degree of student satisfaction of the connection M_1 with M_2 is 70%(although there is no good satisfaction between the two professors in terms of communication, the student has a strong desire to establish contact with the two professors.). In this problem, we want to find the high degree in communication with a colleague, colleague satisfaction, colleague satisfaction and student satisfaction between of masters.

Component	**Notation**	**Values** (xy **or** x**,** μ**,** ν)
$\mathcal{V}$	P_1	$(M_1, 0.9, 0.1), (M_2, 0.8, 0.2), (M_3, 0.7, 0.3)$
	P_2	$(M_4, 0.6, 0.4), (M_5, 0.7, 0.3)$
	P_3	$(M_6, 0.5, 0.5), (M_7, 0.8, 0.2)$
$\mathcal{D}$	R_1	$(M_1M_2, 0.7, 0.2), (M_2M_3, 0.6, 0.3), (M_1M_3, 0.6, 0.3)$
	R_2	$(M_4M_5, 0.5, 0.4)$
	R_3	$(M_6M_7, 0.4, 0.5)$
$\mathcal{E}$	Q_{12}	$(M_1M_4, 0.2, 0.7), (M_2M_5, 0.3, 0.6), (M_3M_4, 0.1, 0.8)$
	Q_{23}	$(M_4M_6, 0.3, 0.6), (M_5M_7, 0.4, 0.5)$
	Q_{31}	$(M_6M_1, 0.4, 0.5), (M_7M_2, 0.5, 0.3)$

Table 4. An RPFS $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ on $X = \{M_1, M_2, M_3, M_4, M_5, M_6, M_7\}$.

Now, compute the link-degree and superedge-degree of 7 masters in the Table 5.

Customer	$deg_{\mathcal{V}}(x)$	$deg_{\mathcal{E}}(x)$	**Total**
M_1	$(1.3, 0.5)$	$(0.2, 0.7)$	$(1.5, 1.2)$
M_2	$(1.3, 0.5)$	$(0.3, 0.6)$	$(1.6, 1.1)$
M_3	$(1.2, 0.6)$	$(0.1, 0.8)$	$(1.3, 1.4)$
M_4	$(0.5, 0.4)$	$(0.3, 0.6)$	$(0.8, 1.0)$
M_5	$(0.5, 0.4)$	$(0.4, 0.5)$	$(0.9, 0.9)$
M_6	$(0.4, 0.5)$	$(0.4, 0.5)$	$(0.8, 1.0)$
M_7	$(0.4, 0.5)$	$(0.5, 0.3)$	$(0.9, 0.8)$
$deg(H) = (7.8, 7.4)$			

Table 5. link-degree and superedge-degree for the RPFS.

Based the Table 5, the masters M_2 and M_1 have the high communication with a colleague, colleague satisfaction, colleague satisfaction and student satisfaction with other masters.

4. Conclusion

This paper introduced a novel mathematical model for fuzzy data clustering and complex network analysis called refined Pythagorean fuzzy superhypergraphs (RPFS). As an extension of Pythagorean fuzzy graphs, the proposed tool comprises 3 components: Pythagorean fuzzy supervertices ($\mathcal{V}$), Pythagorean fuzzy superedges ($\mathcal{E}$), and Pythagorean fuzzy links ($\mathcal{D}$), all of which are based on a non-empty set X. The main results obtained in this paper are as follows:

(1) We proved that for any $RPFS$, each pair (P_i, R_i) forms a Pythagorean fuzzy graph (PFG) on the support P_i^*. Furthermore, we established cardinality bounds on internal links. For any supervertex with $|P_i^*| \geq 2$, the number of links satisfies $|P_i^*| - 1 \leq |R_i^*| \leq \binom{|P_i^*|}{2}$, ensuring that each supervertex is minimally connected without redundancy.

(2) We introduced a fundamental relation β on X defined by $x\beta y$ if and only if x and y belong to the same supervertex. We proved that the quotient $H/\beta = (P/\beta, Q/\beta)$ is a Pythagorean fuzzy graph on X/β. This result establishes that every $RPFS$ is a *quotientable PFG*, providing a systematic method to reduce a complex hierarchical superhypergraph to a standard PFG while preserving Pythagorean fuzzy membership and non-membership information.

(3) We proved that for any PFG $G = (P, Q)$ on a non-empty set X, there exists at least one $RPFS$ H such that G is isomorphic to the quotient of H under β. This demonstrates that the classes of $RPFS$ and PFG are categorically related, and that $PFGs$ can be viewed as the quotient simplifications of more complex $RPFS$.

(4) We defined the link-degree $deg_{\mathcal{V}}(x)$ and superedge-degree $deg_{\mathcal{E}}(x)$ for any element $x \in X$, and proved that $deg_{H/\beta}(\beta(x)) \leq deg_{\mathcal{E}}(x)$. This

inequality establishes a monotonic relationship between the degree of a vertex in the quotient PFG and the superedge-degree in the $RPFS$.

(5) We defined the direct product $H \times H'$ of two $RPFSs$ and proved that the product is itself an $RPFS$. Moreover, we established the fundamental isomorphism: $H \times H'/\beta_{H\times H'} \cong H/\beta_H \times H'/\beta_{H'}$, showing that the quotient operation commutes with the direct product.

(6) We proved that $(\mathbb{RPFS}(X), \times, \triangle)$ forms a commutative monoid, where $\triangle$ is the singleton $RPFS$, and that there exists a unique homomorphism from any $RPFS$ to $\triangle$. These results lay the foundation for a functorial connection between the category of $RPFS$ and the category of PFG.

Our methodological contributions in this study are based, partition-based clustering(any $RPFS$ partitions the underlying set X into disjoint supervertices P_i^*), quotient construction(the equivalence relation β collapses each supervertex into a single node), cardinality constraints(the lower bound $|P_i^*| - 1$ ensures that the Pythagorean fuzzy link R_i forms a connected spanning Pythagorean fuzzy superhyper subgraph and the upper bound $\binom{|P_i^*|}{2}$ allows for complete internal connectivity), direct product construction (this enables the aggregation of independent $RPFS$ models into a unified framework.).

The significance of this work is threefold. Theoretically, it bridges hierarchical superhypergraph structures with Pythagorean fuzzy uncertainty via the fundamental relation β, establishing a categorical equivalence between $RPFS$ and PFG. Practically, the direct product and degree measures provide aggregation and key element identification tools for collaboration networks. Methodologically, $RPFS$ offers a rigorous approach to fuzzy data clustering that simultaneously handles hierarchical groups, internal links, external superedges, and Pythagorean membership, faithfully representing complex systems.

Future works: We want to work on the fuzzy entropy of the refined Pythagorean fuzzy superhypergraph, make some algorithms for computing the optimal fuzzy entropy of refined Pythagorean fuzzy superhypergraphs, investigate the categorical properties of refined Pythagorean fuzzy superhypergraphs, and the application of refined Pythagorean fuzzy superhypergraphs in complex hypernetworks. Future work will develop the additional theorems and computational procedures needed for an optimal RPFS-based fuzzy clustering algorithm.

5. **Author Contributions**

Conceptualization and methodology, writing—review and editing: M. Hamidi; validation and investigation, writing—original: shohreh shirani Nazhvani.

6. **Data Availability Statement**

Not applicable.

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9. Conflict of interest

The authors declare that they have no conflict interest.

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