Shahid Bahonar University of Kerman
Journal of Mahani Mathematical Research
Print ISSN: 2251-7952 Online ISSN: 2645-4505

COMPLETENESS OF TAYLOR SEQUENCE SPACES

G. Talebi

Article type: Research Article
(Received: 05 March 2026, Received in revised form 14 July 2026)
(Accepted: 25 July 2026, Published Online: 25 July 2026)

Abstract. We prove that the Taylor sequence space t_p^θ is complete for every $1 \leq p \leq \infty$ and $0 \leq \theta < \frac{1}{2}$. Although t_p^θ is naturally isometrically embedded into ℓ_p, the surjectivity of the Taylor operator is not known *a priori*; therefore, completeness does not follow immediately. The main step is to show that the inverse Taylor matrix defines an operator on ℓ_p, which holds precisely for $0 \leq \theta < \frac{1}{2}$. It follows that the Taylor operator is an isometric isomorphism from t_p^θ onto ℓ_p, yielding the desired completeness. As consequences, we deduce several geometric properties of t_p^θ, including reflexivity, the Banach–Saks property, and the fixed point property for nonexpansive mappings.

Keywords: Taylor sequence spaces, completeness, uniform rotundity.
2020 MSC: 46B20, 46B45, 40C05.

1. Introduction

For $0 \leq \theta < 1$, the Taylor method of summability is defined by the upper triangular matrix $T^\theta = (t_{n,k}^\theta)_{n,k\geq 0}$ [4], whose entries are given by

$$t_{n,k}^\theta = \binom{k}{n}(1-\theta)^{n+1}\theta^{k-n}, \qquad k \geq n,$$

and $t_{n,k}^\theta = 0$ otherwise. For general background on matrix domains and summability theory, we refer the reader to [1, 7]. The Taylor sequence spaces t_p^θ $(1 \leq p < \infty)$ and t_∞^θ, introduced in [5], are defined by

$$t_p^\theta = \{x = (x_k) : T^\theta x \in \ell_p\}, \qquad 1 \leq p < \infty,$$

and

$$t_\infty^\theta = \{x = (x_k) : T^\theta x \in \ell_\infty\},$$

both equipped with the norm

$$\|x\|_{t_p^\theta} = \|T^\theta x\|_{\ell_p}.$$

Their basic properties were investigated in [5], while several geometric properties of these spaces were established in [6]. Although the Taylor operator

$$T^\theta : t_p^\theta \longrightarrow \ell_p$$

✉ gh.talebi@vru.ac.ir, ORCID: 0000-0002-6781-440X
https://doi.org/10.22103/jmmr.2026.26959.1949
Publisher: Shahid Bahonar University of Kerman
How to cite: G. Talebi, *Completeness of Taylor sequence spaces*, J. Mahani Math. Res. 2027; 16(1): 243-247.

is an isometric embedding, the completeness of t_p^θ is not immediate, since the surjectivity of T^θ is not known *a priori*. The main difficulty is to show that the inverse Taylor matrix is well defined on ℓ_p. We show that this is precisely the case whenever

$$0 \leq \theta < \frac{1}{2},$$

which is exactly the range for which the inverse Taylor matrix defines an operator on ℓ_p. The main result of this paper is that t_p^θ is complete for every $1 \leq p \leq \infty$ and $0 \leq \theta < \frac{1}{2}$. The proof is based on the explicit inverse Taylor matrix together with the fact that it acts on ℓ_p in this parameter range. Consequently, the Taylor operator is an isometric isomorphism from t_p^θ onto ℓ_p, and hence t_p^θ is a Banach space. Combined with the geometric properties established in [6], our completeness theorem yields reflexivity, the Banach–Saks property, and the fixed point property for nonexpansive mappings whenever $1 < p < \infty$.

2. Completeness

The proof of completeness is based on the inverse Taylor matrix

$$S = (T^\theta)^{-1} = T^{\frac{\theta}{\theta-1}},$$

whose existence is guaranteed by Corollary 6.3 of [4]. The following lemma establishes the properties of S needed in the proof of the main theorem.

Lemma 2.1 (The inverse Taylor transform on ℓ_p)**.** *Let $1 \leq p \leq \infty$ and let $0 \leq \theta < \frac{1}{2}$. Then the following assertions hold.*

(i) *For every $y \in \ell_p$, the series*

$$(Sy)_k = \sum_{j=k}^{\infty} s_{kj} y_j$$

converges absolutely for every $k \geq 0$.

(ii) *For every $y \in \ell_p$,*

$$T^\theta(Sy) = y.$$

(iii) *For every $y \in \ell_p$,*

$$Sy \in t_p^\theta.$$

Equivalently,

$$S : \ell_p \longrightarrow t_p^\theta.$$

Proof. The entries of S are

$$s_{kj} = \begin{cases} 0, & j < k, \\ \binom{j}{k} (-1)^{k+1} (\theta - 1)^{-(j+1)} \theta^{j-k}, & j \geq k. \end{cases}$$

Fix $k \geq 0$. Suppose first that $1 < p < \infty$, and let q denote the conjugate exponent. By Hölder's inequality,

$$\sum_{j=k}^{\infty} |s_{kj} y_j| \leq \left(\sum_{j=k}^{\infty} |s_{kj}|^q \right)^{1/q} \|y\|_{\ell_p}.$$

Hence it suffices to prove that

$$\sum_{j=k}^{\infty} |s_{kj}|^q < \infty.$$

Since

$$\frac{|s_{k,j+1}|^q}{|s_{kj}|^q} = \left(\frac{j+1}{j+1-k} \cdot \frac{\theta}{1-\theta} \right)^q,$$

we obtain

$$\lim_{j \to \infty} \frac{|s_{k,j+1}|^q}{|s_{kj}|^q} = \left(\frac{\theta}{1-\theta} \right)^q < 1,$$

because $0 \leq \theta < \frac{1}{2}$. Therefore,

$$\sum_{j=k}^{\infty} |s_{kj}|^q < \infty$$

by the ratio test. For $p = 1$,

$$\sum_{j=k}^{\infty} |s_{kj} y_j| \leq \sup_{j \geq k} |s_{kj}| \, \|y\|_{\ell_1}.$$

Moreover,

$$\frac{|s_{k,j+1}|}{|s_{kj}|} \longrightarrow \frac{\theta}{1-\theta} < 1,$$

so $|s_{kj}| \to 0$ as $j \to \infty$. Hence $(|s_{kj}|)_{j \geq k}$ is bounded. Finally, if $p = \infty$,

$$\sum_{j=k}^{\infty} |s_{kj} y_j| \leq \|y\|_{\ell_\infty} \sum_{j=k}^{\infty} |s_{kj}|,$$

and the latter series converges by the ratio test. This proves (i).
Assertion (ii) follows immediately from Corollary 6.3 of [4].
Finally, by (ii), for every $y \in \ell_p$,

$$T^\theta(Sy) = y.$$

Hence, by the definition of t_p^θ,

$$Sy \in t_p^\theta.$$

This proves (iii). □

Theorem 2.2. *Let $1 \leq p \leq \infty$ and let $0 \leq \theta < \frac{1}{2}$. Then the Taylor sequence space t_p^θ is complete.*

Proof. Let $(x^{(n)})$ be a Cauchy sequence in t_p^θ. Since

$$\|x\|_{t_p^\theta} = \|T^\theta x\|_{\ell_p},$$

the sequence

$$(T^\theta x^{(n)})$$

is Cauchy in ℓ_p. Because ℓ_p is complete, there exists $y \in \ell_p$ such that

$$T^\theta x^{(n)} \longrightarrow y \qquad \text{in } \ell_p.$$

By Lemma 2.1, define

$$x := Sy.$$

Then

$$x \in t_p^\theta$$

and

$$T^\theta x = T^\theta(Sy) = y.$$

Therefore,

$$\|x^{(n)} - x\|_{t_p^\theta} = \|T^\theta x^{(n)} - T^\theta x\|_{\ell_p} = \|T^\theta x^{(n)} - y\|_{\ell_p} \longrightarrow 0.$$

Hence

$$x^{(n)} \longrightarrow x \quad \text{in } t_p^\theta,$$

and therefore t_p^θ is complete. □

3. Applications

We shall use the following classical consequences of uniform rotundity: every uniformly rotund Banach space is reflexive [3, Theorem 5.2.15], possesses the Banach–Saks property [3, Theorem 2.4.12], and has the fixed-point property for nonexpansive mappings [2, Theorem 5.2]. Together with Theorem 2.2 and the uniform rotundity of t_p^θ established in [6, Theorem 2], these results imply the following theorem.

Theorem 3.1. *Let $1 < p < \infty$ and let $0 \leq \theta < \frac{1}{2}$. Then the Taylor sequence space t_p^θ is reflexive and possesses both the Banach–Saks property and the fixed-point property for nonexpansive mappings.*

Proof. By Theorem 2.2, t_p^θ is a Banach space. Furthermore, by [6, Theorem 2], t_p^θ is uniformly rotund for $1 < p < \infty$. The stated conclusions follow directly from the classical results cited above. □

Remark 3.2. The restriction $1 < p < \infty$ is essential. Indeed, the proof above relies on the uniform rotundity of t_p^θ, which is available only for $1 < p < \infty$ [6, Theorem 2]. For $p = 1$ and $p = \infty$, the spaces t_1^θ and t_∞^θ are not rotund [6, Proposition 3]; therefore, the classical implications of uniform rotundity cannot be applied in these cases.

Acknowledgements

The author would like to thank the anonymous reviewers for their careful reading of the manuscript and for their valuable comments and suggestions, which have significantly improved the presentation of this paper.

References

[1] Başar, F. (2012). *Summability Theory and its Applications.* Bentham Science Publishers, Istanbul.
[2] Kirk, W. A. (1965). A fixed point theorem for mappings which do not increase distances. *Amer. Math. Monthly*, **72**, 1004–1006. https://doi.org/10.2307/2313345
[3] Megginson, R. E. (1998). *An Introduction to Banach Space Theory.* Springer-Verlag, New York.
[4] Natarajan, P. N. (2015). *An Introduction to Ultrametric Summability Theory* (2nd ed.). Forum for Interdisciplinary Mathematics. Springer India, New Delhi. https://doi.org/10.1007/978-81-322-2559-1
[5] Talebi, G. (2017). On the Taylor sequence spaces and upper boundedness of Hausdorff matrices and Nörlund matrices. *Indag. Math.*, **28**(3), 629–636. https://doi.org/10.1016/j.indag.2017.02.001
[6] Talebi, G. (2025). Geometric properties of Taylor sequence spaces. *J. Inequal. Appl.*, **46**, 1–7. https://doi.org/10.1186/s13660-025-03294-7
[7] Wilansky, A. (1984). *Summability through Functional Analysis.* North-Holland, Amsterdam.

Gholamreza Talebi
Orcid number: 0000-0002-6781-440X
Department of Mathematics
Vali-e-Asr University of Rafsanjan
Rafsanjan, Iran
Email address: `Gh.talebi@vru.ac.ir`