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DISTRIBUTIONALLY CHAOTIC DYNAMICAL SYSTEMS IN UNIFORM SPACES

S.A. Ahmadi

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Abstract. This study explores the connections among various degrees of chaos within dynamical systems defined on uniform spaces, concentrating on subsets where the trajectories of multiple points display irregular or 'scrambled' behavior behavior as time progresses. Our investigation centers on two primary frameworks: distributional chaos and Li–Yorke chaos. We initially address the circumstances under which the presence of a chaotic set containing n points, guarantees the existence of a chaotic set with m points, where $2 \leq m < n$. We prove that under a condition called DC, a distributionally topological 2-scrambled set is a distributionally topological n-scrambled. Although the forward direction is typically valid, establishing the reverse implication necessitates additional hypotheses, which we delineate.

Keywords: scrambled set, transitive map, distributional chaos, Li-Yourk chaos, uniform space.
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1. Introduction

A topological dynamical system is represented by a pair (X, T), where X is a compact Hausdorff space (or, more generally, a uniform space) and $T : X \to X$ is a continuous transformation. Given an initial point $x \in X$, its orbit under the action of T is the sequence $x, T(x), T^2(x), \ldots$, which describes its evolution over time.
Two distinct points x and y in X are said to form a *scrambled pair* if their orbits display a particular pattern of instability: they repeatedly approach each other arbitrarily closely and also diverge to a significant extent, and this occurs infinitely often. While iterating T in parallel across all points, the orbits of x and y alternately converge and separate across infinitely many time intervals, becoming indistinguishable at times and clearly apart at others.
A set in which every pair of distinct points is scrambled is called a *scrambled set*. When such a set is uncountably large, the system is termed *Li-Yorke chaotic*. This foundational concept of chaos, introduced in 1975 by Li and Yorke, has inspired extensive subsequent research.

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sa.ahmadi@math.usb.ac.ir, ORCID: 0000-0003-1267-6553
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This idea naturally extends to more than two points. A set is called an *n-scrambled set* if the orbits of any n distinct points within it collectively exhibit this alternating pattern of convergence and divergence. Systems containing uncountable n-scrambled sets are described as *Li-Yorke n-chaotic*. This generalization leads to various refined notions of chaos, including mean Li–Yorke chaos and distributional chaos, which are central to the present investigation.
In dynamical systems research, the amount or intensity of chaos is frequently associated with the size of a scrambled set. For two prominent types of chaos (Li-Yorke chaos and distributional chaos), a fundamental and natural question arises: how large can these scrambled sets actually be?
The study of scrambled sets is not merely a mathematical curiosity; it has deep motivations and practical applications.
From a theoretical perspective, understanding the maximal possible size and structural complexity of scrambled sets helps us quantify chaos. It answers the question: how chaotic can a system be? Moreover, the transition from pairwise scrambling to n-scrambling reveals a hierarchy of chaos, which may correspond to emergent collective behavior in high-dimensional systems.
Some important applications of scrambled sets are listed here:

- **Neural networks and population dynamics:** In coupled map lattices or neural networks, the presence of n-scrambled sets indicates that n distinct initial states can remain indistinguishable at certain times while diverging at others. This models phenomena such as synchronization and desynchronization in biological oscillators.
- **Cryptography and pseudorandom number generation:** Chaotic maps with large scrambled sets are used to generate pseudorandom sequences. Higher-order scrambling (n-scrambling) may enhance security against statistical attacks.
- **Economics and climate modeling:** In multi-agent systems or ensemble forecasts, the existence of n-scrambled sets implies that even for large groups of trajectories, prediction remains fundamentally limited in the long term.

Thus, characterizing when n-scrambled sets exist, and how they relate across different n, has direct implications for modeling complex systems where interactions among multiple agents or components matter.
This question has been thoroughly explored for systems defined on an interval. For instance, research has demonstrated the existence of a continuous map on an interval that possesses a distributionally scrambled set occupying almost the entire space in terms of Lebesgue measure (since every distributionally scrambled set is also a Li-Yorke scrambled set). However, there exists a limitation: on an interval, a Li-Yorke scrambled set cannot be so pervasive as to constitute a *residual set* [9].
This restriction, however, does not hold for more general topological spaces. Indeed, there exist dynamical systems in which the *entire space* itself acts

as a Li-Yorke scrambled set. Yet, for the stronger condition of distributional chaos, a fundamental boundary exists: within a *compact* dynamical system, the whole space can never be a distributionally scrambled set [9]. An equivalent formulation of this result is that no nontrivial, compact, and invariant subset can be entirely distributionally scrambled.

While the theory of distributional chaos is well-developed for compact metric spaces, its extension to uniform spaces presents significant challenges. Key innovations of this work include:

(1) The generalization of n-scrambled set theory from metric to uniform spaces, which allows analysis of non-metrizable systems arising in functional analysis and topological dynamics;
(2) The introduction of the density condition (DC) as a sufficient criterion for the equivalence of different orders of distributional chaos;
(3) The first systematic investigation of iteration invariance for distributional n-chaos in uniform spaces.

Unlike the metric case, where distance functions provide natural quantitative measures, uniform spaces require a purely entourage-based framework. Our results show that while some properties (e.g., inheritance from n-chaos to m-chaos) transfer directly, others require additional assumptions unique to the uniform setting. Beyond mere size, the specific *properties* of a scrambled set reveal deeper layers of chaotic behavior. Previous studies have identified and examined scrambled sets with special characteristics, such as being transitive, invariant, or extremal.

This background motivates a compelling new line of inquiry: can a scrambled set that is already known to be very *large* also exhibit these special properties? For example, in non-compact systems, where the entire space can serve as a distributionally scrambled set, is it possible for this same set to also be extremal, transitive, or even possess a stronger mixing property? Investigating these questions promises to significantly enrich our understanding of the intricate structure underlying distributional chaos.

A pivotal issue explored in this manuscript pertains to the interconnection among distinct hierarchical tiers of chaos: given a system that demonstrates chaos involving n points (i.e., n-chaos), does it necessarily follow that chaos involving a smaller number of points, say m points, where $m < n$, is also present? Although it is broadly recognized that n-chaos entails m-chaos for $m < n$, the reverse direction does not hold in general. We investigate the precise hypotheses under which these varying chaos levels become equivalent.

A further principal objective of this paper is to identify criteria that ensure the existence of such chaotic sets. Prior research has revealed links between chaotic dynamics and several systems attributes, including entropy, transitivity, and the density of periodic points. For instance, positive topological entropy frequently indicates the presence of distributional chaos.

A robust theory of topological dynamics is fundamentally grounded in the premise that the phase space is not only metric but also compact. The majority of general findings related to chaotic behavior (including those associated with positive entropy) are obtained under this compactness assumption [10, 11, 17]. However, there is often a practical and theoretical need to examine dynamical systems on more general topological spaces that may not be metrizable. A substantial difficulty emerges because many core concepts in topological dynamics are formulated using notions of distance between points or sets. In the absence of a metric, such distance-based constructs cannot be defined without imposing additional structure beyond the topology itself. This impediment can be circumvented by considering completely regular topological spaces endowed with a uniformity. A uniformity furnishes a coherent, topological method for gauging proximity between points, effectively acting as a generalization of a metric.

The generalization of dynamical concepts from metric to uniform spaces has constituted an active research frontier. Das and collaborators [4] extended foundational definitions such as expansivity for topological dynamical systems from metric to uniform topological spaces. In our earlier research [15], we formulated and examined topological analogs of weak rigidity and sensitivity, providing equivalent descriptions for uniform rigidity. We subsequently proved [16] that a topologically transitive dynamical system must be either sensitive or almost equicontinuous.

More recently [2], we successfully extended the notions of entropy points, expansivity, and the shadowing property to uniform spaces, revealing a connection between the topological shadowing property and positive uniform entropy. Shah and associates [13] showed that a topologically shadowing, mixing, and expansive system on a totally bounded uniform space possesses the topological specification property. We also demonstrated [1] that systems with ergodic shadowing are topologically chain transitive, and that numerous chain transitive properties of a topological dynamical system are preserved under iteration [3].

In a notable contribution, Good and Macías [6] furnished equivalent characterizations and established the iteration invariance of various shadowing definitions within the setting of compact uniformities, thereby generalizing previous results from the compact metric case. Their work was later advanced in [7]. Good et al. [5] further explored diverse shadowing properties in the compact uniform context. For a more comprehensive overview of results pertaining to shadowing, transitivity, and chain properties, we direct the reader to [11, 12, 14] and the references cited therein.

In this paper, we generalize well-established theorems concerning scrambled sets [18] to the uniform space setting.

Structure of the paper:

- Section 2 introduces the necessary definitions, notation, and background material.
- Section 3 analyzes the equivalence between different levels of chaos in uniform dynamical systems and establishes criteria for inheritance under iteration.
- Section 4 provides conclusions, open problems, and directions for future research.

2. **Basic definitions and preliminaries**

This section establishes the foundational concepts needed for our main results. We begin with uniform spaces and then define distributional chaos. Consider a non-empty set X. A collection $\mathscr{U}$ of subsets of $X \times X$ is termed a *uniformity* on X provided it fulfills these axioms:

(1) For any $E_1, E_2 \in \mathscr{U}$, the intersection $E_1 \cap E_2$ is also contained in $\mathscr{U}$, and if $E_1 \subset E_2$ and $E_1 \in \mathscr{U}$, then $E_2 \in \mathscr{U}$;
(2) Every set $E \in \mathscr{U}$ contains the diagonal $\Delta_X = \{(x, x) : x \in X\}$;
(3) If $E \in \mathscr{U}$, then $E^{-1} = \{(y, x) : (x, y) \in E\} \in \mathscr{U}$;
(4) For any $E \in \mathscr{U}$ there exists $\hat{E} \in \mathscr{U}$ such that $\hat{E} \circ \hat{E} \subset E$, where

$$\hat{E} \circ \hat{E} = \{(x, y) : (x, z) \in \hat{E}, (z, y) \in \hat{E}, \text{ for some } z \in X\}.$$

The pair $(X, \mathscr{U})$, consisting of the set X equipped with a uniformity $\mathscr{U}$, is known as a *uniform space* [8]. A subfamily $\mathscr{B}$ of $\mathscr{U}$ forms a basis for $\mathscr{U}$ if every element of $\mathscr{U}$ contains an element of $\mathscr{B}$. It is straightforward to verify that the symmetric entourages within $\mathscr{U}$ serve as such a basis. Every compact topological space admits a uniform structure, specifically the one generated by its open coverings. This construction facilitates the examination of uniform continuity for functions defined on compact spaces. Members of the uniformity $\mathscr{U}$ are referred to as *entourages* of X. An entourage E is called *symmetric* whenever $E = E^{-1}$.
Let $(X, \mathscr{U})$ be a uniform space. For a point $x \in X$ and an entourage $E \in \mathscr{U}$, we define $E[x] = \{y \in X : (x, y) \in E\}$. Consider $\tau_{\mathscr{U}} = \{A \subset X :$ for each $a \in A$, there exists $E \in \mathscr{U}$ with $E[a] \subset A\}$. Then $\tau_{\mathscr{U}}$ constitutes a topology on X induced by the uniformity $\mathscr{U}$, termed the *uniform topology.*
A function $f : (X, \mathscr{U}) \longrightarrow (Y, \mathscr{V})$ is *uniformly continuous* provided that for every entourage E of Y, the inverse image $(f \times f)^{-1}(E)$ is an entourage of X. By a topological dynamical system (X, T) we mean a uniform space X and a uniformly continuous map $T : X \to X$.
Suppose (X, T) is a topological dynamical system and $n \geq 2$. The product system is given by $(X^{(n)}, T^{(n)})$, where $X^{(n)} = X \times X \times \cdots \times X$ (n-times) and $T^{(n)} = T \times \cdots \times T$ (n-times). The set $\triangle^n$ is defined as $\triangle^n = \{(x_1, x_2, \cdots, x_n) \in X^{(n)} : \exists\ 1 \leq i < j \leq n \text{ with } x_i = x_j\}$.
A topological dynamical system (X, T) is *topologically transitive* when, for any pair of nonempty open subsets U and V in X, one can find $k \in \mathbb{N}$ such that

$T^k(U) \cap V \neq \emptyset$. It exhibits *sensitivity* if there exists an entourage $E \in \mathscr{U}$ with the property that for any point $x \in X$ and any entourage $F \in \mathscr{U}$, there is a point $y \in F[x]$ and a positive integer $k \in \mathbb{N}$ for which $(T^k(x), T^k(y)) \notin E$. The system is *weakly mixing* if, given any four nonempty open sets U_1, U_2, V_1, V_2, there exists a positive integer $k \in \mathbb{N}$ satisfying $T^k(U_i) \cap V_i \neq \emptyset$ for each $i = 1, 2$. A point $x \in X$ is an *n-periodic point* of T provided $T^n(x) = x$ for some $n \in \mathbb{N}$ and $T^k(x) \neq x$ for all $1 \leq k \leq n-1$. The case $n = 1$ corresponds to a fixed point. A topological dynamical system (X, T) is considered *Devaney chaotic* when it is topologically transitive, sensitively dependent on initial conditions, and its periodic points form a dense subset of X.

To clarify the above definitions, we provide several concrete examples.

Example 2.1 (Metric space as a uniform space). *Let (X, d) be a metric space. For each $\varepsilon > 0$, define $E_\varepsilon = \{(x, y) : d(x, y) < \varepsilon\}$. Then $\{E_\varepsilon\}_{\varepsilon>0}$ forms a for a uniformity on X, called the metric uniformity. A map $T : X \to X$ is uniformly continuous in the metric sense if and only if it is uniformly continuous with respect to this uniformity.*

Example 2.2 (Shift space). *basis Define $\Sigma_2 = \{0, 1\}^{\mathbb{N}} = \{(x_0, x_1, x_2, \cdots) : x_l \in \{0, 1\}\}$. A uniformity $\mathscr{U}$ on Σ_2 is generated by the basis of entourages $V_r = \{(x, y) : \max\{\frac{1}{l+1} : x_l \neq y_l\} < r\}$ for $r > 0$. The shift transformation $\sigma : \Sigma_2 \to \Sigma_2$ is given by $\sigma(x) = (x_1, x_2, \cdots)$. This system is uniformly continuous and serves as a canonical example of chaos.*

3. Main results: equivalence and inheritance of distributional n-chaos

This section analyzes the equivalence of distributionally topological n-scrambled sets and the conditions under which uncountable distributionally topological n-scrambled sets are inherited. We also address the converse of the main theorems.

Definition 3.1 ([18]). Given a topological dynamical system (X, T) and an integer $n \geq 2$, a subset $D \subseteq X$ is said to be *distributionally topological n-scrambled* provided that for any selection of n distinct points $x_1, x_2, x_3, \ldots, x_n \in D$ and for every entourage $E \in \mathscr{U}$,

$$F^*_{x_1,\ldots,x_n}(E, T) = \limsup_{N\to\infty} \frac{1}{N} \sharp \left\{ 1 \leq k \leq N : \bigcup_{1\leq i<j\leq n} (T^k x_i, T^k x_j) \in E \right\} = 1,$$

and, for some entourage $F \in \mathscr{U}$,

$$F_{x_1,\ldots,x_n}(F, T) = \liminf_{N\to\infty} \frac{1}{N} \sharp \left\{ 1 \leq k \leq N : \exists_{1\leq i<j\leq n} (T^k x_i, T^k x_j) \notin F \right\} = 0,$$

where $\sharp$ denotes the cardinality of a set.

The notation $\bigcup_{1\leq i<j\leq n}(T^k x_i, T^k x_j) \in E$ signifies that for every pair $1 \leq i < j \leq n$, the condition $(T^k x_i, T^k x_j) \in E$ holds. Conversely, $\exists_{1\leq i<j\leq n}(T^k x_i, T^k x_j) \notin F$ means there exists at least one pair $1 \leq i < j \leq n$ for which $(T^k x_i, T^k x_j) \notin F$.

A set of points $x_1, x_2, x_3, \ldots, x_n$ satisfying these conditions is termed a *distributionally topological n-scrambled tuple*. The system (X, T) is described as *distributionally topological n-chaotic* when it possesses an uncountable distributionally topological n-scrambled set. The case $n = 2$ corresponds to the standard notion of distributional chaos.

The following result follows directly from the definition.

Proposition 3.2. *Consider a topological dynamical system (X, T). If a subset $D \subseteq X$ is not distributionally topological 2-scrambled for T, then it cannot contain any distributionally topological n-scrambled tuple for integers $n > 2$. Moreover, if D is uncountable, a map T that fails to be distributionally topological 2-chaotic cannot exhibit distributional n-chaos for any $n > 2$.*

A collection $\mathcal{F}$ of subsets of $\mathbb{N}_0$ is a *Furstenburg family* if it is upward closed; that is, if $\mathbb{S} \in \mathcal{F}$ and $\mathbb{S} \subset \mathbb{S}'$, then $\mathbb{S}' \in \mathcal{F}$. For any subset $\mathbb{S} \subset \mathbb{N}_0$, its upper density is

$$\overline{d}(\mathbb{S}) := \limsup_{n\to\infty} \frac{1}{n} |\mathbb{S} \cap \{0, 1, \ldots, n-1\}|$$

and its lower density is

$$\underline{d}(\mathbb{S}) := \liminf_{n\to\infty} \frac{1}{n} |\mathbb{S} \cap \{0, 1, \ldots, n-1\}|.$$

When $\overline{d}(\mathbb{S}) = \underline{d}(\mathbb{S}) = d(\mathbb{S})$, we say the set $\mathbb{S}$ has density $d(\mathbb{S})$.

For a set $G \subseteq \mathbb{N}$, its upper and lower densities are defined respectively as:

$$\overline{d}(G) = \limsup_{n\to\infty} \frac{\sharp(G \cap \{1, 2, ..., n\})}{n}, \quad \underline{d}(G) = \liminf_{n\to\infty} \frac{\sharp(G \cap \{1, 2, ..., n\})}{n}.$$

Let $\overline{M}(s) = \{G \subseteq \mathbb{N} : \overline{d}(G) \geq s\}$ for any $s \in [0, 1]$. Consequently,

$$\overline{M}(1) = \left\{ G \subseteq \mathbb{N} : \limsup_{n\to\infty} \frac{\sharp(G \cap \{1, ..., n\})}{n} = 1 \right\}.$$

The following observation follows from the definition.

Remark 3.3. (1) For any entourage $E \in \mathscr{U}$,

$$F^*_{x_1,\cdots,x_n}(E, T) = 1 \iff \left\{ k \in \mathbb{N} : \bigcup_{1\leq i<j\leq n} (T^k x_i, T^k x_j) \in E \right\} \in \overline{M}(1).$$

(2) For some entourage $F \in \mathscr{U}$,

$$F_{x_1,\cdots,x_n}(F, T) = 0 \iff \left\{ k \in \mathbb{N} : \forall_{1\leq i<j\leq n}(T^k x_i, T^k x_j) \notin F \right\} \in \overline{M}(1).$$

Theorem 3.4. *Let (X, T) be a topological dynamical system. If $D \subseteq X$ constitutes a distributionally topological n-scrambled set for T, then it is also a distributionally topological m-scrambled set for any $2 \leq m < n$.*

Proof. Given that D is a distributionally topological n-scrambled set of T, for any selection of n distinct points $x_1, \ldots, x_n \in D$ the following holds:

$$(a)\ \forall E \in \mathscr{U},\ F^*_{x_1,\ldots,x_n}(E,T) = 1, \qquad (b)\ \exists F \in \mathscr{U},\ F_{x_1,\ldots,x_n}(F,T) = 0.$$

By the remark in conjunction with condition (a), the equality $F^*_{x_1,\cdots,x_n}(E,T) = 1$ corresponds to

$$\left\{ k \in \mathbb{N} : \bigcup_{1 \leq i < j \leq n} (T^k x_i, T^k x_j) \in E \right\} \in \overline{M}(1).$$

Now, for an arbitrary $k \in \mathbb{N}$, observe that the condition

$$\bigcup_{1 \leq i < j \leq n} (T^k x_i, T^k x_j) \in E$$

implies $(T^k x_i, T^k x_j) \in E$ holds for all pairs $1 \leq i < j \leq n$. Consequently, $(T^k x_i, T^k x_j) \in E$ is also valid for any $1 \leq i < j \leq m$ with $2 \leq m < n$. It follows that $\bigcup_{1 \leq i < j \leq m} (T^k x_i, T^k x_j) \in E$, leading to the set inclusion

$$\left\{ k \in \mathbb{N} : \bigcup_{1 \leq i < j \leq n} (T^k x_i, T^k x_j) \in E \right\} \subseteq \left\{ k \in \mathbb{N} : \bigcup_{1 \leq i < j \leq m} (T^k x_i, T^k x_j) \in E \right\}.$$

Hence,

$$\left\{ k \in \mathbb{N} : \bigcup_{1 \leq i < j \leq m} (T^k x_i, T^k x_j) \in E \right\} \in \overline{M}(1),$$

which is itself equivalent to the statement $F^*_{x_1,\cdots,x_m}(E,T) = 1$.

From condition (b) and the remark, $F_{x_1,\ldots,x_n}(F,T) = 0$ corresponds to

$$\left\{ k \in \mathbb{N} : \forall_{1 \leq i < j \leq n} (T^k x_i, T^k x_j) \notin F \right\} \in \overline{M}(1).$$

For any fixed $k \in \mathbb{N}$, the relation $\forall_{1 \leq i < j \leq n} (T^k x_i, T^k x_j) \notin F$ necessarily implies $\forall_{1 \leq i < j \leq m} (T^k x_i, T^k x_j) \notin F$. This yields the containment

$$\left\{ k \in \mathbb{N} : \forall_{1 \leq i < j \leq n} (T^k x_i, T^k x_j) \notin F \right\} \subseteq \left\{ k \in \mathbb{N} : \forall_{1 \leq i < j \leq m} (T^k x_i, T^k x_j) \notin F \right\}.$$

Using the upward closure property of $\overline{M}(1)$, we obtain

$$\left\{ k \in \mathbb{N} : \forall_{1 \leq i < j \leq m} (T^k x_i, T^k x_j) \notin F \right\} \in \overline{M}(1),$$

which consequently gives $F_{x_1,\ldots,x_m}(F,T) = 0$. Thus, D is verified to be a distributionally topological m-scrambled set for all integers $2 \leq m < n$. $\square$

We now examine the converse direction: does the existence of m-scrambled sets for all $m < n$ imply the existence of an n-scrambled set? In general, the answer is negative without additional assumptions.

The difficulty arises because for an n-scrambled tuple, we need the chaotic behavior to be *synchronized*: the same times k must witness divergence for *all* pairs simultaneously. For m-scrambled sets, the witnessing times may differ for different pairs. To overcome this obstruction, we introduce a density condition that ensures synchronization across all pairs in the set.

Remark 3.5 (Density Condition (DC)). Consider a subset $D \subseteq X$. Suppose there exist subsets $A, B \subseteq \mathbb{N}$ with upper densities $\overline{d}(A) = \overline{d}(B) = 1$ such that for every scrambled pair $(x, y) \in D \times D$ the following conditions are met:

$$\forall E \in \mathscr{U},\ A \subseteq \left\{k \in \mathbb{N} : (T^k x, T^k y) \in E\right\} \text{ and } \exists F \in \mathscr{U},\ B \subseteq \left\{k \in \mathbb{N} : (T^k x, T^k y) \notin F\right\},$$

then we say D satisfies the **density condition (DC)**.

The intuition behind (DC) is that all scrambled pairs in D share a common set of times A when they are simultaneously close, and a common set of times B when they are simultaneously far apart. This uniformity is what allows us to stitch together pairwise chaotic behavior into n-wise chaos. As we will see in Section 4, this condition is satisfied in natural examples such as the full shift.

Theorem 3.6. *Let $D \subseteq X$ be a distributionally topological 2-scrambled set of T satisfying density condition (DC). Then D is a distributionally topological n-scrambled set of T for every $n \geq 2$.*

Proof. Suppose $D \subset X$ is a distributionally topological 2-scrambled set for T, with subsets $A, B \subset \mathbb{N}$ satisfying $\overline{d}(A) = \overline{d}(B) = 1$. For any integer $n \geq 2$, consider the following arrangement of distinct point pairs:

$$(x_1, x_2), (x_1, x_3), \ldots, (x_1, x_n); (x_2, x_3), (x_2, x_4), \ldots, (x_2, x_n);$$
$$(x_3, x_4), (x_3, x_5), \ldots, (x_3, x_n); \ldots; (x_{n-2}, x_{n-1}), (x_{n-2}, x_n); (x_{n-1}, x_n).$$

Our objective is to demonstrate that D is also a distributionally topological n-scrambled set. This requires establishing the existence of an entourage $F \in \mathscr{U}$ such that

(1)
$$F_{x_1,\ldots,x_n}(F, T) = \liminf_{N\to\infty} \frac{1}{N} \sharp \left\{1 \leq k \leq N : \forall_{1\leq i<j\leq n} (T^k x_i, T^k x_j) \notin F\right\} = 0$$

and, for every entourage $E \in \mathscr{U}$, that

(2)
$$F^*_{x_1,\ldots,x_n}(E, T) = \limsup_{N\to\infty} \frac{1}{N} \sharp \left\{1 \leq k \leq N : \bigcup_{1\leq i<j\leq n} (T^k x_i, T^k x_j) \in E\right\} = 1.$$

We begin by verifying condition (1). For any selection of n distinct points $x_1, x_2, \ldots, x_n \in D$, consider the pairwise combinations outlined above. Focusing first on the pairs (x_1, x_j) with $1 < j \leq n$, the hypothesis guarantees the

existence of a set A with $\overline{d}(A)=1$ and, for each j, an entourage $F_{1,j}\in\mathscr{U}$ such that
$$B\subseteq\left\{k\in\mathbb{N}:(T^kx_1,T^kx_j)\notin F_{1,j}\right\}\text{ for all }1<j\leq n.$$
For the pairs (x_2,x_j) where $2<j\leq n$, one can find corresponding entourages $F_{2,j}\in\mathscr{U}$ satisfying
$$B\subseteq\left\{k\in\mathbb{N}:(T^kx_2,T^kx_j)\notin F_{2,j}\right\}\text{ for all }2<j\leq n.$$
This reasoning extends to the remaining pairs. Specifically, for (x_3,x_j) with $3<j\leq n$, and so forth up to (x_{n-1},x_n), there exist entourages
$$F_{3,j}\in\mathscr{U}\ (3<j\leq n),\ \cdots,\ F_{n-2,j}\in\mathscr{U}\ (n-2<j\leq n),\ F_{n-1,j}\in\mathscr{U}\ (n-1<j\leq n)$$
such that
$$B\subseteq\left\{k\in\mathbb{N}:(T^kx_3,T^kx_j)\notin F_{3,j}\right\}\text{ for all }3<j\leq n,$$
$$\cdots\cdots$$
$$B\subseteq\left\{k\in\mathbb{N}:(T^kx_{n-2},T^kx_j)\notin F_{n-2,j}\right\}\text{ for all }n-2<j\leq n,$$
$$B\subseteq\left\{k\in\mathbb{N}:(T^kx_{n-1},T^kx_j)\notin F_{n-1,j}\right\}\text{ for all }n-1<j\leq n.$$
In consequence, for every pair x_i,x_j with $1\leq i<j\leq n$, one can associate an entourage $F_{i,j}\in\mathscr{U}$ fulfilling
$$B\subseteq\left\{k\in\mathbb{N}:(T^kx_i,T^kx_j)\notin F_{i,j}\right\}\text{ for all }1\leq i<j\leq n.$$
Defining the entourage $F=\bigcap_{1\leq i<j\leq n}F_{i,j}$, we obtain the stronger inclusion
$$B\subseteq\left\{k\in\mathbb{N}:(T^kx_i,T^kx_j)\notin F\right\}\text{ for all }1\leq i<j\leq n.$$
Let $P=\bigcap_{i=1}^{n-1}\bigcap_{j=i+1}^{n}\left\{k\in\mathbb{N}:(T^kx_i,T^kx_j)\notin F\right\}$. It then follows that
$$B\subseteq P\subseteq\left\{k\in\mathbb{N}:\forall_{1\leq i<j\leq n}(T^kx_i,T^kx_j)\notin F\right\}.$$
Given that $\overline{d}(B)=1$, we deduce
$$\overline{d}\left(\left\{k\in\mathbb{N}:\forall_{1\leq i<j\leq n}(T^kx_i,T^kx_j)\notin F\right\}\right)=1,$$
which necessarily implies
$$\underline{d}\left(\left\{k\in\mathbb{N}:\forall_{1\leq i<j\leq n}(T^kx_i,T^kx_j)\notin F\right\}\right)=0,$$
thereby confirming $F_{x_1,\dots,x_n}(F,T)=0$.
We now turn to condition (2). Let $E\in\mathscr{U}$ be an arbitrary entourage. Since each pair x_i,x_j (for $1\leq i<j\leq n$) is distributionally topological 2-scrambled, there exists a set A with $\overline{d}(A)=1$ such that
$$A\subseteq\left\{k\in\mathbb{N}:(T^kx_i,T^kx_j)\in E\right\}\text{ for all }1\leq i<j\leq n.$$
Define $Q=\bigcap_{i=1}^{n-1}\bigcap_{j=i+1}^{n}\left\{k\in\mathbb{N}:(T^kx_i,T^kx_j)\in E\right\}$. This yields the containment
$$A\subseteq Q\subseteq\left\{k\in\mathbb{N}:\bigcup_{1\leq i<j\leq n}(T^kx_i,T^kx_j)\in E\right\}.$$

Again, employing the fact that $\overline{d}(A) = 1$, we have

$$\overline{d}\left(\left\{k \in \mathbb{N} : \bigcup_{1\le i<j\le n} (T^k x_i, T^k x_j) \in E\right\}\right) = 1,$$

which can be expressed as $F^*_{x_1,\ldots,x_n}(E,T) = 1$. Thus, D is verified to be a distributionally topological n-scrambled set for T. □

Remark 3.7. For a topological dynamical system (X,T) possessing an uncountable distributionally topological n-scrambled set for each $n \ge 2$, the statements of Theorems 3.4 and 3.6 can be reformulated for any integers $2 \le m < n$ as follows:

(1) Distributional n-chaos for T implies distributional m-chaos.
(2) If T exhibits distributional 2-chaos and condition (DC) holds, then T is distributionally topological n-chaotic for every $n \ge 2$.

We continue by showing that distributional n-chaos of a topological dynamical system is inherited by its iteration, and vice versa.

Remark 3.8. We define the following notations.

$$\xi_N(T,x_1,\ldots,x_n,E) = \sum_{k=1}^{N} \chi_E\Big(\bigcup_{1\le i<j\le n} (T^k x_i, T^k x_j)\Big),$$

$$\xi^c_N(T,x_1,\ldots,x_n,E) = \sum_{k=1}^{N} \chi_{E^c}\Big(\exists_{1\le i<j\le n}(T^k x_i, T^k x_j)\Big),$$

$$\eta_N(T,x_1,\ldots,x_n,E) = \sum_{k=1}^{N} \chi_E\Big(\exists_{1\le i<j\le n}(T^k x_i, T^k x_j)\Big),$$

$$\eta^c_N(T,x_1,\ldots,x_n,E) = \sum_{k=1}^{N} \chi_{E^c}\Big(\bigcup_{1\le i<j\le n} (T^k x_i, T^k x_j)\Big).$$

Then one can see that

$$F^*_{x_1,\ldots,x_n}(E,T) = \limsup_{N\to\infty} \frac{1}{N}\xi_N(T,x_1,\ldots,x_n,E)$$

and

$$F_{x_1,\ldots,x_n}(E,T) = \liminf_{N\to\infty} \frac{1}{N}\eta_N(T,x_1,\ldots,x_n,E).$$

Many researchers examine conditions under which key properties including topological transitivity, mixing, recurrence, and shadowing are preserved when passing from a dynamical system (X,f) to its iterates (X,f^n) for $n \ge 2$. Several classical results are extended, and new criteria are established that characterize when such properties are inherited. In particular, it is shown that while topological mixing propagates to all iterates, transitivity and recurrence require additional assumptions, such as surjectivity or minimality. The work of

Wang established that distributional chaos can propagate from a system to its iterates. The proposition below demonstrates that, conversely, the property of distributional n-chaos in an iterated system ensures its presence in the original system.

Proposition 3.9. *Let (X,T) be a topological dynamical system. Then for any selection of points $x_1, x_2, \ldots, x_n \in X$ and any positive integer $p > 0$,*

(a) $F_{x_1,x_2,\ldots,x_n}(E, T^p) = 0$ for some entourage $E \in \mathscr{U}$ implies $F_{x_1,x_2,\ldots,x_n}(F,T) = 0$ for some entourage $F \in \mathscr{U}$.

(b) $F^*_{x_1,x_2,\ldots,x_n}(E, T^p) = 1$ for any entourage $E \in \mathscr{U}$ implies $F^*_{x_1,x_2,\ldots,x_n}(F,T) = 1$ for any entourage $F \in \mathscr{U}$.

Proof. (a) The assumption $F_{x_1,x_2,\ldots,x_n}(E, T^p) = 0$ for some $E \in \mathscr{U}$ guarantees the existence of an increasing sequence $\{n_l\}_{l=1}^{\infty} \subseteq \mathbb{N}$ for which

$$\frac{1}{n_l}\eta_{n_l}(T^p, x_1, x_2, \ldots, x_n, E) \to 0 \quad \text{as} \quad l \to \infty. \tag{3}$$

By virtue of the compactness of X and the continuity of T, each map T^s for $s \in \{0, 1, \ldots, p-1\}$ is uniformly continuous. Consequently, for the given entourage $E \in \mathscr{U}$ one can select an entourage $F \in \mathscr{U}$ such that for any n distinct points $x_1, x_2, \ldots, x_n \in X$ and any $s \in \{0, 1, \ldots, p-1\}$ the following implication holds:

$$\forall_{1 \le i < j \le n}(T^p x_i, T^p x_j) \notin E \Rightarrow \forall_{1 \le i < j \le n}(T^s x_i, T^s x_j) \notin F.$$

This leads to the inequality

$$p(\eta^c_{n_l}(T^p, x_1, x_2, \ldots, x_n, E) - 1) \le \eta^c_{pn_l}(T, x_1, x_2, \ldots, x_n, F). \tag{4}$$

Set $m_l = pn_l$. Combining inequality (4) with the identity

$$\frac{1}{n_l}\eta_{n_l}(T, x_1, x_2, \ldots, x_n, F) + \frac{1}{n_l}\eta^c_{n_l}(T, x_1, x_2, \ldots, x_n, F) = 1 \quad \text{for each } n_l,$$

yields the estimate

$$\frac{1}{m_l}\eta_{m_l}(T, x_1, x_2, \ldots, x_n, F) \le \frac{1}{n_l}\eta_{n_l}(T^p, x_1, x_2, \ldots, x_n, E) + \frac{1}{n_l}. \tag{5}$$

From condition (3) together with inequality (5), it follows that

$$\frac{1}{m_l}\eta_{m_l}(T, x_1, x_2, \ldots, x_n, F) \to 0 \quad \text{as} \quad l \to \infty,$$

which establishes $F_{x_1,x_2,\ldots,x_n}(F,T) = 0$ for some entourage $F \in \mathscr{U}$.

(b) Given that $F^*_{x_1,x_2,\ldots,x_n}(E, T^p) = 1$ for all entourages $E \in \mathscr{U}$, for an arbitrary entourage $F \in \mathscr{U}$ we can choose an entourage $E_0 \in \mathscr{U}$ with the property that for any n distinct points $x_1, x_2, \ldots, x_n \in X$ satisfying $\bigcup_{1 \le i < j \le n}(x_i, x_j) \in E_0$,

$$\bigcup_{1 \le i < j \le n}(T^s(x_i), T^s(x_j)) \in F \text{ for any } s \in \{0, 1, \ldots, p-1\}.$$

For this particular $E_0 \in \mathscr{U}$, we have $F^*_{x_1,x_2,\ldots,x_n}(E_0, T^p) = 1$. Hence, there exists an increasing sequence $\{n_l\}_{l=1}^{\infty} \subseteq \mathbb{N}$ such that

$$\frac{1}{n_l}\xi_{n_l}(T^p, x_1, x_2, \ldots, x_n, E_0) \to 1 \quad \text{as} \quad l \to \infty. \tag{6}$$

Letting $m_l = pn_l$, we deduce $p(\xi_{n_l}(T^p, x_1, x_2, \ldots, x_n, E_0) - 1) \leq \xi_{m_l}(T, x_1, x_2, \ldots, x_n, F)$. Furthermore,

$$\frac{1}{n_l}\xi_{n_l}(T^p, x_1, x_2, \ldots, x_n, E_0) \leq \frac{1}{m_l}\xi_{m_l}(T, x_1, x_2, \ldots, x_n, F),$$

which, in conjunction with condition (6), gives

$$\frac{1}{m_l}\xi_{m_l}(T, x_1, x_2, \ldots, x_n, F) \to 1 \quad \text{as} \quad l \to \infty.$$

Therefore, $F^*_{x_1,x_2,\ldots,x_n}(F, T) = 1$ for any entourage $F \in \mathscr{U}$. □

Employing reasoning analogous to that found in the proof of Lemma 3.2 in Wang's work, one obtains the following result.

Proposition 3.10. *Let (X, T) be a topological dynamical system. Then for any $x_1, x_2, \ldots, x_n \in X$, any entourage $E \in \mathscr{U}$, and any positive integer $p > 0$,*
(a) $F_{x_1,x_2,\ldots,x_n}(E, T) = 0$ implies $F_{x_1,x_2,\ldots,x_n}(E, T^p) = 0$.
(b) $F^*_{x_1,x_2,\ldots,x_n}(E, T) = 1$ implies $F^*_{x_1,x_2,\ldots,x_n}(E, T^p) = 1$.

Theorem 3.11. *Let (X, T) be a topological dynamical system. Then T exhibits distributional n-chaos if and only if T^p exhibits distributional n-chaos for every positive integer $p \geq 2$ and for all $n > 2$.*

Corollary 3.12. *For a topological dynamical system (X, T) and any fixed integer $n > 2$, the following statements are equivalent:*

(1) *T is distributionally n-chaotic.*
(2) *For every integer $p \geq 2$, the iterate T^p is distributionally topological n-chaotic.*
(3) *There exists some integer $p \geq 2$ such that T^p is distributionally topological n-chaotic.*

Examples

Example 3.13 (Full shift scrambled set)**.** *Let $\Sigma_2 = \{0, 1\}^{\mathbb{N}}$ be the space of all one-sided infinite sequences of 0's and 1's, equipped with the product topology and consider the shift map $\sigma : \Sigma_2 \to \Sigma_2$ by $\sigma(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots)$. The pair (Σ_2, σ) is a topological dynamical system. Moreover, Σ_2 is a subshift of finite type (SFT) with alphabet $\{0, 1\}$ and empty set of forbidden blocks; it is known as the* full 2-shift*. The system is topologically mixing (hence transitive) and has positive topological entropy.*

We now exhibit an uncountable 2*-scrambled set. For each* $t \in (0,1)$*, construct a point* $x(t) \in \Sigma_2$ *as follows. Let* $(w_n)_{n\geq 1}$ *be an enumeration of all finite words over* $\{0,1\}$*. Define* $x(t)$ *by concatenating the words*

$$w_1,\ w_2,\ \ldots,\ w_k,\ \ldots$$

but insert a block of 0*'s of length* $\lfloor k \cdot t \rfloor$ *after each* w_k*. One can verify that for any distinct* $t_1, t_2 \in (0,1)$*, the pair* $(x(t_1), x(t_2))$ *satisfies*

$$\liminf_{n\to\infty} d(\sigma^n(x(t_1)), \sigma^n(x(t_2))) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d(\sigma^n(x(t_1)), \sigma^n(x(t_2))) > 0.$$

Thus $S = \{x(t) : t \in (0,1)\}$ *is an uncountable subset of* Σ_2 *in which every pair of distinct points forms a scrambled pair. Hence* S *is an uncountable* 2*-scrambled set, and* (Σ_2, σ) *is Li-Yorke chaotic.*

Example 3.14 (Non-metrizable uniformity)**.** *Let* $\Sigma = \{(x_j) : x_j \in \{0,1\},\ j = 0,1,2,\cdots\}$*. For a subset* $\mathbb{J}$ *of* $\mathbb{N}_0$*, the upper density* $\overline{d}$ *of* $\mathbb{J}$ *is defined by*

$$\overline{d}(\mathbb{J}) = \limsup_{n\to\infty} \frac{1}{n} |\mathbb{J} \cap \{0,1,\cdots,n-1\}|.$$

For each $\epsilon > 0$*, let*

$$U_\epsilon = \{\big((x_j),(y_j)\big) \in \Sigma \times \Sigma :\ \overline{d}\,(\{j \geq 0 :\ x_j \neq y_j\}) < \epsilon\}.$$

Then the family $\mathscr{B} = \{U_\epsilon : \epsilon > 0\}$ *is a basis for a non-metrizable uniformity* $\mathscr{U}$ *on* Σ*. It is easy to see that any distinct pair* (x,y) *cannot be a distributionally topological* 2*-scrambled pair.*

Example 3.15 (Non-existence of 3-scrambled tuple)**.** *Take the topological dynamical system* (Σ_2, σ) *defined earlier and define the points*

$$x_1 = 00000000\cdots, \quad x_2 = 101010101010\cdots, \quad x_3 = 0101010101\cdots.$$

Clearly, no pair (x_i, x_j) *with* $1 \leq i < j \leq 3$ *is distributionally topological* 2*-scrambled. Therefore, the triple* (x_1, x_2, x_3) *cannot be a distributionally topological* 3*-scrambled tuple.*

Remark 3.16 (Concrete Example Satisfying Condition (DC))*.* The density condition (DC) is a strong sufficient condition. We now provide a concrete example of a dynamical system where (DC) holds.

Let (Σ_2, σ) be the full shift on two symbols, where $\Sigma_2 = \{0,1\}^{\mathbb{N}_0}$ with the metric

$$d(x,y) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^{i+1}}.$$

For each $t \in (0,1)$, let $\{w_n\}_{n=1}^{\infty}$ be an enumeration of all finite binary words. Define

$$x^{(t)} = w_1 0^{\lfloor t\cdot 1\rfloor} w_2 0^{\lfloor t\cdot 2\rfloor} w_3 0^{\lfloor t\cdot 3\rfloor} \cdots$$

where 0^m denotes a block of m zeros. Let

$$D = \{x^{(t)} : t \in (0,1)\}.$$

Let $A = \mathbb{N}$ (trivially has upper density 1). For B, define

$$B = \left\{ k \in \mathbb{N} : k \text{ falls within a zero block of length at least } \lfloor \sqrt{k} \rfloor \right\}.$$

This set has upper density 1 because the zero blocks grow in length.

For any entourage $E \in \mathscr{U}$, there exists $N \in \mathbb{N}$ such that

$$E_N = \{(x, y) : x_i = y_i \text{ for all } i = 0, 1, \ldots, N-1\} \subseteq E.$$

For any two distinct points $x^{(s)}, x^{(t)} \in D$ with $s \neq t$, and any $k \in \mathbb{N}$, if k falls within a region where both points have the same finite word w_n (which occurs infinitely often by construction), then for those k,

$$(\sigma^k x^{(s)}, \sigma^k x^{(t)}) \in E_N \subseteq E.$$

The set of such k has upper density 1 uniformly in s, t, because the blocks w_n are exhaustive and appear periodically. Thus

$$A = \mathbb{N} \subseteq \{k \in \mathbb{N} : (\sigma^k x^{(s)}, \sigma^k x^{(t)}) \in E\}$$

for all $E \in \mathscr{U}$.

Fix an entourage F such that

$$F = E_1 = \{(x, y) : x_0 = y_0\}$$

(i.e., agreement on the first coordinate). For any two distinct $x^{(s)}, x^{(t)} \in D$ with $s \neq t$, consider the times k when the shift is positioned at the beginning of a zero block of length $\lfloor t \cdot n \rfloor$ vs. $\lfloor s \cdot n \rfloor$ for some n. At such positions, since $s \neq t$, the lengths of the zero blocks differ, so there exists a coordinate where one point has 0 and the other has 1. Hence

$$(\sigma^k x^{(s)}, \sigma^k x^{(t)}) \notin F.$$

The set of such k has upper density 1 uniformly in s, t, because the zero blocks grow without bound and occur with frequency 1 in the upper density sense. Therefore

$$B \subseteq \{k \in \mathbb{N} : (\sigma^k x^{(s)}, \sigma^k x^{(t)}) \notin F\}.$$

Thus, for this scrambled set D in the full shift (Σ_2, σ), we have $A = \mathbb{N}$ with $\overline{d}(A) = 1$ and B with $\overline{d}(B) = 1$ such that for all $x, y \in D$:

$$\forall E \in \mathscr{U}, \quad \mathbb{N} \subseteq \{k : (\sigma^k x, \sigma^k y) \in E\}$$

and

$$\exists F \in \mathscr{U}, \quad B \subseteq \{k : (\sigma^k x, \sigma^k y) \notin F\}.$$

Therefore, D satisfies the density condition (DC).

Discussion. We emphasize that (DC) is a sufficient condition whose necessity is unknown. The condition is quite restrictive, as it requires the same density-one sets A and B to work uniformly for all entourages and all scrambled pairs. However, as the example above shows, there are natural systems where (DC) is satisfied. The key insight is that the construction ensures that the structural features (separators and zero blocks) are synchronized across all points in D, allowing uniform sets A and B to work for all pairs simultaneously.

Whether (DC) is necessary for the equivalence of 2- and n-chaos remains an open question for future investigation.

4. Conclusions and future directions

This paper contributes to the understanding of n-scrambled sets within the context of topological dynamical systems on uniform spaces. Our main achievements include:

- Generalizing the theory of distributional n-scrambled sets from metric spaces to uniform spaces, thereby allowing applications to non-metrizable topological spaces.
- Proving that distributional n-chaos implies distributional m-chaos for all $2 \leq m < n$ (Theorem 3.4).
- Establishing a sufficient condition (density condition DC) under which the converse holds: if a system is m-chaotic for all $m < n$ and satisfies DC, then it is n-chaotic (Theorem 3.6).
- Demonstrating that distributional n-chaos is preserved under iteration and, conversely, that if an iterate is n-chaotic then the original system is n-chaotic (Theorem 3.11).

Several natural questions remain unresolved:

- *Converse of Theorem 3.4:* Does distributional m-chaos for all $m < n$ imply distributional n-chaos without assuming condition (DC)? We suspect the answer is negative in general, and constructing a counterexample remains an open problem.
- *Necessity of (DC):* Is condition (DC) also necessary for the converse? That is, if every m-scrambled set (for all $m < n$) forces an n-scrambled set, must (DC) hold? This requires further investigation.
- *Uniform versus metric:* Are there properties of scrambled sets that are strictly stronger in uniform spaces than in metric spaces? For instance, can we have an n-scrambled set in a non-metrizable uniform space that is not measurable in any compatible metric?

We propose the following avenues for future work:

- *Entropy and n-scrambling:* Investigate whether positive topological uniform entropy implies the existence of uncountable n-scrambled sets for all n, extending known results for metric spaces.
- *Shadowing and n-scrambling:* Given the known relation between shadowing and chaos, examine how uniform shadowing properties (e.g., L-shadowing, asymptotic shadowing) interact with higher-order scrambling.
- *Mean Li-Yorke n-chaos:* Extend our results to mean Li-Yorke n-chaos and compare with distributional n-chaos in uniform spaces.

The results presented here lay a foundation for further exploration of higher-order chaos in uniform spaces, bridging abstract topological dynamics with potential applications in complex systems where non-metrizable structures naturally arise.

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References

[1] Ahmadi, S. A. (2017). Shadowing, ergodic shadowing and uniform spaces. *Filomat*, 31(16):5117–5124. DOI: https://doi.org/10.2298/FIL1716117A

[2] Ahmadi, S. A., Wu, X., Feng, Z., Ma, X., & Lu, X. (2018). On the entropy points and shadowing in uniform spaces. *Int. J. Bifurcation and Chaos*, 28(12):1850155 (10 pages). DOI: https://doi.org/10.1142/S0218127418501555

[3] Darban Maghami, N., Ahmadi, S. A., & Shabani, Z. (2026). A note on sensitivity and chaos on hyperspaces of uniform spaces. *J. Mahani Math. Res.*, 15(1):239–250. DOI: https://doi.org/10.22103/jmmr.2025.24567.1234

[4] Das, T., Lee, K., Richeson, D., & Wiseman, J. (2013). Spectral decomposition for topologically anosov homeomorphisms on noncompact and non-metrizable spaces. *Topology Appl.*, 160(1):149–158. DOI: https://doi.org/10.1016/j.topol.2012.10.009

[5] Good, C., Leek, R., & Mitchell, J. (2020). Equicontinuity, transitivity and sensitivity: The auslander-yorke dichotomy revisited. *Discrete Contin. Dyn. Syst.*, 40(4):2441, 33. DOI: https://doi.org/10.3934/dcds.2020127

[6] Good, C. & Macías, S. (2018). What is topological about topological dynamics? *Discrete Contin. Dyn. Syst.*, 38(3):1007–1031. DOI: https://doi.org/10.3934/dcds.2018045

[7] Good, C., Mitchell, J., & Thomas, J. (2020). Preservation of shadowing in discrete dynamical systems. *J. Math. Anal. Appl.*, 485(1):123767, 39. DOI: https://doi.org/10.1016/j.jmaa.2019.123767

[8] Kelley, J. L. *General topology*. Springer-Verlag, New York-Berlin, 1975. DOI: https://doi.org/10.1007/978-1-4684-9919-1

[9] Liao, G., Wang, L., & Duan, X. (2007). A chaotic function with a distributively scrambled set of full Lebesgue measure. *Nonlinear Analysis: Theory, Methods and Applications*, 66(10), 2274–2280. DOI: https://doi.org/10.1016/j.na.2006.03.021

[10] Kawaguchi, N. (2016). Entropy points of continuous maps with the sensitivity and the shadowing property. *Topology Appl.*, 210:8–15. DOI: https://doi.org/10.1016/j.topol.2016.06.001

[11] Moothathu, T. K. S. (2011). Implications of pseudo-orbit tracing property for continuous maps on compacta. *Topology Appl.*, 158(16):2232–2239. DOI: https://doi.org/10.1016/j.topol.2011.06.032

[12] Richeson, D. & Wiseman, J. (2008). Chain recurrence rates and topological entropy. *Topology Appl.*, 156(2):251–261. DOI: https://doi.org/10.1016/j.topol.2008.06.014

[13] Shah, S., Das, R., & Das, T. (2016). Specification property for topological spaces. *J. Dyn. Control Syst.*, 22(4):615–622. DOI: https://doi.org/10.1007/s10883-015-9300-3

[14] Wu, X., Liang, S., Ma, X., Lu, T., & Ahmadi, S. A. (2020). The mean sensitivity and mean equicontinuity on uniform spaces. *Int. J. Bifurcation and Chaos*, 30(8):2050122 (11 pages). DOI: https://doi.org/10.1142/S0218127420501224

[15] Wu, X., Luo, Ma, X., & Lu, T. (2019). Rigidity and sensitivity on uniform spaces. *Topology Appl.*, 252:145–157. DOI: https://doi.org/10.1016/j.topol.2018.12.005

[16] Wu, X., Ma, X., Zhu, Z., & Lu, T. (2018). Topological ergodic shadowing and chaos on uniform spaces. *Int. J. Bifurcation and Chaos*, 28(3):1850043 (9 pages). DOI: https://doi.org/10.1142/S0218127418500438

[17] Wu, X., Oprocha, P., & Chen, G. (2016). On various definitions of shadowing with average error in tracing. *Nonlinearity*, 29(7):1942–1972. DOI: https://doi.org/10.1088/0951-7715/29/7/1942

[18] Yang, Q. & Yang, X. (2024). On n-scrambled sets. *Topology and its Applications*, 344:108814. DOI: https://doi.org/10.1016/j.topol.2023.108814

Seyyed Alireza Ahmadi
Orcid number: 0000-0003-1267-6553
Department of Mathematics
University of Sistan and Baluchestan
Zahedan, Iran
Email address: `sa.ahmadi@math.usb.ac.ir, sa.ahmdi@gmail.com`