Shahid Bahonar University of Kerman

Journal of Mahani Mathematical Research

JMMR 2012

Print ISSN: 2251-7952 Online ISSN: 2645-4505

NEIGHBORHOOD SYSTEMS AND IRREDUCIBLE FILTERS IN L-ALGEBRAS

M. Ebrahimi ✉ and C. Park

Article type: Research Article

(Received: 01 February 2026, Received in revised form 24 July 2026)

(Accepted: 26 August 2026, Published Online: 26 August 2026)

Abstract. In this respect, the concept of an L-algebra L is revisited and after that, the article is prepared to deal with the notion of irreducible filters of L. Some basic properties of irreducible filters are investigated. In the following, the concept of a neighborhood of a point in an L-algebra is defined, and its connection with the notion of a filter is investigated. The article establishes conditions under which a filter can be expressed as a union of these neighborhoods and in special cases, every filter is itself a neighborhood. These neighborhoods can also form the foundation of a topological structure on L-algebras that a person can pursue in the future.

Keywords: L-algebra; filter; irreducible filter; neighborhood; congruence.
2020 MSC: 06A06, 03B80, 06B10, 32C09

1. Introduction

L-algebras are a class of algebraic structures that emerged from the study of certain types of groups and their connections to the quantum Yang-Baxter equation [7]. They were formally introduced by Rump [9], and then further works were done, even in the field of topology on L-algebras [8]. Additionally, certain algebraic structures such as CL-algebras or KL-algebras were introduced [5], and even their relationships with other structures like MV-algebras were investigated [2]. L-algebras provide a unifying framework for various algebraic structures, including those found in non-classical logic [1,11].
A common definition of a bounded L-algebra $(L, \to, 0, 1)$ involves a set L with a binary operation $\to$ (implication) and two distinguished elements 0 (bottom) and 1 (top). In many contexts, L-algebras are also equipped with a partial order $\leq$ such that $x \leq y$ if and only if $x \to y = 1$. Although this order can make $(L, \leq)$ a bounded lattice, it does not hold true in general. We will investigate this matter in the next section. The implication operation is often related to the relative pseudo-complement (as seen in Heyting algebras) [4].
Filters in L-algebras are special subsets that capture the notion of truth or distinguished elements within the algebraic structure. Their definition is closely tied to the implication operation. Filters are fundamental for defining congruence relations on L-algebras. A congruence relation is an equivalence relation

✉ mohamad_ebrahimi@uk.ac.ir, ORCID: 0000-0003-4830-7100
https://doi.org/10.22103/jmmr.2026.26754.1934

Publisher: Shahid Bahonar University of Kerman
How to cite: M. Ebrahimi, C. Park, *Neighborhood systems and irreducible filters in L-algebras*, J. Mahani Math. Res. 2027; 16(1): 289-303.

that is compatible with the algebraic operations. The quotient algebra formed by factoring an L-algebra by a congruence relation induced by a filter is also an L-algebra. This is a standard construction in universal algebra.
The primary motivation for introducing neighborhoods into L-algebras is the structural unification of discrete logic and continuous space. While L-algebras successfully model the quantum Yang-Baxter equation, they are traditionally treated as discrete sets. By implementing open sets, we can transition from a purely logical analysis to a functional one. Endowing an L-algebra with the concepts of neighborhoods and open sets opens several avenues for practical and theoretical applications. Since L-algebras are rooted in the Yang-Baxter equation, an L-algebra together with these neighborhoods can be used to model topological quantum computation. By combining the congruence relations (which provide the algebraic skeleton) with neighborhood systems (which provide the topological skin), this article moves toward a synthesis where L-algebras are no longer just symbols on a page, but are the building blocks of a new class of geometric spaces. This L-geometry provides the necessary language to explore the intersection of quantum symmetry and topological invariance [10].

2. **Preliminaries**

In this section one can become familiar with the notion of L-algebras and some their basic properties.

An algebra $(L, \to, 1)$ of type $(2, 0)$ is said to be an *L-algebra* [9], if for all $x, y, z \in L$, it satisfies the following conditions:

(1) $x \to x = 1$,
(2) $x \to 1 = 1$,
(3) $1 \to x = x$,
(4) $(x \to y) \to (x \to z) = (y \to x) \to (y \to z)$,
(5) if $x \to y = y \to x = 1$, then $x = y$.

The conditions (1), (2) and (3) satisfy that, 1 is a logical unit when the condition (4) is related to the quantum Yang-Baxter equation.

Proposition 2.1. [9] *A logical unit is unique and it is an element of the L-algebra L.*

If $(L, \to, 1)$ is an L-algebra, then there is a partially order on L, defined by:

$$x \leq y \text{ if and only if } x \to y = 1, \text{ for all } x, y \in L.$$

If L has a smallest element 0, then L is called a bounded L-algebra.
As mentioned above a bounded L-algebra is not a Lattice. One can see the following L-algebra which is not a Lattice.

Definition 2.2. [9] For an element $x \in L$, define the downset $\downarrow x$ by

$$\downarrow x = \{y \in L : \ y \leq x\}.$$

$\to$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	0	1	0	0	0	1	1	1
b	0	0	1	0	0	1	1	1
c	0	0	0	1	0	1	1	1
d	0	0	0	0	1	1	1	1
e	0	a	b	c	d	1	e	1
f	0	a	b	c	d	e	1	1
1	0	a	b	c	d	e	f	1

TABLE 1. Cayley table for the L-algebra structure which is not a Lattice.

Definition 2.3. [9] An L-algebra L is said to be a self-similar L-algebra if for each $x \in L$, the map $\phi_x : L \to L$ defined by $\phi_x(y) = x \to y$ induces a bijection $\downarrow x \to L$.

Example 2.4. *Let $L = (-\infty, 0]$ and define $x \to y = min\{0, y - x\}$. Define $\Phi : \downarrow x \to L$ by $\Phi(y) = \phi_x(y) = x \to y$. One can find that $(L, \to, 0)$ is a self-similar L-algebra.*

Corollary 2.5. *[9] Let L be an L-algebra. Then the followings are satisfied for all $x, y, z \in L$,*
(1) $(x \to y) \to 1 = (x \to 1) \to (y \to 1)$, $1 \to (x \to y) = (1 \to x) \to (1 \to y)$,
(2) $x \leq y$ implies that $z \to x \leq z \to y$.
If L is a self-similar L-algebra, then
(3) $x \to (y \to x) = y \to (x \to y)$,

Corollary 2.6. *[9] Let L be an L-algebra and $x, y, z \in L$, then the followings are equivalent:*

(1) $x \leq y \to x$,
(2) $x \leq z$ *implies* $z \to y \leq x \to y$,
(3) $((x \to y) \to z) \to z \leq ((x \to y) \to z) \to ((y \to x) \to z)$.

Definition 2.7. [9] Let L be an L-algebra. A nonempty subset K of L is said to be an *L-subalgebra* if the followings hold:

(1) $1 \in K$,
(2) $x \to y \in K$, for all $x, y \in K$.

Definition 2.8. [9] An L-algebra L is said to be a *commutative* L-algebra if for each $x, y \in L$ the following holds:

$$(x \to y) \to y = (y \to x) \to x.$$

An L-algebra L is said to be *self-distributive* if for each $x, y, z \in L$ the following hold:

$$x \to (y \to z) = (x \to y) \to (x \to z).$$

Definition 2.9. [6] An L-algebra L is said to be a CL-algebra if for each $x, y, z \in L$:

$$x \to (y \to z) = y \to (x \to z).$$

An L-algebra L is said to be a KL-algebra if the following holds:

$$x \le a \to x, \text{ for each } x, a \in L.$$

Definition 2.10. [3] Let L and K be two L-algebras. A map $f : L \to K$ is said to be an L-algebra *homomorphism* (L-homomorphism) if

(1) $f(1) = 1$,
(2) $f(x \to y) = f(x) \to f(y)$, for all $x, y \in L$.

Definition 2.11. [9] Let L be an L-algebra. A subset F of L is said to be a *filter* of L if the followings hold:
1) $1 \in F$,
2) if $x \in F$ and $x \to y \in F$ then $y \in F$, for all $x, y \in L$.
A filter F of L is called a *proper filter* if $F \subset L$.

[5] A relation θ on a set X is said to be an *equivalence relation* if the following conditions are satisfied:
1) $(x, x) \in \theta$, for each $x \in X$, (reflexive)
2) if $(x, y) \in \theta$, then $(y, x) \in \theta$, for each $x, y \in X$, (symmetric)
3) if $(x, y) \in \theta$ and $(y, z) \in \theta$, then $(x, z) \in \theta$, for each $x, y, z \in X$, (transitive).
[5] An equivalence relation θ on an L-algebra L is said to be a *congruence relation* if the following holds:

if $(x, y) \in \theta$ and $(z, w) \in \theta$, then $(x \to z, y \to w) \in \theta$, for each $x, y, z, w \in L$.

3. Irreducible filters

We begin this section by defining the concept of an irreducible filter on an L-algebra. Thereafter, the criteria for a filter to be irreducible are explored, and several sufficient conditions are derived.

Definition 3.1. Let L be an L-algebra. A proper filter F of L is said to be an *irreducible filter* if, for any two proper filters A and B of L, the following holds:

$$F = A \cap B \text{ implies that } F = A \text{ or } F = B.$$

Example 3.2. *Let $L = \{1, a, b, c\}$ and consider Table 2. Then $(L, \to, 1)$ is an L-algebra.*

$F = \{1\}$, $F = \{1, b\}$ and $F = \{1, b, c\}$ are irreducible filters of L.

Proposition 3.3. *Let F be a proper filter of L. If for $a, b \notin F$ there exists $c \notin F$ such that $a \le c$ and $b \le c$, then F is an irreducible filter of L.*

Proof. Suppose that $F = A \cap B$ for two proper filters A, B of L. If $F \ne A$ and $F \ne B$, then there exist $a \in A - F$ and $b \in B - F$. By hypothesis for $a, b \notin F$

$\to$	1	a	b	c
1	1	a	b	c
a	1	1	1	a
b	1	a	1	c
c	1	a	1	1

TABLE 2. The binary operation $\to$ implies Definition 3.1, Example 2.

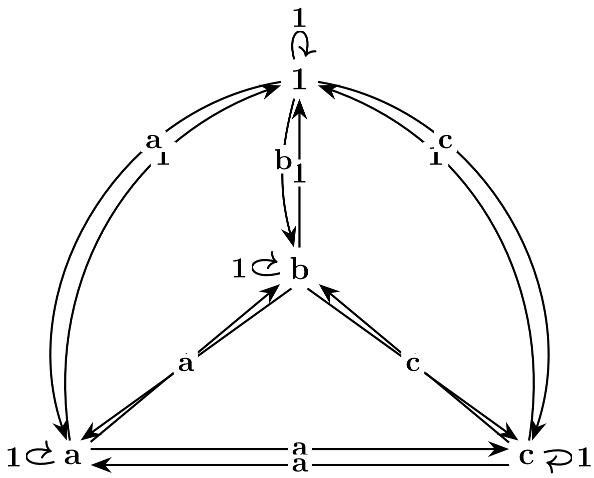

FIGURE 1. The binary operation $\to$ implies Definition 3.1, Example 2.

there exists $c \in L - F$ such that $a \leq c$ and $b \leq c$. Since $a \in A$ and $b \in B$, $c \in A$ and $c \in B$. Thus $c \in A \cap B = F$ that is a contradiction. $\square$

Corollary 3.4. *A proper filter of L is irreducible if and only if for each $a, b \notin F$ there exists $c \notin F$ such that $a \to c \in F$ and $b \to c \in F$.*

Proof. Assume that for each $a, b \notin F$ there exists $c \notin F$ such that $a \to c \in F$ and $b \to c \in F$. If F is not irreducible then there are proper filters A, B of L such that $F = A \cap B$ while $F \neq A$ and $F \neq B$. Then there exists $a \in A - F$ and $b \in B - F$. By hypothesis there exist $c \in L - F$ such that $a \to c \in F$ and $b \to c \in F$. Since A, B are filters of L and $a \in A$ and $b \in B$, $c \in F = A \cap B$ that is a contradiction. Conversely, assume that $F = A \cup B$ where A and B are proper filters of L such that $F \neq A$ and $F \neq B$. Then there exist $a \in A - F$ and $b \in B - F$. By hypothesis there exists $c \in L - F$ such that $a \to c \in F$ and $b \to c \in F$. Since $c \notin F = A \cup B$, Without loss of generality we assume that $c \notin A$. Now $a \in A$ and $a \to c \in A$ implies that $c \in A$ which is a contradiction. □

Definition 3.5. A filter M of L is said to be a maximal filter if for any filter F of L the following holds:

$$M \subseteq F \subseteq L \text{ implies that } F = M \text{ or } F = L.$$

Example 3.6. *Let $L = \{1, a, b, c\}$ as in Example 2, then $F = \{1, b, c\}$ is a maximal filter of L.*

Lemma 3.7. *Every maximal filter of L is an irreducible filter.*

Proof. Let M be a maximal filter of L and $M = A \cap B$ for proper filters A and B of L. Then $M \subseteq A \subset L$ and $M \subseteq B \subset L$. Since M is maximal, $M = A = B$. Therefore M is an irreducible filter. □

The converse of Lemma 3.5 is not true in general case.

Example 3.8. *Let $L = \{1, a, b, c\}$ be as in Example 2. One can see that $\{1, b\}$ is an irreducible filter but it is not a maximal filter of L.*

Then $(L, \to, 1)$ is an L-algebra and $F = \{1, a\}$ is an irreducible filter but it is not a maximal filter.

Proposition 3.9. *Let F be a proper filter of L and $x \in L - F$. Then there exists an irreducible filter A of L such that $F \subseteq A$ and $x \notin A$.*

Proof. Let $x \in L - F$ and define:

$$\mathfrak{F}(x) = \{G : \ G \ is \ a \ filter \ of \ L, \ F \subseteq G, \ x \notin G\}.$$

Since $F \in \mathfrak{F}(x)$, $\mathfrak{F}(x) \neq \emptyset$. Now let $\{G_i\}_{i \in N}$ be a chain of elements of $\mathfrak{F}(x)$. Since $x \notin G_i$ for any $i \in N$, $x \notin \bigcup_{i \in N} G_i$ and furthermore $F \subseteq \bigcup_{i \in N} G_i$. Hence $\bigcup_{i \in N} G_i$ is an upper bound of $\{G_i\}_{i \in N}$.
By Zorn's Lemma, $\mathfrak{F}(x)$ has a maximal element $A \in \mathfrak{F}(x)$. One can show that A is an irreducible filter of L, for this aim suppose that if $A = B \cap C$ for proper filters B and C of L. Since $x \notin A$, $x \notin B$ or $x \notin C$, $F \subseteq A = B \cap C$ implies that $F \subseteq B$ and $F \subseteq C$. Without loss of generality one may assume that $x \notin B$ and $F \subseteq B$. Then $B \in \mathfrak{F}(x)$. But $A \subseteq B$ makes a contradiction with maximality of A in $\mathfrak{F}(x)$. Thus $A = B$ and hence $A \in \mathfrak{F}(x)$ is an irreducible filter of L and $x \notin A$. □

Proposition 3.10. *Every proper filter F of L can be characterized as the intersection of all irreducible filters of L, containing F.*

Proof. Let F be a proper filter of L. One has to prove that

$$F = \bigcap\{A : \ A \ is \ an \ irreducible \ filter \ of \ L, \ F \subseteq A\}.$$

Now suppose that $x \notin F$, then by Proposition 3.6 there exists an irreducible filter B of L such that $F \subseteq B$ and $x \notin B$. Then $x \notin \bigcap\{A : \ A \ is \ an \ irreducible \ filter \ of \ L, \ F \subseteq A\}$. Therefore

$$\bigcap\{A:\ A\ is\ an\ irreducible\ filter\ of\ L,\ F\subseteq A\}\subseteq F.$$

The converse is obvious. □

Lemma 3.11. *Let F be a filter of a CL-algebra L. Then the following holds:*

$$y\in F \text{ implies that } (y\to x)\to x\in F, \text{ for each } x\in L.$$

Proof. Let $y\in F$ and $x\in L$ be arbitrary. $y\to((y\to x)\to x)=(y\to x)\to(y\to x)=1$. Since F is a filter and $y\in F$, $(y\to x)\to x\in F$. □

Lemma 3.12. *Let $\Phi: L\to L'$ be an L-homomorphism of L-algebras and F' be a filter of L'. Then $F=\Phi^{-1}(F')$ is a filter of L.*

Proof. Since $\Phi(1)=1\in F'$, $1\in\Phi^{-1}(F')=F$. Now let $x, x\to y\in F$, then $\Phi(x)\in F'$ and $\Phi(x\to y)\in F'$. $\Phi(x\to y)=\Phi(x)\to\Phi(y)\in F'$ and $\Phi(x)\in F'$ implies that $\Phi(y)\in F'$. Thus $y\in F=\Phi^{-1}(F')$. □

Corollary 3.13. *Let L be a commutative CL-algebra and $\Phi: L\to L'$ be an L-epimorphism that maps filters of L to filters of L'. If F' is an irreducible filter of L', then $F=\Phi^{-1}(F')$ is an irreducible filter of L.*

Proof. By Lemma 3.9, $F=\Phi^{-1}(F')$ is a filter of L. Now if $F=A\cap B$ for two proper filters of L, then $F'=\Phi(F)=\Phi(A\cap B)$. In this step, one can claim that $\Phi(A\cap B)=\Phi(A)\cap\Phi(B)$. It is clear that $\Phi(A\cap B)\subseteq\Phi(A)\cap\Phi(B)$. Now let $y\in\Phi(A)\cap\Phi(B)$, then there exist $a\in A$ and $b\in B$ such that $y=\Phi(a)$ and $y=\Phi(b)$. By Lemma 3.8 $(a\to b)\to b\in A$ and $(b\to a)\to a\in B$.
Thus $(a\to b)\to b=(b\to a)\to a\in A\cap B$. $y=1\to y=(y\to y)\to y=(\Phi(a)\to\Phi(b))\to\Phi(b)=\Phi(a\to b)\to\Phi(b)=\Phi((a\to b)\to b)\in\Phi(A\cap B)$. Hence $\Phi(F)=F'=\Phi(A)\cap\Phi(B)$ and then $F'=\Phi(A)$ or $F'=\Phi(B)$. Therefore $F=\Phi^{-1}(F')=\Phi^{-1}(\Phi(A))=A$ or $F=\Phi^{-1}(F')=\Phi^{-1}(\Phi(B))=B$. □

Proposition 3.14. *Let $\mathfrak{F}(L)$ be the set of all proper filters of an L-algebra L. Then $\mathfrak{F}(L)$ is a linearly ordered set if and only if every proper filter of L is an irreducible filter of L.*

Proof. Suppose that every proper filter of L is an irreducible filter. Let $F\in\mathfrak{F}(L)$ and $G\in\mathfrak{F}(L)$, then $E=F\cap G\in\mathfrak{F}(L)$ and so by hypothesis $E=F$ or $E=G$. Thus $F\subseteq G$ or $G\subseteq F$. Therefore $\mathfrak{F}(L)$ is a linearly ordered set.
For the converse, suppose that $\mathfrak{F}(L)$ is a linearly ordered set and $F=A\cap B$ for $F,A,B\in\mathfrak{F}(L)$. Since $A,B\in\mathfrak{F}(L)$, $A\subseteq B$ or $B\subseteq A$. Hence $F=A\cap B=A$ or $F=A\cap B=B$. □

4. Neighborhoods in an L-algebra

In this section the concept of positive neighborhoods around an element of an L-algebra and negative neighborhoods around an element of a bounded L-algebra L is defined. Then the connection between some neighborhoods and filters of L is investigated. One can see [10] for more information.

Definition 4.1. Let L be a bounded L-algebra and $a \in L$. Define the negative neighborhood of a by

$$U^{-}(a) = \{x \in L :\ x \to a = 1\} = \{x \in L :\ x \leq a\} = [0, a].$$

The positive neighborhood of a is defined for all L-algebras not necessary bounded:

$$U^{+}(a) = \{x \in L :\ a \to x = 1\} = \{x \in L :\ a \leq x\} = [a, 1].$$

The neighborhood of a in a bounded L-algebra L is defined as follow:

$$U(a) = U^{-}(a) \cup U^{+}(a).$$

Example 4.2. *Let $L = \{0, c, a, b, 1\}$ and consider the following table.*

$\to$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

TABLE 3. Definition of $\to$ implies Definition 4.1, Example 5.

Then $(L, \to, 1)$ is an L-algebra.

$$U^{-}(a) = [0, a] = \{0, a, c\} \text{ and } U^{+}(a) = [a, 1] = \{a, 1\},$$
$$U^{-}(c) = [0, c] = \{0, c\} \text{ and } U^{+}(c) = [c, 1] = \{a, b, c, 1\}.$$

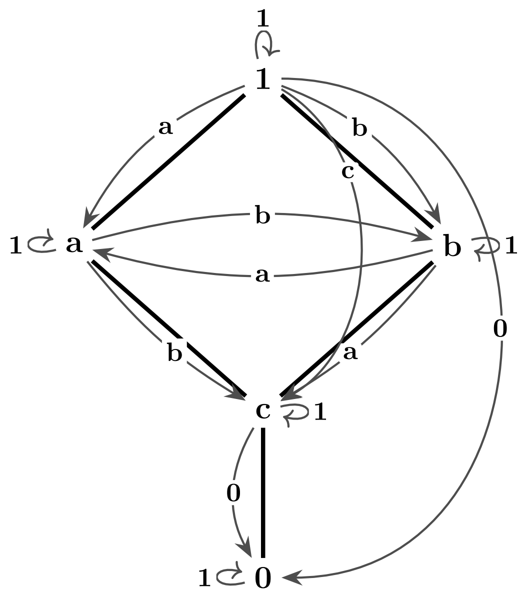

FIGURE 2. Geometric representation of the L-algebra from Example 5. Bold lines represent the order $x \leq y$, while arrows indicate the result of the operation $x \to y$.

Proposition 4.3. *Let L be a KL-algebra and $a \in L$. Then $U^+(a)$ is a subalgebra of L.*

Proof. Let $a \in L$ be fixed and arbitrary. It is clear that $1 \in U^+(a)$. Assume that $x, y \in U^+(a)$. Hence $a \leq x$ and $a \leq y$. Now $a \leq x \to a \leq x \to y$. Then $x \to y \in U^+(a)$. Therefore $U^+(a)$ is a subalgebra of L. □

Proposition 4.4. *The union of two positive neighborhoods of a KL-algebra L is a subalgebra of L.*

Proof. Let $a, b \in L$, clearly $1 \in U^+(a) \cup U^+(b)$. Now let $x, y \in U^+(a) \cup U^+(b)$. Assume that $x \in U^+(a)$ and $y \in U^+(b)$, then $a \leq x \leq y \to x$ and hence $y \to x \in U^+(a) \subseteq U^+(a) \cup U^+(b)$. If $x, y \in U^+(a)$ the proof is clear. □

If L is a KL-algebra and $a \in L$, it is not necessary that $U^+(a)$ is a filter of L. It is enough to consider the following example.

Example 4.5. *Let $L = \{0, a, b, 1\}$ and consider the following table.*

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

TABLE 4. Definition of $\to$ implies Example 6.

Then $(L, \to, 1)$ is a KL-algebra. $U^+(a) = [a, 1] = \{a, b, 1\}$ is not a filter of L because $a \in U^+(a)$, $a \to 0 = b \in U^+(a)$, but $0 \notin U^+(a)$.

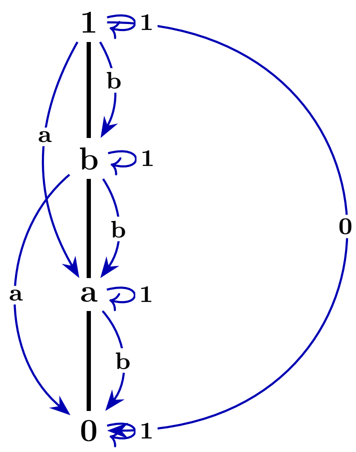

FIGURE 3. Geometric skin of the L-algebra Example 6. The long left arc denotes the mapping $1 \to 0 = 0$.

The following proposition represents the connection between filters of L and some neighborhoods in L.

Proposition 4.6. *Every filter of an L-algebra L can be written as $F = \bigcup_{a \in F} U^+(a)$.*

Proof. Let $a \in F$ and consider $x \in U^+(a)$. Then $x \in F$ and hence $U^+(a) \subseteq F$, for each $a \in F$. Therefore $\bigcup_{a \in F} U^+(a) \subseteq F$. Now let $x \in F$, since $x \leq x$, $x \in U^+(x) \subseteq \bigcup_{a \in F} U^+(a)$. Then $F = \bigcup_{a \in F} U^+(a)$. □

In general case the converse of Proposition 4.4 is not true. Let $L = \{0, a, b, 1\}$ as in Example 5, and $F = \{a, b, 1\}$.

$$U^+(a) = \{a, b, 1\},\ U^+(b) = \{b, 1\} \text{ and } U^+(1) = \{1\}.$$

One can check that $F = U^+(a) \cup U^+(b) \cup U^+(1)$. But F is not a filter of L.

Corollary 4.7. *Let L be an L-algebra and F be a filter of L. Then the followings are satisfied:*
1) $F = \bigcup_{a \in F} U^+(a)$,
2) if $U^+(a) = U^+(b)$ and $a \in F$, then $b \in F$ for each $a, b \in L$,
3) if $a \in F$, then $U^+(a) \subseteq F$ for each $a \in L$.

Proof. The proof is clear. □

One remembers that if L is an L-algebra, then $L \times L$ is an L-algebra. It is enough to consider the following operation:

$$(x, y) \to (z, w) = (x \to z, y \to w).$$

Proposition 4.8. *Let $a, b \in L$, then $U^+(a) \times U^+(b) = U^+((a, b))$.*

Proof. $(x, y) \in U^+(a) \times U^+(b)$ if and only if $a \leq x \leq 1$ and $a \leq y \leq 1$ if and only if $(a, b) \leq (x, y) \leq (1, 1)$ if and only if $(x, y) \in U^+((a, b))$. □

The proposition is true for negative neighborhoods in bounded L-algebras also.

Proposition 4.9. *Let L be a CL-algebra and A be a nonempty subset of L. Then*

$$F = \{x \in L :\ a_n \to (\ldots \to (a_1 \to x) \ldots) = 1,\ \mathit{for\ some}\ a_1, ..., a_n \in A,\ n \in N\}$$

is the smallest filter of L containing A.

Proof. It is obvious that $1 \in F$. Let $x, x \to y \in F$, then there exist $a_1, ..., a_n \in A$ and $b_1, ..., b_m \in A$ such that $a_n \to (\ldots \to (a_1 \to x) \ldots) = 1$ and $b_m \to (\ldots \to (b_1 \to (x \to y))...) = 1$. Now consider the following:

$$1 = b_m \to (\ldots \to (b_1 \to (x \to y)...) = b_m \to (\ldots \to (x \to (b_1 \to y))...) = ... = x \to (b_m \to (b_{m-1} \to ... \to (b_1 \to y))...).$$

Thus $x \leq b_m \to (b_{m-1} \ldots \to (b_1 \to y)...)$.
Now one has the following inequality:

$$1 = a_n \to (... \to (a_1 \to x)...) \leq a_n \to (... \to (a_1 \to (b_m \to (b_{m-1} \to ... \to (b_1 \to y))))...).$$

Then $a_n \to (... \to (b_m \to (... \to (b_1 \to y)...)))...) = 1$, and hence $y \in F$. Therefore F is a filter of L.
If $x \in A$, then $x \to (... \to (x \to x)...) = 1$ and hence $x \in F$. Thus $A \subseteq F$. For the third aim of the proposition assume that E is a filter of L containing A and $x \in F$. Then there exist $a_1, ..., a_n \in A$ such that $a_n \to (... \to (a_1 \to x)...) = 1$. Since $A \subseteq E$, $a_1, ..., a_n \in E$, now $a_n \to (... \to (a_1 \to x)...) = 1 \in E$ and $a_n \in E$ implies that $a_{n-1} \to (... \to (a_1 \to x)...) \in E$. By continuing this way one has $a_1, a_1 \to x \in E$. Therefore $x \in E$ and hence $F \subseteq E$. □

The smallest filter F containing $A \subseteq L$, is called the filter generated by A and is denoted by $F =< A >$.

Corollary 4.10. *Let L be a CL-algebra and $a \in L$, then*

$$< a >= \{x \in L : \ a^n \to x = 1, \ for\ some\ n \in N\}. \ Note\ that$$
$$a^n \to x = a \to (a^{n-1} \to x) \ and\ a^1 \to x = a \to x.$$

Corollary 4.11. *Let L be a self-distributive CL-algebra and $a \in L$, then*

$$< a >= \{x \in L : \ a \to x = 1\}$$

Proposition 4.12. *Let L be a CL-algebra and $a \in L$. Then $U^+(a) \subseteq< a >$.*

Proof. Suppose that $x \in U^+(a)$, then $a \leq x$ and so $a \to x = 1$. Then for $n = 1$, $x \in< a >$. □

Proposition 4.13. *Let L be a CL-algebra and $a \in L$, then the followings are equivalent:* 1) if $a \leq a \to x$, then $a \leq x$ for each $x \in L$,
2) $U^+(a) =< a >$,
3) $U^+(a)$ is a filter of L.

Proof. 1) $\Rightarrow$ 2): By Proposition 4.10, $U^+(a) \subseteq< a >$. Assume that $x \in< a >$, then $a^n \to x = 1$ for some $n \in N$. Then $a \to (a^{n-1} \to x) = 1$ and hence $a \leq a^{n-1} \to x = a \to (a^{n-2} \to x)$. By hypothesis, $a \leq a^{n-2} \to x$. If one continues this way, then it is concluded that $a \leq a \to x$ and hence $a \leq x$. Therefore $x \in U^+(a)$ and $U^+(a) =< a >$.
2) $\to$ 3): it is obvious.
3) $\to$ 1): Suppose that $U^+(a)$ is a filter of L and $a \leq a \to x$, for some $x \in L$. Then $a \to x \in U^+(a)$. On the other hand $a \in U^+(a)$ implies that $x \in U^+(a)$. Then $a \leq x$. □

Corollary 4.14. *Let L be a CL-algebra and $1 \neq a \in L$. If $U^+(a)$ is a filter of L, then $a \to x \neq a$, for each $x \in L$.*

Proof. Suppose that $U^+(a)$ is a filter of L, for $a \neq 1$. If $a \to x = a$, then $a \to x \in U^+(a)$ and hence $a \leq a \to x$. By Proposition 4.11, we obtain that $a \leq x$. Thus $a \to x = 1$ and hence $a = 1$ that is a contradiction. □

Proposition 4.15. *Let L be a self-distributive CL-algebra and $a \in L$. Then $U^+(a)$ is a filter of L.*

Proof. Let $x, x \to y \in U^+(a)$. Then $a \leq x$ and $a \leq x \to y$.

$$a \leq x \Rightarrow x \to y \leq a \to y.$$

Then $a \to (a \to y) = 1$. But

$$a \to y = 1 \to (a \to y) = (a \to a) \to (a \to y) = a \to (a \to y) = 1.$$

Thus $a \leq y$ and hence $y \in U^+(a)$. Therefore $U^+(a)$ is a filter of L. □

Definition 4.16. Let L be an L-algebra and $a \in L$. Define a binary relation θ_a on L by

$$\theta_a = \{(x, y) \in L \times L : \ a \leq x \to y \ and \ a \leq y \to x\}.$$

Example 4.17. *Let* $L = \{0, a, b, 1\}$ *be as in Example 6.*
$\theta_a = \{(0,0),(a,a),(b,b),(1,1),(0,a),(a,0),(0,b),(b,0),(a,b),(b,a),(a,1),(1,a),(b,1),(1,b)\} = L \times L - \{(0,1),(1,0)\}$.

Proposition 4.18. *Let* L *be a self-distributive* CL*-algebra and* $a \in L$*. Then* θ_a *is a congruence relation on* L*.*

Proof. One knows that $U^+(a)$ is a filter of L. Clearly θ_a is reflexive and symmetric. Now let $(x, y) \in \theta_a$ and $(y, z) \in \theta_a$. Then $a \leq x \to y$ and $a \leq y \to x$. Furthermore $a \leq y \to z$ and $a \leq z \to y$. Since L is a CL-algebra, $a \leq y \to z \leq x \to (y \to z) = (x \to y) \to (x \to z) \leq a \to (x \to z)$. By Proposition 4.11, $a \leq x \to z$. Similarly one can prove that $a \leq z \to x$. Therefore $(x, z) \in \theta_a$. Then θ_a is an equivalence relation. Now let $(x, y) \in \theta_a$ and $(u, v) \in \theta_a$. Then $a \leq x \to y$ and $a \leq y \to x$. Furthermore $a \leq u \to v$ and $a \leq v \to u$.
Then $a \leq x \to y \leq u \to (x \to y) = (u \to x) \to (u \to y)$. Similarly one can show that $a \leq (u \to y) \to (u \to x)$. Hence $(u \to x, u \to y) \in \theta_a$. Since L is a self-distributive CL-algebra (and then self-distributive KL-algebra),

$$(u \to v) \to ((v \to y) \to (u \to y)) = (v \to y) \to ((u \to v) \to (u \to y)) = (v \to y) \to (u \to (v \to y)) = 1.$$

$$1 = (u \to v) \to (u \to v) \leq (u \to v) \to ((v \to y) \to (u \to y)) = (v \to y) \to ((u \to v) \to (u \to y)).$$

Then $(v \to y) \to ((u \to v) \to (u \to y)) = 1$. Since $a \leq u \to v$,

$$a \leq 1 = (v \to y) \to ((u \to v) \to (u \to y)) \leq (v \to y) \to (a \to (u \to y)) = a \to ((v \to y) \to (u \to y)).$$

Now one has

$$a \leq (v \to y) \to (u \to y) \text{ and } a \leq (u \to y) \to (v \to y).$$

Hence $(u \to y, v \to y) \in \theta_a$.

Since θ_a is an equivalence relation, we conclude that $(u \to x, v \to y) \in \theta_a$. Therefore θ_a is a congruence relation on L. □

5. Conclusion

In this work, we have bridged the gap between the discrete, symbolic world of L-algebras and the continuous, geometric nature of topological spaces. By carefully re-evaluating the foundational properties of L-algebras, we established a rich framework for irreducible filters and demonstrated their deep, structural synergy with point neighborhood systems. Crucially, our findings show

how filters can be reconstructed through unions of these localized neighborhoods and under specific conditions, how filters themselves manifest directly as neighborhoods. By endowing an L-algebra with the notions of neighborhoods and open sets, this paper moves beyond classical algebraic logic toward a true L-geometry. Unifying the "algebraic skeleton" provided by filter-induced congruences with the topological skin of neighborhood systems creates a powerful, hybrid toolkit. Because L-algebras are intrinsically tied to the quantum Yang-Baxter equation, this topological evolution paves the way for concrete applications in topological quantum computation, quantum symmetry, and invariant theory. Looking forward, this L-geometry opens several compelling avenues for future research. Immediate next steps include fully characterizing the topological properties such as separation axioms, compactness, and connectedness induced by these neighborhood foundations, as well as exploring continuous homomorphisms between L-algebraic spaces. Ultimately, by shifting L-algebras from static symbolic systems to dynamic geometric building blocks, we invite further exploration at the vibrant intersection of non-classical logic, quantum topology, and abstract geometry.

6. Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

References

[1] Aaly Kologani, M. (2023). Relations between L-algebras and other logical algebras. *J. Algebr. Hyperstruct. Log. Algebras*, **4**(1), 27–46. https://doi.org/10.61838/kman.jahla.4.1.3

[2] Chakraborty, M. K., & Sen, J. (1998). MV-algebra embedded in a CL-algebra. *Int. J. Approx. Reasoning*, **18**(3–4), 217–229. https://doi.org/10.1016/S0888-613X(98)00007-3

[3] Ciungu, L. C. (2022). The Category of L-algebras. *Trans. Fuzzy Sets Syst.*, **1**(2), 142–159. https://doi.org/10.52547/tfss.1.2.142

[4] Heyting, A. (1930). Die formalen Regeln der intuitionistischen Logik. *Sitzungsberichte der Preußischen Akademie der Wissenschaften, Physikalisch-Mathematische Klasse*, 42–56, 57–71, 158–169.

[5] Hua, X. (2021). State L-algebras and derivations of L-algebras. *Soft Comput.*, **25**(6), 4201–4212. https://doi.org/10.1007/s00500-020-05437-0

[6] Mao, L., Xin, X., & Zhang, S. (2023). The extension of L-algebras and states. *Fuzzy Sets Syst.*, **455**, 35–52. https://doi.org/10.1016/j.fss.2022.10.007

[7] Rump, W. (2005). A decomposition theorem for square-free unitary solutions of the quantum Yang–Baxter equation. *Adv. Math.*, **193**(1), 40–55. https://doi.org/10.1016/j.aim.2004.03.019

[8] Rump, W. (2020). L-algebras in logic, algebra, geometry, and topology. *Soft Comput.*, **24**(5), 3077–3085. https://doi.org/10.1007/s00500-019-04176-x

[9] Rump, W. (2008). L-algebras, self-similarity, and l-groups. *J. Algebra*, **320**(6), 2328–2348. https://doi.org/10.1016/j.jalgebra.2008.06.012

[10] Rump, W., & Yang, Y. (2012). Intervals in l-groups as L-algebras. *Algebra Universalis*, **67**(2), 121–130. https://doi.org/10.1007/s00012-012-0176-x

[11] Wang, J., Wu, Y., & Yang, Y. (2020). Basic algebras and L-algebras. *Soft Comput.*, **24**(19), 14327–14332. https://doi.org/10.1007/s00500-020-04797-1

Mohamad Ebrahimi
Orcid number: 0000-0003-4830-7100
Department of Mathematics
Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
Email address: mohamad_ebrahimi@uk.ac.ir

Choonkil Park
Orcid number: 0000-0001-6329-8288
Research Institute for Convergence of Basic Science
Hanyang University
Seoul, Korea
Email address: baak@hanyang.ac.kr